ON THE CAUCHY–NICOLETTI WEIGHTED PROBLEM FOR HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. The unimprovable in a certain sense conditions are established which, respectively, ensure the solvability and well-posedness of the weighted Cauchy–Nicoletti problem for higher order nonlinear singular differential equations.

Let $-\infty < a < b < +\infty$, $n \geq 2$ be a natural number and $f$ be an operator defined on some set $D(f) \subset C^{n-1}([a, b])$ and mapping $D(f)$ onto $L([a, b])$. We consider the functional differential equation

$$u^{(n)}(t) = f(u)(t)$$

with the Cauchy–Nicoletti weighted conditions

$$\limsup_{t \to t_i} \left( \frac{|u^{(i-1)}(t)|}{\rho_i(t)} \right) < +\infty \quad (i = 1, \ldots, n).$$

Here $t_i \in [a, b] \; (i = 1, \ldots, n)$ and $\rho_i : [a, b] \to [0; +\infty] \; (i = 1, \ldots, n)$ are continuous functions such that

$$\rho_n(t_n) = 0, \quad \rho_n(t) > 0 \quad \text{for} \quad t \neq t_n, \quad \rho_i(t_i) = 0,$$

$$\left| \int_{t_i}^{t} \rho_{i+1}(s) \, ds \right| \leq \rho_i(t) \quad \text{for} \quad a \leq t \leq b \; (i = 1, \ldots, n-1).$$

By $C^{n-1}_{\rho_1, \ldots, \rho_n}([a, b])$ we denote a set of functions $u \in C^{n-1}([a, b])$ such that

$$\mu(u) = \max \{ \mu_1(u), \ldots, \mu_n(u) \} < +\infty,$$

where
\[ \mu_i(u) = \sup \left\{ \frac{|u^{(i-1)}(t)|}{\rho_i(t)} : a \leq t \leq b, \ t \neq t_i \right\}. \]

For an arbitrary \( x > 0 \), assume
\[ C_{\rho_1, \ldots, \rho_n}^{n-1}(\[a, b\]) = \left\{ u \in C_{\rho_1, \ldots, \rho_n}(\[a, b\]) : \mu(u) \leq x \right\}. \]

We investigate the problem (1), (2) in the case, where
\[ f^*(\rho_1, \ldots, \rho_n; x)(t) = \sup \left\{ |f(u)(t)| : u \in C_{\rho_1, \ldots, \rho_n}^{n-1}(\[a, b\]) \right\}. \]

and for any \( x > 0 \) the conditions
\[ f : C_{\rho_1, \ldots, \rho_n}^{n-1}(\[a, b\]) \rightarrow L(\[a, b\]) \] is continuous
are fulfilled.

Of special interest is the case, where
\[ D(f) \neq C^{n-1}(\[a, b\]). \]

In this sense the equation (1) is singular one.

In the case, where \( f \) is the Nemytski’s operator, i.e., when
\[ f(u)(t) = f_0(t, u(t), \ldots, u^{(n-1)}(t)), \]
where \( f : ([a, b] \setminus \{t_1, \ldots, t_n\}) \times \mathbb{R}^n \rightarrow \mathbb{R} \) is the function satisfying the local Carathéodory conditions, the problems of the type (1), (2) are investigated thoroughly (see [1]–[6] and references therein). The problem (1), (2) is also investigated in the case, where
\[ f(u)(t) = f_0(t, u(t_1), \ldots, u^{(n-1)}(t_n)); \]
\[ t_1 = \cdots = t_n \text{ and } \rho_{i+1}(t) = \rho'_i(t) \ (i = 1, \ldots, n) \]
(see [7]–[9]).

However, the problem mentioned above remains still little studied in a general case. Just this case we consider in the present paper.

The function \( u \in D(f) \) with an absolutely continuous \( (n-1) \)th derivative is said to be a solution of the equation (1) if it almost everywhere on \([a, b]\) satisfies this equation.

A solution of the equation (1) satisfying the boundary conditions (2) is called a solution of the problem (1), (2).
Theorem 1. Let the conditions (3) and (4) be fulfilled, and there exist constants \( \alpha \in [0, 1] \) and \( x_0 > 0 \) such that
\[
\left| \int_{t_n}^t f^*(\rho_1, \ldots, \rho_n; x)(s) \, ds \right| \leq \alpha \rho_n(x) \quad \text{for } a \leq t \leq b, \ x \geq x_0. \tag{5}
\]
Then the problem (1), (2) has at least one solution.

Corollary 1. Let there exist integrable functions \( p \) and \( q : [a, b] \to [0; +\infty[ \) such that
\[
\sup \left\{ \left| \int_{t_n}^t p(s) \, ds \right| / \rho_n(t) : \ a \leq t \leq b, \ t \neq t_n \right\} < 1, \tag{6}
\]
\[
\sup \left\{ \left| \int_{t_n}^t q(s) \, ds \right| / \rho_n(t) : \ a \leq t \leq b, \ t \neq t_n \right\} < +\infty \tag{7}
\]
and for any \( u \in C^{n-1}_{\rho_1, \ldots, \rho_n}([a, b]) \) almost everywhere on \( [a, b] \) the condition
\[
|f(u)(t)| \leq \rho(t) \mu(u) + q(t)
\]
is fulfilled. Then the problem (1), (2) has at least one solution.

Along with the problem (1), (2) we consider the perturbed problem
\[
v^{(n)}(t) = f(v)(t) + h(t), \tag{8}
\]
\[
\limsup_{t \to t_i} \left( \frac{|v^{(i-1)}(t)|}{\rho_n(t)} \right) < +\infty \ (i = 1, \ldots, n), \tag{9}
\]
where \( h : [a, b] \to \mathbb{R} \) is the integrable function such that
\[
\mu_0(h) = \sup \left\{ \left| \int_{t_n}^t h(s) \, ds \right| / \rho_n(t) : \ a \leq t \leq b, \ t \neq t_n \right\} < +\infty. \tag{10}
\]

Definition 1. The problem (1), (2) is said to be well-posed if for any integrable function \( h : [a, b] \to \mathbb{R} \) satisfying the condition (10), the problem (8), (9) is uniquely solvable, and there exists an independent of \( h \) positive constant \( r \) such that
\[
\mu(u - v) \leq r \mu_0(h),
\]
where \( u \) and \( v \) are, respectively, the solutions of the problems (1), (2) and (8), (9).

Theorem 2. Let there exist an integrable function \( p : [a, b] \to [0, +\infty[ \) satisfying the inequality (6) such that for any \( u \) and \( v \in C^{n-1}_{\rho_1, \ldots, \rho_n}([a, b]) \) almost everywhere on \( [a, b] \) the condition
\[
|f(u)(t) - f(v)(t)| \leq p(t) \mu(u - v)
\]
is fulfilled. If, moreover, the inequality (7), where \( q(t) \equiv |f(0)(t)| \), is fulfilled, then the problem (1), (2) is well-posed.
Note that the condition (5) in Theorem 1, where \( \alpha \in [0, 1] \), is unimprovable and it cannot be replaced by the condition
\[
\left| \int_{t_n}^{t} f^*(\rho_1, \ldots, \rho_n; x)(s) \, ds \right| \leq \rho_n(t)x \quad \text{for } a \leq t \leq b, \ x \geq x_0.
\]

Similarly, in Corollary 1 and in Theorem 2, the strict inequality (6) cannot be replaced by the nonstrict inequality
\[
\sup \left\{ \left| \int_{t_n}^{t} p(s) \, ds \right| / \rho_n(t) : \ a \leq t \leq b, \ t \neq t_n \right\} \leq 1.
\]

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