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ON A NONLOCAL BOUNDARY VALUE PROBLEM FOR
TWO-DIMENSIONAL NONLINEAR SINGULAR
DIFFERENTIAL SYSTEMS

Abstract. For two-dimensional nonlinear differential systems with strong
singularities with respect to a time variable, unimprovable sufficient con-
ditions for solvability and well-posedness of the Nicoletti type nonlocal
boundary value problem are established.

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Let \(-\infty < a < b < +\infty\), \(C([a,b])\) be the space of continuous functions
\(u : [a,b] \rightarrow \mathbb{R}\) with finite right and left limits \(u(a+)\) and \(u(b-)\) at the points \(a\)
and \(b\) and with the norm \(\|u\|_C = \sup \{|u(t)| : a < t < b\}\), and let \(L^2([a,b])\) be the space of square integrable functions \(u : [a,b] \rightarrow \mathbb{R}\) with the norm
\[\|u\|_{L^2} = \left( \int_a^b u^2(t) \, dt \right)^{1/2}.\]

By \(C^{1,2}_0([a,b]; \mathbb{R}^2)\) we denote the space of vector functions \((u_1, u_2) : [a,b] \rightarrow \mathbb{R}^2\) with continuously differentiable components \(u_1\) and \(u_2\), satisfying the conditions
\[u_1(a+) = 0, \quad \int_a^b (u_1^2(t) + u_2^2(t)) \, dt < +\infty.\]

We consider the nonlinear differential system
\[\frac{du_1}{dt} = f_1(t, u_2), \quad \frac{du_2}{dt} = f_2(t, u_1)\]
with the Nicoletti type nonlocal boundary conditions
\[u_1(a+) = 0, \quad u_2(b-) = \varphi(u_1, u_2).\]
Here \( f_1 : [a, b] \times \mathbb{R} \to \mathbb{R} \) and \( f_2 : [a, b] \times \mathbb{R} \to \mathbb{R} \) are continuous functions, and \( \varphi : C([a, b]) \times L^2([a, b]) \to \mathbb{R} \) is a continuous functional.

A vector function \((u_1, u_2) : [a, b] \to \mathbb{R}^2\) is said to be a solution of the problem (1), (2) if:

(i) \( u_1 \) and \( u_2 \) are continuously differentiable and satisfy the system (1) at every point of the interval \([a, b]\);

(ii) \( u_1 \in C([a, b]), \ u_2 \in L^2([a, b]) \), and the equalities (2) are satisfied.

In the present paper, unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability of (1), (2) in the space \( C^{1,2}_0([a, b]; R^2) \) and the stability of its solution with respect to small perturbations of right-hand sides of (1) and the functional \( \varphi \). In contrast to the results from [2]–[6], concerning the solvability and well-posedness of the Nicoletti type problems, the theorems below cover the case, where the system (1) with respect to a time variable has a strong singularity at the point \( a \) in the Agarwal–Kiguradze sense [1], i.e., the case, where

\[
\int_a^b (t-a) \left( |f_2(t,x)| - f_2(t,x) \text{sgn}(x) \right) dt = +\infty \quad \text{for} \quad x \neq 0.
\]

Along with the problem (1), (2) we consider the auxiliary problem

\[
\begin{align*}
\frac{du_1}{dt} &= \lambda f_1(t, u_2), \\
\frac{du_2}{dt} &= \lambda \delta(t) f_2(t, u_1),
\end{align*}
\]

(3)

dependent on a parameter \( \lambda \in [0, 1] \) and an arbitrary continuous function \( \delta : [a, b] \to [0, 1] \).

**Theorem 1 (A principle of a priori boundedness).** Let there exist a nonnegative function \( g \in L^2([a, b]) \) and a positive constant \( \rho \) such that

\[
|f_1(t, x)| \leq g(t)(1 + |x|) \quad \text{for} \quad a < t < b, \ x \in \mathbb{R},
\]

and for any number \( \lambda \in [0, 1] \) and an arbitrary continuous function \( \delta : [a, b] \to [0, 1] \) every solution \((u_1, u_2)\) of the problem (3), (4) admits the estimate

\[
\|u_1\|_{L^2} + \|u_2\|_{L^2} < \rho.
\]

Then the problem (1), (2) has at least one solution in the space \( C^{1,2}_0([a, b]; R^2) \).

Consider now the case, where

\[
\varphi(u_1, u_2) \text{sgn}(u_1(1-)) \leq \alpha_0 + \alpha_1 \|u_1\|_{L^2} + \alpha_2 \|u_2\|_{L^2} \quad \text{for} \quad (u_1, u_2) \in C^{1,2}_0([a, b]; R^2),
\]

(5)

and in the domain \([a, b] \times \mathbb{R}\) the inequalities

\[
\ell_0 |x| \leq \left| f_1(t, x) - f_1(t, 0) \right| \text{sgn}(x) \leq \ell_1 |x|,
\]

(6)

\[
\left| f_2(t, x) - f_2(t, 0) \right| \text{sgn}(x) \geq -\frac{\ell}{(t-a)^2} |x|
\]

(7)

are fulfilled.
On the basis of Theorem 1, the following theorem can be proved.

**Theorem 2.** Let

\[
\int_a^b f_1(t,0) \, dt < +\infty, \quad \int_a^b (t-a)^{1/2} |f_2(t,0)| \, dt < +\infty,
\]

and let the conditions (5)–(7) hold, where \(\alpha_i \geq 0 \ (i = 0, 1, 2)\), \(\ell_k > 0 \ (k = 0, 1)\), and \(\ell \geq 0\) are constants such that

\[
(b-a)^{1/2} (\alpha_1 \ell_1 + \alpha_2) \ell_1 + 4\ell_1^2 \ell < \ell_0.
\]

Then the problem (1), (2) has at least one solution in the space \(C^{1,2}_0[a,b;\mathbb{R}^2]\).

Particular case of the boundary conditions (2) are the multi-point boundary conditions

\[
u_1(a+) = 0, \quad \nu_2(b-) = \sum_{k=1}^{n-1} \beta_k \nu_1(t_k) + \beta_n \nu_1(b-) + \beta_0,
\]

where \(\beta_k \in R \ (k = 0, \ldots, n)\).

Suppose

\[
[\beta_n]_+ = \frac{1}{2} (|\beta_n| + \beta_n).
\]

From Theorem 2 it follows

**Corollary 1.** Let the conditions (6)–(8) be satisfied, where \(\ell_k > 0 \ (k = 0, 1)\) and \(\ell \geq 0\) are constants such that

\[
2 \pi (b-a)^{1/2} \left( \sum_{k=1}^{n-1} |\beta_k| (t_k - a)^{1/2} + [\beta_n]_+ (b-a)^{1/2} \right) \ell_1^2 + 4\ell_1^2 \ell < \ell_0.
\]

Then the problem (1), (10) has at least one solution in the space \(C^{1,2}_0[a,b;\mathbb{R}^2]\).

Now we consider the perturbed problem

\[
\begin{align*}
\frac{dv_1}{dt} &= f_1(t, v_2) + q_1(t), \\
\frac{dv_2}{dt} &= f_2(t, v_1) + q_2(t),
\end{align*}
\]

and we introduce the following

**Definition.** The problem (1), (2) is said to be well-posed in the space \(C^{1,2}_0[a,b;\mathbb{R}^2]\) if it has a unique solution \((u_1, u_2)\) in that space and there exists a positive constant \(r\) such that for any continuous functions \(q_1, q_2 : a, b \rightarrow R \ (i = 1, 2)\), satisfying the condition

\[
\nu(q_1, q_2) = \left( \int_a^b q_1^2(t) \, dt \right)^{1/2} + \int_a^b (t-a)^{1/2} |q_2(t)| \, dt < +\infty,
\]

and for any real number \(\alpha\), the problem (12), (13) has at least one solution \((v_1, v_2) \in C^{1,2}_0[a,b;\mathbb{R}^2]\), and every such solution admits the estimate

\[
\|v_1 - u_1\|_{L^2} + \|v_2 - u_2\|_{L^2} \leq r (\nu(q_1, q_2) + |\alpha|).
\]
Theorem 3. Let
\[ \varphi(u_1, u_2) \text{sgn}(u_1(b-)) \leq \alpha_1 \|u_1\|_{L^2} + \alpha_2 \|u_2\|_{L^2} \]
for \((u_1, u_2) \in C^{1,2}_0([a, b]; \mathbb{R}^2)\), and let in the domain \([a, b] \times \mathbb{R}\) the conditions
\[ \ell_0|x| \leq f_1(t, x) \text{sgn}(x) \leq \ell_1|x|, \]
\[ f_2(t, x) \text{sgn}(x) \geq -\frac{\ell}{(t-a)^2} |x| \]
be fulfilled, where \(\alpha_i \geq 0 \ (i = 1, 2)\), \(\ell_k > 0 \ (k = 0, 1)\), and \(\ell \geq 0\) are constants, satisfying the inequality (9). Then the problem (1), (2) is well-posed in the space \(C^{1,2}_0([a, b]; \mathbb{R}^2)\).

In the case, where the boundary conditions (2) have the form
\[ u_1(a+) = 0, \quad u_2(b-) = \sum_{k=1}^{n-1} \beta_k u_1(t_k) + \beta_n u_1(b-), \]
(16)
then Corollary 2 yields
\[ \text{Corollary 2. Let in the domain \([a, b] \times \mathbb{R}\) the conditions (14) and (15) be satisfied, where \(\ell_k > 0 \ (k = 0, 1)\) and \(\ell \geq 0\) are constants, satisfying the inequality (11). Then the problem (1), (16) is well-posed in the space \(C^{1,2}_0([a, b]; \mathbb{R}^2)\).} \]

As an example, we consider the differential system
\[ \frac{du_1}{dt} = p_1(t, u_2)u_2, \quad \frac{du_2}{dt} = \frac{p_2(t, u_1)}{(t-a)^2} u_1, \]
(17)
where \(p_1 : [a, b] \times \mathbb{R} \to \mathbb{R}\) and \(p_2 : [a, b] \times \mathbb{R} \to \mathbb{R}\) are continuous functions. For this system from Corollary 2 it follows
\[ \text{Corollary 3. Let in the domain \([a, b] \times \mathbb{R}\) the conditions} \]
\[ \ell_0 \leq \ell_1 \leq \ell_2, \quad p_2(t, x) \geq -\ell \]
be satisfied, where \(\ell_i > 0 \ (i = 0, 1)\) and \(\ell \geq 0\) are constants, satisfying the inequality (11). Then the problem (17), (16) is well-posed.

Remark 1. If the conditions of Corollary 3 are satisfied and in the domain \([a, b] \times \mathbb{R}\) the inequality
\[ p_2(t, x) \leq -\ell \]
holds, where \(\ell\) is a positive constant, then the system (17) with respect to a time variable has a strong singularity at the point \(a\) in the Agarwal–Kiguradze sense.

Remark 2. The condition (9) in Theorems 2 and 3 is unimprovable and it cannot be replaced by the condition
\[ (b-a)^{1/2}(\alpha_1 \ell_1 + \alpha_2)\ell_1 + 4\ell^2 \ell \leq \ell_0. \]
Also, the strict inequality (11) in Corollaries 1–3 cannot be replaced by the non-strict one
\[
\frac{2}{\pi} (b-a)^{1/2} \left( \sum_{k=1}^{n-1} |\beta_k|(t_k-a)^{1/2} + [\beta_n]+(b-a)^{1/2} \right) \ell^2 + 4\ell^2 \ell \leq \ell_0.
\]

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