RAINBOW HAMILTONIAN PATHS AND CANONICALLY COLORED SUBGRAPHS IN INFINITE COMPLETE GRAPHS

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Abstract: A sufficient condition is given for the existence of a Hamiltonian path all of whose edges have a distinct color, in edge-colored infinite complete graphs. Also, a variant of the Erdős-Rado theorem is presented for canonically colored subgraph.

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0. Introduction

In this note we consider complete graphs $K = (X, E)$ with an infinite vertex set $X$ and edge set $E = \{xx' : x, x' \in X\}$. For a given coloring $\varphi$ of the edge set, a subgraph $G \subset K$ is called a rainbow subgraph of $K$ if $G|_{\varphi}$, the coloring of $G$ induced by $\varphi$, contains no monochromatic pair of edges.

If $Y \subset X$ and $G$ is the complete subgraph induced by $Y$ in $K$, then we write $Y|_{\varphi}$ instead of $G|_{\varphi}$.

Our first aim is to find a condition ensuring the existence of a rainbow Hamiltonian path (i.e., a path visiting all vertices of $K$) when $X$ is countable. As shown in Theorem 1, it is enough to exclude canonically colored infinite subgraphs (see definition below) from $K|_{\varphi}$, provided that at each vertex, each color class has a finite or 0-measure infinite degree. This result generalizes a theorem of Hahn and Thomassen [6]. Examples show that the condition in Theorem 1 is nearly the best possible; it would be interesting, however, to see an "if and only if"-type characterization, in terms of forbidden subgraphs (cf. Problem 6).

In the second part of the paper we investigate the question how large canonically colored subgraphs exist in $K$ when $X$ is an ordered set of arbitrary cardinality. We consider a particular class of (so-called "properly ordered") colorings and show that if rainbow triangles are forbidden in $K|_{\varphi}$ then there can be found a canonically colored complete subgraph on a vertex set of cardinality $|X|$ (Theorem 3). The exclusion of a rainbow $K_4$, however, is not sufficient, as shown by a suitable coloring for $X = \mathbb{R}$ (the set of real numbers).

1. Rainbow Hamiltonian paths in $K_\omega$

Throughout this section, $K$ denotes the countable complete graph with vertex set $X = \{x_1, x_2, \ldots\}$ and edge set $E = \{x_ix_j : i \neq j\}$. We assume there is a 0-1 measure $\mu$ on $X$, i.e., for every $Y \subset X$, $\mu(Y) \in \{0, 1\}$, $\mu$ is finitely additive, $\mu(X) = 1$, and $\mu(Y) = 0$ for all
finite $Y \subset X$.

For convenience, we denote the colors by integers $1, 2, \ldots$. Two colorings $\varphi, \varphi'$ of a graph $G$ are said to be isomorphic if $\varphi'$ can be obtained from $\varphi$, as well as $\varphi$ from $\varphi'$, by renumbering (but not indentifying) the colors. In this sense, two edge-colored graphs $G_1, G_2$ are isomorphic if for their colorings $\varphi_1, \varphi_2$ we have $G_1|_{\varphi_1} \cong G_2|_{\varphi_2}$, i.e., there is a one-to one mapping between the vertex sets $V(G_1)$ and $V(G_2)$, yielding the isomorphism of $\varphi_1$ und $\varphi_2$.

Denote by $Z^*$ the complete graph with a countable vertex set $\{z_0, z_1, z_2, \ldots \}$ and having the (canonical) edge coloring in which $z_i z_j$ has color $j$ whenever $i < j$.

**Theorem 1.** Suppose $\varphi : E \to \mathbb{N}$ is a coloring of $K$, such that for each vertex $x_i$ and each color $j$, the vertices adjacent to $x_i$ by an edge of color $j$ form a set of measure 0. If $K|_{\varphi}$ contains no subgraph isomorphic to $Z^*$ then $K$ has a one-way infinite and a two-way infinite rainbow Hamiltonian path.

**Proof.** We construct a sequence $P_1, P_2, \ldots$ of (finite) rainbow paths with the following properties: $x_i \in P_i$ for all $i \geq 1$, and $P_i \subset P_{i+1}$ in the sense that all edges of $P_i$ are edges of $P_{i+1}$ too. This clearly implies that $\cup P_i$ is a rainbow Hamiltonian path of $K$.

Let $P_1 = (x_1), P_2 = (x_1 x_2)$. If the Hamiltonian path to be found is one-way infinite then we extend $P_i$ at the end different from $x_1$; if it should be two-way infinite, we extend $P_i$ at the end being closer to $x_1$.

Suppose $P_i$ is a rainbow path covering $\{x_1, \ldots, x_i\}$. If $x_{i+1} \in P_i$ define $P_{i+1} = P_i$. Otherwise, denote by $y_j$ the $j^{th}$ vertex of $P_i$, i.e., $P_i = (y_1 y_2 \ldots y_k)$ where $k = |P_i|$. Set $Y = X \setminus (\{x_{i+1}\} \cup \{y_1, \ldots, y_k\})$.

Delete all vertices $y$ from $Y$, for which $\varphi(y y)$ or $\varphi(x_{i+1} y)$ appears on some edge of $P_i$. The resulting vertex set $Y'$ has $\mu(Y') = 1$, since each of the $k - 1$ colors appearing in $P_i$ defines a neighborhood of $x_{i+1}$ and $y_k$ of measure 0 (and $\mu$ is finitely additive). If there is a $y \in Y'$ such that $\varphi(x_{i+1} y) \neq \varphi(y y)$ then $P_{i+1} = (y_1 \ldots y_k y x_{i+1})$ is a rainbow path containing $x_{i+1}$. Otherwise, $\varphi(x_{i+1} y) = \varphi(y y)$ for all $y \in Y'$.

Let $Y_1 \cup Y_2 \cup \ldots = Y'$ be the partition of $Y'$ in which two vertices $y$ and $y'$ belong to the same class if and only if $\varphi(y y) = \varphi(y' y')$. Then $\mu(Y_m) = 0$ for all $m \geq 1$.

Choose an arbitrary $y' \in Y'$, and delete all $y$ from $Y'$ for which $\varphi(y' y)$ appears in $P_i$ or is identical to $\varphi(y y')$. The set of those $y$
is of measure 0, so that the resulting set $Y''$ has $\mu(Y'') = 1$. If, for some $y'' \in Y''$, $\varphi(y''y') \neq \varphi(y''y_k)$ then $P_{i+1} = (y_1 \ldots y_k y'' y' x_{i+1} )$ is a rainbow path containing $x_{i+1}$ and we are home. Otherwise, choose a $y'' \in Y''$ and repeat the same argument. Either a rainbow $P_{i+1}$, containing $x_{i+1}$, is found after a finite number of steps, or an infinite sequence $y', y'', y''', \ldots$ of vertices is defined with the property that $\varphi(y^{(p)}y^{(q)}) = \varphi(y_k y^{(q)})$ for all $p < q$. In the latter case, however, those vertices would induce a subgraph isomorphic to $Z^*$, contradicting our assumptions, so that $P_i$ can be extended to a rainbow path $P_{i+1}$, for all $i$.

\[ \Diamond \]

**Corollary 1.1.** (Hahn and Thomassen [6]) *If all monochromatic subgraphs are locally finite in a $Z^*$-free coloring of $K$, then $K$ contains a rainbow Hamiltonian path.*

\[ \Diamond \]

An interesting particular case is when any two edges of the same color in $K|_\varphi$ are vertex-disjoint. Such a $\varphi$ is called a **proper edge coloring** of $K$.

**Corollary 1.2.** *Every proper edge coloring of $K$ contains a rainbow Hamiltonian path.*

\[ \Diamond \]

Though $Z^*$ itself contains a rainbow Hamiltonian path, it is very close to being non-Hamiltonian in the following sense. Denote by $Z^\Delta$ the graph which is obtained from $Z^*$ by recoloring the edge $z_0 z_1$ to color 2.

**Proposition 2.** *The graph $Z^\Delta$ contains no rainbow Hamiltonian paths.*

Based on a similar idea, the following more general class of examples can be given. Consider an arbitrary complete graph $K_n$ on $n$ vertices, with a coloring $\varphi_n$ which does not contain a rainbow Hamiltonian path. Suppose $\varphi_n$ uses colors $1', 2', \ldots$, none of them appearing among the colors $1, 2, \ldots$. Replace $z_0$ by $K_n|_{\varphi_n}$ in $Z^*$, and define the edge $z_i y$ to have color $i$, whenever $y \in V(K_n)$ and $i \geq 1$. Denote this edge-colored graph by $Z^*(\varphi_n)$. Now Proposition 2 can be stated in the following stronger form.
Proposition 2'. If $K_n|\varphi_n$ contains no rainbow Hamiltonian path then neither does $Z^*(\varphi_n)$.

Proof. Suppose to the contrary that $P$ is a rainbow Hamiltonian path in $Z^*(\varphi_n)$. Then the vertices of $K_n$ induce at least two subpaths $P_1, P_2$ (both maximal under inclusion) in $P$. We may assume all vertices between $P_1$ and $P_2$ belong to $Z^* z_0$. Let $z_m$ be the vertex between $P_1$ and $P_2$ in $P$ having maximum subscript. Then the two neighbors of $z_m$ in $P$ are adjacent to $z_m$ by edges of color $m$, contradicting the assumption that $P$ is rainbow.

In particular, any coloring of $K_n$ with at most $n-2$ colors satisfies the assumptions on $\varphi_n$.

2. Canonically colored subgraphs

In this section we consider infinite complete graphs $K = (X, E)$ with a vertex set $X$ of arbitrary cardinality. We assume there is an ordering $<$ given on $X$.

Erdős and Rado [2] proved that every coloring $\varphi$ of $K$ contains an infinite $Y \subset X$ such that $Y|\varphi$ is rainbow or monochromatic or, $\varphi(yy') = \varphi(yy'')$ either for all $y < y' < y''$ or for all $y'' < y' < y$ ($y, y', y'' \in Y$).

Call a $Y \subset X$ canonically colored if for all $y, y', y'' \in Y$, $y < y' < y''$, $\varphi(yy') = \varphi(yy'')$. We are interested in the question how large canonically colored complete subgraphs must exist in $K|\varphi$. The following particular class of colorings will be considered. We say that $\varphi$ is properly ordered if $\varphi(xx') \neq \varphi(xx'')$ whenever $x'' < x' < x(x, x', x'' \in X)$.

Theorem 3. Let $\varphi$ be a properly ordered coloring of $K$, not containing rainbow triangles. Then there is a $Y \subset X$, $|Y| = |X|$, such that $Y|\varphi$ is canonically colored.

Proof. For any three elements $x, y, z \in X$, $x < y < z$, either $\varphi(xy) = \varphi(xz)$ or $\varphi(xy) = \varphi(yz)$, since $\varphi(xz) \neq \varphi(yz)$.

If $X$ contains a maximum element $z_0$ then set $X' = X\{z_0\}$; otherwise, $X' = X$. Now any two monochromatic edges of $X'$ share a
vertex. Indeed, suppose \( \varphi(uv) = \varphi(yz) \). Choose an \( x \in X \) such that \( x > \max(u,v,y,z) \). Then there is an edge of color \( \varphi(uv) \) that joins \( x \) to \( uv \) and also to \( yz \). Those two edges must coincide, however, since we have a properly ordered coloring.

Thus, each monochromatic subgraph of \( X'|_\varphi \) is a star, since monochromatic triangles cannot occur in properly ordered colorings.

Call a monochromatic star non-trivial if it contains at least two edges. Such a star has a (unique) centre, the common vertex of its edges. Observe that every \( x \in X' \) is the centre of at most one (non-trivial) star. Otherwise, let \( \varphi(xy) = \varphi(xy') \neq \varphi(xz) = \varphi(xz') \). Choose a \( w \in X, w > \max(x,y,y',z,z') \). Then \( \varphi(xy) = \varphi(xw) = \varphi(xz) \) should hold, a contradiction. Since each triangle contains a pair of monochromatic edges, there are at most two vertices \( x', x'' \) that are not centres of some star. Set \( X'' = X'\setminus\{x', x''\} \).

Thus, each \( x \in X'' \) is the centre of exactly one non-trivial star \( S_x \). Renumbering the colors, if necessary, we may assume \( S_x \) is colored by color \( x \). We define a partition \( X_1 \cup X_2 = X'' \) as follows: \( x \in X_1 \) if \( y < x \) implies \( \varphi(xy) \neq x \); \( x \in X_2 \) if there is a \( y < x \) with \( \varphi(xy) = x \). The proof will be done if we show \( X_1|_\varphi \) and \( X_2|_\varphi \) are both canonically colored.

Suppose \( x \in X_2 \). If there were a \( z > x \) such that \( \varphi(xz) \neq x \) then \( \varphi(yz) = x \) would follow for any \( y, \varphi(xy) = x \), a contradiction as \( S_y \) cannot have color \( x \). Hence, \( X_2 \) is canonically colored, and \( y \in X_1 \) whenever \( \varphi(xy) \neq y, y < x \).

Suppose \( X_1 \) is not canonically colored, i.e., there are three elements \( x,y,z \in X_1, x < y < z, \varphi(xy) = a \neq b = \varphi(xz) \). Then \( \varphi(yz) = a \) (since \( \varphi \) is properly ordered), so that \( y \in X_2 \), contrary to our assumption.

We note that the above argument yields the following result for the finite case.

**Theorem 3'.** Every properly ordered coloring of \( K_n \) with no rainbow triangle contains a canonically colored \( K_{[n/2]-1} \).

Instead of \( K_3 \), the exclusion of a rainbow \( K_4 \) is not sufficient in Theorem 3. This fact can be proved in the following stronger form. (\( \mathbb{R} \) denotes the set of real numbers.)
Theorem 4. For $X = \mathbb{R}$, there exists a properly ordered coloring $\varphi$ with the following properties:

(i) Every canonically colored $Y$ is countable;
(ii) $X|_{\varphi}$ contains no rainbow finite subgraphs of minimum degree greater than 2. (In particular, $X|_{\varphi}$ is rainbow-$K_4$-free.)

Proof. First, consider the properly ordered (canonical) coloring $\varphi^+$ defined by $\varphi^+(xy) = x$ for all $x < y$. We modify $\varphi^+$ by splitting each color class into two parts, and replacing each color $x$ by two colors $x'$, $x''$. (Clearly, after any kind of splitting, the obtained coloring remains properly ordered.)

The splitting is based on idea due to Sierpiński [5]. Consider a well-ordering $<_L$ of $\mathbb{R}$. For $x < y$, define $\varphi(xy)$ to be $x'$ if $x <_L y$ and to be $x''$ if $y <_L x$. Let $Y|_{\varphi}$ be canonically colored, for some $Y \subset X = \mathbb{R}$. We show $Y$ is countable.

Set $E_x = \{xy : x < y \in Y\}$ for $x \in Y$. If $Y$ is canonically colored then each $E_x$ is monochromatic. Divide $Y$ into two (disjoint) parts $Y_1$, $Y_2$ as follows: $x \in Y_1$ if $E_x$ has color $x'$ and $x \in Y_2$ if $E_x$ has color $x''$. By the definition of $<_L$, for each $x \in Y_1$, the set $\{y \in Y_1 : y > x\}$ contains a minimum element $y_x$. Picking a rational number from the interval $[x, y_x)$, it follows that $Y_1$ is countable. By a similar argument, considering the sets $\{y \in Y_2 : y < x\}$ and the intervals $(y_x, x]$, it follows that $Y_2$ is countable.

Let $G$ be a finite rainbow subgraph of $X|_{\varphi}$, with a vertex set $\{x_1, \ldots, x_n\}$. Then $x = \min x_i$ has degree at most 2, since all edges incident to $x$ in $G$ have color $x'$ or $x''$.

\[\diamond\]

3. Concluding remarks

I. Corollary 1.2 is much easier to prove than Theorem 1. As a matter of fact, in a proper edge coloring, $P_i$ can be extended to a suitable $P_{i+1}$ by adding $x_{i+1}$ and at most one extra vertex. The finite version of Corollary 1.2, however, is unknown. A nice construction of Maamoun and Meyniel [3] shows there is a proper edge coloring of the complete graph $K_n$ on $n = 2^k$ vertices (for all $k \geq 2$) not containing a rainbow Hamiltonian path. It would be interesting to see such colorings for all
even $n$.

On the other hand, Andersen [1] conjectures that every proper edge coloring of $K_n$ contains a rainbow path covering all vertices but one. Some lower bounds on the length of a maximum rainbow path are given by Rödl und Tuza [4]. Here we raise the following related question.

**Problem 5.** Find the minimum number $f(n)$ of colors, such that every proper edge coloring of $K_n$ by at least $f(n)$ colors contains a rainbow Hamiltonian path.

The examples of [3] show $f(n) \leq n - 1$ does not hold in general. It seems to be reasonable to conjecture, however, that $f(n)$ is very close (or, perhaps, equal) to $n$.

II. All our examples for colorings of a countable complete graph without a rainbow Hamiltonian path have a canonical structure (cf. Proposition 2). Now the following two problems arise.

**Problem 6.** (a) Find a class $\mathcal{F}$ of edge-colored countable complete graphs with the following properties:

(i) No $F \in \mathcal{F}$ contains a rainbow Hamiltonian path.

(ii) All infinite complete subgraphs of $K|_\varphi$ have a rainbow Hamiltonian path if and only if $K|_\varphi$ contains no subgraph isomorphic to any $F \in \mathcal{F}$.

(b) Do all $f \in \mathcal{F}$ have a canonical structure?

III. It is easy to show there is a subset $\{a_1, a_2, \dotsc\}$ of the natural numbers such that every positive integer occurs exactly once among the numbers and $|a_i - a_j|, 1 \leq i < j$. In other words, if color $|i - j|$ is assigned to edge $x_i x_j$ then in this coloring of $K_\omega$ some rainbow complete subgraph contains all colors. This observation leads to the following questions.

**Problem 7.** (a) Under what conditions does a countable (or an arbitrary infinite) complete graph $K$ contain a rainbow complete subgraph involving all colors that appear in $K$?

(b) Find theorems of this type for finite complete graphs.

(c) Let $0 < a_1 < a_2 < \dotsc < a_k$, and suppose that for each integer $i, 1 \leq i \leq n$, there is exactly one pair $j, m(1 \leq j < m \leq k)$ such that
$a_m - a_j = i$. Find $a(n) = \min a_k$. Also, find the minimum value of $k = k(n)$, for which such a sequence $a_1, \ldots, a_k$ exists.

Note that a greedy argument shows $a(n) \leq 0(n^3)$. In fact, there exists an infinite sequence $a_1, a_2, \ldots$ with $a_n \leq cn^3$ (for some constant $c$), whose $(2n)^{th}$ slice satisfies the requirements, for all $n \geq 1$.

IV. Concerning Theorem 3, one should ask that, instead of triangles, what sort of rainbow subgraphs $F$ can be excluded so that $X|_\varphi$ must contain a canonically colored subgraph $Y$ of cardinality $|Y| = |X|$. Theorem 4 shows $F$ always has minimum degree at most 2 (when $F$ is finite).

References


