ON AN INTEGRAL INEQUALITY FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract: Let $g$ be an analytic function on the unit disc $U = \{ z; |z| < 1 \}$, with $g(0) = g'(0) - 1 = 0$ and let $f(z) = \int_0^z \frac{g(t)}{t} dt$. It is shown that if $g$ satisfies the inequality $|g'(z) - 1| < 8/(2 + \sqrt{15}) = 1.362 \ldots$ for $z \in U$, then $|zf'(z)/f(z) - 1| < 1$, which is equivalent to $\text{Re} \int_0^1 [g(uz)/ug(z)] du > 1/2$, for $z \in U$.

1. Introduction

Let $A$ denote the class of functions $f$, which are analytic on the unit disc $U = \{ z; |z| < 1 \}$, with $f(0) = 0$ and $f'(0) = 1$. In a recent paper we obtained the following result [3, Corollary 4.2].

If $g \in A$ satisfies $|g'(z) - 1| < 1$, for $z \in U$, then

$$\text{Re} \int_0^1 \frac{g(uz)}{ug(z)} du > \frac{1}{2}, \quad \text{for } z \in U.$$
If we let
\[ f(z) = \int_0^1 \frac{g(uu)}{u} \, du, \]
then this last inequality is equivalent to
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad \text{for } z \in U. \]

In the present paper we improve the above result, by showing that the same conclusion holds under the less restrictive condition \(|g'(z) - 1| < 8/(2 + \sqrt{15}) = 1.362\ldots\)

2. Preliminaries

If \( f \) and \( g \) are analytic functions on \( U \), then we say that \( f \) is subordinate to \( g \), written \( f \prec g \), or \( f(z) \prec g(z) \), if \( g \) is univalent, \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

We shall use the following lemmas to prove our results.

**Lemma 1** [1,p.192]. Let \( h \) be a convex function on \( U \) (i.e. \( h \) is univalent and \( h(U) \) is a convex domain). If \( p \) is analytic in \( U \) and satisfies the differential subordination
\[ p(z) +zp'(z) \prec h(z), \]
then
\[ p(z) \prec \frac{1}{z} \int_0^zh(t) \, dt. \]

**Lemma 2** [2,p.201]. Let \( E \) be a set in the complex plane \( \mathbb{C} \) and let \( q \) be an analytic and univalent function on \( U \). Suppose that the function \( H: \mathbb{C} \times U \rightarrow \mathbb{C} \) satisfies
\[ H[q(\zeta), \zeta q'(\zeta); z] \not\in E, \]
whenever \( m \geq 1, |\zeta| = 1 \) and \( z \in U \). If \( p \) is analytic on \( U \), and satisfies \( p(0) = q(0) \) and
\[ H[p(z), zp'(z); z] \in E, \quad \text{for } z \in U, \]
then $p < q$.

For use in Section 4 we need the following elementary sharp inequalities.

**Lemma 3.** If $z \in \mathbb{C}$ then $|\sin z| \leq \text{sh} |z|$; if $z \in \mathbb{C}$ and $|z| < \pi/2$ then $|\tan z| \leq \tan |z|$.

### 3. Main results

**Theorem 1.** If $f \in A$ satisfies

\begin{equation}
|f'(z) + zf''(z) - 1| < M, \quad z \in U,
\end{equation}

where $M \leq M_0 = 8/(2 + \sqrt{15}) = 1.362\ldots$, then

\begin{equation}
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad z \in U.
\end{equation}

**Proof.** Since the inequality (1) can be rewritten as

\[ f'(z) + zf''(z) < 1 + Mz, \]

by using Lemma 1, we deduce $f'(z) < 1 + Mz/2$ and

\begin{equation}
\frac{f(z)}{z} < 1 + \frac{Mz}{4}.
\end{equation}

Let $p(z) = zf'(z)/f(z)$ and $P(z) = f(z)/z$. Since (3) implies $P(z) \neq 0$, the function $p$ is analytic in $U$ and the inequality (1) becomes

\begin{equation}
|P(z)[zp'(z) + p^2(z)] - 1| < M, \quad z \in U.
\end{equation}

The inequality (2) is equivalent to

\begin{equation}
p(z) < 1 + z
\end{equation}

and in order to show that (5) holds, by Lemma 2, it is sufficient to check the inequality

\begin{equation}
|P(z)[m\zeta + (1 + \zeta)^2] - 1| \geq M,
\end{equation}

where $m$ and $\zeta$ are constants.
for all $m \geq 1$, $|\zeta| = 1$ and $z \in U$.

If we let $\zeta = e^{i\theta}$, then

\[
L(m, \theta, z) \equiv |P(z)[m\zeta + (1 + \zeta)^2] - 1|^2 =
\]
\[
= |P(z)\zeta(\zeta + \bar{\zeta} + m + 2) - 1|^2 =
\]
\[
= (2 \cos \theta + m + 2)((2 \cos \theta + m + 2)|P(z)|^2 -
\]
\[- 2Re[e^{i\theta}P(z)]\} + 1.
\]

From (3) we deduce $|P(z) - 1| < M/4$ and $|P(z)| > 1 - M/4$. For $m \geq 1$ we have

\[
\frac{\partial L}{\partial m} = (2 \cos \theta + m + 2)|P(z)|^2 - Re[e^{i\theta}P(z)] =
\]
\[
= (m + 2)|P(z)|^2 - Re\{e^{i\theta}P(z)[2P(z) - 1]\} \geq
\]
\[
\geq |P(z)|\{(3|P(z)| - |2P(z) - 1|) \geq |P(z)|(2 - \frac{5M}{4}) > 0,
\]

which shows that $L$ is an increasing function of $m$. Hence we deduce

\[
L(m, \theta, z) \geq L(1, \theta, z) = (2 \cos \theta + 3)[3|P|^2 - 2Re[e^{i\theta}P(\bar{P} - 1)] + 1
\]
\[
\geq (2 \cos \theta + 3)|P|[3|P| - 2|P - 1|] + 1 \geq
\]
\[
\geq \left(1 - \frac{M}{4}\right)\left[3\left(1 - \frac{M}{4}\right) - \frac{M}{2}\right] + 1 \equiv K(M).
\]

Since $0 < M \leq M_0$, where $M_0$ is the positive root of the equation $K(M) = M^2$, we deduce $L(m, \theta, z) \geq M^2$, which yields (6). Hence the subordination (5) holds and we obtain (2), which completes the proof of Theorem 1.

The following two theorems are integral versions of Theorem 1.

**Theorem 2.** If $g \in A$ satisfies $|g'(z) - 1| < M_0 = 8/(2 + \sqrt{15})$ then

\[
\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad \text{for } z \in U,
\]

where

\[
f(z) = \int_0^z \frac{g(t)}{t}\ dt = \int_0^1 \frac{g(uz)}{u}\ du.
\]
Theorem 3. If $g \in A$ satisfies $|g'(z) - 1| < M_0 = 8/(2 + \sqrt{15)}$ then
\[
\text{Re} \int_0^1 \frac{g(uz)}{ug(z)} \, du > \frac{1}{2}, \text{ for } z \in U.
\]

4. Examples

Example 1. If we let $g(z) = (\sin \lambda z)/\lambda$, where
\[
|\lambda| \leq \ln[1 + M_0 + \sqrt{M_0(M_0 + 2)}] = 1.504 \ldots
\]
then, by using Lemma 3, we have
\[
|g'(z) - 1| = 2|\sin^2 \frac{\lambda z}{2}| \leq 2 \sin^2 \frac{|\lambda z|}{2} < 2 \sin^2 \frac{|\lambda|}{2} \leq M_0,
\]
for $z \in U$ and by Theorem 3 we deduce
\[
\text{Re} \frac{\text{Si}(z)}{\sin z} > \frac{1}{2}, \text{ for } |z| < 1.504 \ldots
\]
where
\[
\text{Si}(z) = \int_0^1 \frac{\sin uz}{u} \, du = \int_0^z \frac{\sin t}{t} \, dt.
\]

Example 2. If we let $g(z) = (e^{\lambda z} - 1)/\lambda$, where
\[
|\lambda| \leq \ln(1 + M_0) = 0.859 \ldots
\]
then $|g'(z) - 1| \leq M_0$, for $z \in U$ and by Theorem 3 we deduce
\[
\text{Re} \int_0^1 \frac{e^{uz} - 1}{u(e^z - 1)} \, du > \frac{1}{2}, \text{ for } |z| < 0.859 \ldots
\]

Example 3. If we let $g(z) = [\ln(1 + \lambda z)]/\lambda$, where
\[
|\lambda| \leq \frac{M_0}{1 + M_0} = 0.576 \ldots
\]
then \(|g'(z) - 1| < M_0\), for \(z \in U\) and by Theorem 3 we deduce

\[
Re \int_0^1 \frac{\ln(1 + uz)}{u \ln(1 + z)} \, du > \frac{1}{2}, \quad \text{for} \quad |z| < 0.576\ldots
\]

**Example 4.** If we let \(g(z) = (\tan \lambda z)/\lambda\), where

\[|\lambda| \leq \arctan \sqrt{M_0} = 0.862\ldots\]

then, by Lemma 3, we have

\[|g'(z) - 1| = |\tan^2 \lambda z| \leq \tan^2 |\lambda z| < \tan^2 |\lambda| \leq M_0,\]

for \(z \in U\) and by Theorem 3 we deduce

\[
Re \int_0^1 \frac{\tan uz}{u \tan z} \, du > \frac{1}{2}, \quad \text{for} \quad |z| < 0.862\ldots
\]

**References**

