A GENERALIZATION OF CARISTI’S FIXED POINT THEOREM

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Abstract: General common fixed and periodic point theorems are proven for a class of selfmaps of a quasi–metric space which satisfy the contractive conditions (1), or (7), or (8), or (10) below. Presented theorems generalize and extend Caristi’s Theorem [2]. Two examples are constructed to show that an introduced class of selfmaps is indeed wider than a class of selfmaps which satisfy Caristi’s contractive definition (C) below.

1. Introduction. Let $X$ be a non–void set and $T : X \to X$ a selfmap. A point $x \in X$ is called a periodic point for $T$ iff there exists a positive integer $k$ such that $T^k x = x$. If $k = 1$, then $x$ is called a fixed point for $T$.

J. Caristi [2] proved the following an important contraction fixed point theorem.

Theorem 1 (Caristi [2]). Suppose $T : X \to X$ and $\phi : X \to [0, \infty)$, where $X$ is a complete metric space and $\phi$ is lower semi–continuous. If for each $x$ in $X$

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\[ d(x, Tx) \leq \phi(x) - \phi(Tx), \]

then \( T \) has a fixed point.

Th. 1 is sometimes called a Caristi–Kirk–Browder theorem (see [5]). Recently A. Bollenbacker and T. Hicks [1] revisited Th. 1. Various proofs of Th. 1 were presented later in [11, 13, 15]. It is known that Caristi's theorem is essentially equivalent to Ekelend's variational principle [5]. Up to new many extensions of Caristi's result have been obtained [6, 7, 8, 9].

The purpose of this paper is to introduce and investigate a class of selfmaps which satisfy a contractive condition weaker than (C) and still have a fixed or periodic point.

2. Main results. We begin with some notation needed in the sequel. A pair \((X, d)\) of a set \(X\) and a mapping \(d\) from \(X \times X\) into the real numbers is said to be a \textit{quasi-metric space} iff for all \(x, y, z \in X\):

\begin{enumerate}
  \item \(d(x, y) \geq 0\) and \(d(x, y) = 0\) iff \(x = y\),
  \item \(d(x, z) \leq d(x, y) + d(y, z)\).
\end{enumerate}

Let \(d_x : X \to [0, +\infty)\) be defined by \(d_x(y) = d(x, y)\). Let \(N\) denotes the set of all positive integers.

A sequence \(\{x_n\}\) in \(X\) is said to be a \textit{left \(k\)-Cauchy} sequence if for each \(k \in N\) there is one \(N_k\) such that \(d(x_n, x_m) < 1/k\) for all \(m \geq n \geq N_k\). A quasi-metric space is a \textit{left \(k\)-sequentially complete} if each left \(k\)-Cauchy sequence is convergent (compare [12, 14]).

Now we are in position to state the following result.

\textbf{Theorem 2.1.} Let \((X, d)\) be a left \(k\)-complete quasi-metric space and let for each \(x \in X\) a function \(d_x\) be lower semi-continuous \textit{(l.s.c)} on \(X\). Let \(F\) be a family of mappings \(f : X \to X\). If there exists \textit{l.s.c.} function \(\phi : X \to [0, \infty)\) such that for each \(x \in X\):

\[ d(x, fx) \leq \phi(x) - \phi(fx) \text{ for all } f \in F, \]

then for each \(x \in X\) there is a common fixed point \(u\) of \(F\) such that

\[ d(x, u) \leq \phi(x) - s, \text{ where } s = \inf\{\phi(x) : x \in X\}. \]

\textbf{Proof.} For any \(x \in X\) denote
\[ S(x) = \{ y \in X : d(x, y) \leq \phi(x) - \phi(y) \}, \]
\[ a(x) = \inf \{ \phi(y) : y \in S(x) \}. \]

As \( x \in S(x) \), \( S(x) \) is not empty and \( 0 \leq a(x) \leq \phi(x) \).

Let \( x \in X \) be arbitrary. Put \( x_1 = x \). Now we shall choose a sequence \( \{ x_n \} \) in \( X \) as follows: when \( x_1, x_2, \ldots, x_n \) have been chosen, choose \( x_{n+1} \in S(x_n) \) such that \( \phi(x_{n+1}) \leq a(x_n) + 1/n \). In doing so, one obtains a sequence \( \{ x_n \} \) such that

\[ (2) \quad d(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}); \quad a(x_n) \leq \phi(x_{n+1}) \leq a(x_n) + 1/n. \]

Then, as \( \{ \phi(x_n) \} \) is a decreasing sequence of reals, there is some \( a \geq 0 \) such that

\[ (3) \quad a = \lim_n \phi(x_n) = \lim_n a(x_n). \]

Let now \( k \in \mathbb{N} \) be arbitrary. From \( (3) \) there exists some \( N_k \) such that \( \phi(x_n) < a + 1/k \) for \( n = N_k \). Thus, by monotonocity of \( \{ \phi(x_n) \} \) for \( m \geq n \geq N_k \) we have \( a \leq \phi(x_m) \leq \phi(x_n) < a + 1/k \) and hence

\[ (4) \quad \phi(x_n) - \phi(x_m) < 1/k \text{ for all } m \geq n \geq N_k. \]

From \( (ii) \) and \( (2) \) we get

\[ (5) \quad d(x_n, x_m) \leq \sum_{s=n}^{m-1} d(x_s, x_{s+1}) \leq \phi(x_n) - \phi(x_m). \]

Then by \( (4) \) we have

\[ d(x_n, x_m) < 1/k \text{ for all } m \geq n \geq N_k. \]

Therefore, \( \{ x_n \} \) is a \( k \)-Cauchy sequence and, by completeness of \( X \), it converges to some \( u \in X \).

Since \( d \) and \( \phi \) are l.s.c. functions, by \( (5) \) we have

\[
\begin{align*}
d(x_n, u) & \leq \lim_m \inf \, d(x_n, x_m) \leq \lim_m \sup \, d(x_n, x_m) \\
& \leq \phi(x_n) + \lim_m \sup \, [\phi(x_m)] = \phi(x_n) - \lim_m \inf \phi(x_m) \\
& \leq \phi(x_n) - \phi(u).
\end{align*}
\]

Thus \( u \in S(x_n) \) for all \( n \in \mathbb{N} \) and hence \( a(x_n) \leq \phi(u) \). So by \( (3) \), \( a \leq \phi(u) \). On the other hand, by l.s.c. of \( \phi \) and \( (3) \), we have \( \phi(u) \leq \lim_n \inf \phi(x_n) = a \). Therefore, \( \phi(u) = a \).
Now we shall show that $fu = u$ for all $f \in F$. Suppose not and let $f \in F$ be such that $fu \neq u$. Then (1) implies $\phi(fu) < \phi(u) = a$. Hence, by (3), there is a $n \in \mathbb{N}$ such that

$$\phi(fu) < a(x_n).$$

Since $u \in S(x_n)$ for all $n \in \mathbb{N}$, we have

$$d(x_n, fu) \leq d(x_n, u) + d(u, fu) \leq [\phi(x_n) - \phi(u)] + [\phi(u) - \phi(fu)] = \phi(x_n) - \phi(fu).$$

Hence we conclude that $fu \in S(x_n)$. Hence $\phi(fu) \geq a(x_n)$, which is a contradiction with (6). Therefore, $fu = u$ for all $f \in F$. Since $u \in S(x_n)$ implies

$$d(x_n, u) \leq \phi(x_n) - \phi(u) \leq \phi(x) - \inf \{\phi(y) : y \in X\} = \phi(x) - s. \quad \Diamond$$

The following result contains the above theorem.

**Theorem 2.2.** Let $E$ be a set, $(X, d)$ as in Th. 2.1, $g : E \to X$ a surjective mapping and $F = \{f\}$ a family of arbitrary mappings $f : E \to X$. If there exists a l.s.c. function $\phi : X \to [0, \infty)$, such that

$$d(ga, fa) \leq \phi(ga) - \phi(fa) \text{ for all } f \in F$$

and each $a \in E$, then $g$ and $F$ has a common coincidence point, that is, for some $v \in E$ $gv = fv$ for all $f \in F$.

**Proof.** Let $x \in X$ be arbitrary and $u \in X$ as in Th. 2.1. Since $g$ is surjective, for each $x \in X$ there is some $a = a(x)$ such that $ga = x$. Let $f \in F$ be a fixed mapping. Define by $f$ a mapping $h = h(f)$ of $X$ into itself such that $hx = fa$, where $a = a(x)$, that is, $ga = x$. Let $H$ be a family of all mappings $h = h(f)$. Then (7) implies

$$d(x, hx) \leq \phi(x) - \phi(hx) \text{ for all } h \in H.$$

Thus, by Th. 2.1, $u = hu$ for all $h \in H$. Hence $gv = fv$ for all $f \in F$, where $v = v(u)$ is such that $gv = u$. \quad \Diamond

The following result is related to periodic points.

**Theorem 2.3.** Let $(X, d)$ and $\phi$ be as in Th. 2.1. Let $T : X \to X$ be an arbitrary mapping. If for each $x \in X$ there is $n(x)$ in $\mathbb{N}$ such that

$$d(x, T^{n(x)}x) \leq \phi(x) - \phi(T^{n(x)}x),$$

then $T$ has a periodic point.
Proof. Define \( f : X \to X \) by \( fx = T^n(x)x \). Then by Th. 2.1 (with \( F \) singleton) \( fu = u \) for some \( u \in X \). Hence \( T^n(x)u = u \) that is, \( u \) is a periodic point of \( T \). \( \Box \)

**Remark 2.1.** Example 2 below shows that a periodic point in Th. 2.3 need not be a fixed point. Therefore, one must add some hypothesis in order to ensure that \( T \) possesses a fixed point.

**Theorem 2.4.** Let \((X,d)\) and \( \phi \) be as in Th. 2.1 and let \( T : X \to X \) be a mapping. If for each \( x \in X \), with \( Tx \neq x \), there is \( n(x) \in \mathbb{N} \) and a real number \( C(x) > 0 \) such that

\[
\max\{d(x, T^n(x)x), C(x) \cdot d(x, Tx)\} \leq \phi(x) - \phi(T^n(x)x),
\]

then \( T \) has a fixed point.

**Proof.** If we suppose that \( T^n x \neq x \) for all \( n \in \mathbb{N} \), then we can choose \( C(x) \) such that (10) reduces to (9). Then by the proof of Th. 2.3 \( T^n(x)u = u \) for some \( u \in X \). Therefore, from (10) we have

\[
\max\{0, C(u) \cdot d(u, Tu)\} \leq \phi(u) - \phi(u) = 0.
\]

If we suppose that \( u \neq Tu \), then \( C(u) > 0 \) and so we have \( C(u) \cdot d(u, Tu) \leq 0 \), a contradiction. Therefore \( Tu = u \). \( \Box \)

**Remark 2.2.** It is clear that if \( T \) satisfies (C), then \( T \) satisfies (10) with \( n(x) = 1 \) and, for instance, \( C(x) = 1 \). Therefore, Th. 1 is a special case of Th. 2.1, even if \((X,d)\) in Th. 2.1 is a metric space. Example 1 below shows that Th. 2.1 is a proper generalization of Caristi's Th. 1.

**Remark 2.3.** In [14] is given an example of a quasi-metric space \((X,d)\) with \( d_x \) continuous for each \( x \) that is not metrizable.

### 3. Examples

1. Let \( X = \{0\} \cup \{\pm 1/n : n = 1, 2, \ldots\} \) with the usual metric. Define \( T : X \to X \) by \( T(1/n) = -1/(n+1), T(-1/n) = 1/(n+1) \) and \( T(0) = 0 \). Define \( \phi : X \to [0, +\infty) \) by \( \phi(x) = d(x, Tx) \). Then for \( x = \pm 1/n \) we have

\[
d(x, Tx) = 1/n + 1/(n+1) : d(x, T^2x) = 1/n = 1/(n+2).
\]

Hence

\[
d(x, T^2x) = 1/n - 1/(n+2) < 1/n + 1/(n+1) - [1/(n+2) + 1/(n+3)] = \phi(x) - \phi(T^2x).
\]

Since for each \( x = \pm 1/n \) we can choose \( C(\pm 1/n) \leq 2(n+1)/(n+2)^2 \),
we conclude that $T$ satisfies (10) for each $x$ in $X$ with $n(x) = 2$ (and $n(0) = 1$). As $X$ is a complete metric space and $\phi(x) = |x| + |x|/(1+|x|)$ is continuous on $X$, we conclude that Th. 2.4 can be applied and $x = 0$ is a fixed point.

To show that Caristi’s theorem is not applicable, we shall show that there is not a function $\phi : X \to [0, \infty)$ such that $T$ satisfies (C). We pointed out [4] that such a function exists if and only if the series $\sum_{n=0}^{\infty} d(T^n x, T^{n+1} x)$ converges for all $x \in X$. Since in our example for any fixed $x = \pm 1/m_0$ we have

$$d(T^n x, T^{n+1} x) = 1/(n + m_0) + 1/(n + 1 + m_0) > 2/(n + 1 + m_0),$$

we conclude that the above series is divergent and hence there is no function $\phi$ such that (C) holds for any $x = \pm 1/n$ in $X$.

2. Let $X = [-2, -1] \cup [1, 2]$ with the usual metric. Define $T : X \to X$ by $Tx = -x$. Then $T$ satisfies (9) with $n(x) = 2$ for any (continuous) function $\phi : X \to [0, +\infty)$.

References


