w-JORDAN NEAR-RINGS I

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Abstract: Let $N$ be a zero-symmetric near-ring with an invariant series whose factors are $N$-simple. We prove that the radical $J_2 (N)$ is nilpotent and the factor $N/J_2 (N)$ is a direct sum of a finite number of $A$-simple and strongly monogenic near-rings. Moreover we characterize nilpotent near-rings with invariant series whose factors are of prime order.

Introduction and general results

Many authors have studied near-rings containing particular chains of ideals (see [5,8,10]) and have often shown the existence of links between these chains of ideals and the structure of the near-rings under consideration. In this paper we begin a study of near-rings with an invariant series whose factors belong to certain given classes. In particular we study here the zero-symmetric case; the general case and the construction of finite near-rings satisfying these conditions will be covered in future papers.

For the zero-symmetric near-rings with an invariant series whose factors are $N$-simple, we obtain a result analogous to the Artin-Noether theorem. We prove that a zero-symmetric near-ring $N$ with an invariant
series whose factors are $N$-simple has the radical $J_2(N)$ nilpotent and
the factor $N/J_2(N)$ is a direct sum of $A$-simple and strongly monogenic
near-rings. Moreover we discuss the finite case and characterize the
near-rings with an invariant series whose factors are of prime order.
We prove a necessary and sufficient condition so that $N$ is nilpotent
and we establish a link between the nilpotence index and the length of
the series. In the case in which index and length coincide, we prove
that the order of $N$ is a prime power.

In the following we will often refer to [12] without express recall.
Let $N$ be a left near-ring. A finite system of subnear-rings of $N$
contained in one another
\[ N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\} \]
is called a normal series of $N$ if every subnear-ring $N_i$, $i \in \{1, 2, \ldots, n\}$, is a proper ideal in $N_{i-1}$, an invariant series of $N$ if every subnear-ring $N_i$, $i \in \{1, 2, \ldots, n\}$, is a proper ideal of $N$. The factor-near-rings $N_i/N_{i+1}$ are called principal factors of the invariant series. For invariant
series, in the following, we will indicate $N_i/N_{i+1}, N_i/N_{i+2}, \ldots, N_i/N_{i+k}$
respectively with $N'_i, N''_i, \ldots, N^k_i$ and with $f'_i, f''_i, \ldots, f^k_i$ the corresponding
canonical epimorphisms.

Let us consider now the following classes of near-rings:

- $S_0$: class of simple near-rings;
- $S_1$: class of simple and strongly monogenic near-rings;
- $S_2$: class of $N_0$-simple near-rings\(^{(1)}\);
- $S_3$: class of near-rings without proper subnear-rings;
- $S_4$: class of near-rings of prime order.

**Definition 1.** A near-ring $N$ is a w-Jordan near-ring \((wJ\text{-near-ring})\) if it has an invariant series whose factors belong to $S_w$ \((w \in \{0, 1, 2, 3, 4\})\).

We can observe that in near-ring-theory the classes $S_i$ \((i \in \{0, 1, 2, 3, 4\})\) never coincide without further conditions while in ring-theory, for instance, $S_1$ and $S_2$ coincide.

In order to establish relationships between the classes $S_w$, let us state some results that concern the near-rings belonging to $S_2$. We
recall that: A near-ring $N$ is $N_0$-simple if it is without proper additive subgroups $S$ such that $SN_0 \subseteq S$.

\(^{(1)}\) We observe that if $N$ is zero-symmetric, $N_0$-simplicity and $N$-simplicity coincide.
Definition 2. A zero-symmetric near-ring \( N \) is \( A \)-simple if it is without non-zero \( N \)-subgroups \( H \) such that \( HN = \{0\} \).

Theorem 1. A near-ring \( N \) belongs to \( S_2 \) iff \( N \) is a zero-ring of prime order, a constant near-ring of prime order or an \( A \)-simple and strongly monogenic near-ring.

Proof. Let \( N \) be an \( N_0 \)-simple near-ring. The constant and the zero-symmetric parts are both \( N_0 \)-subgroups of \( N \), hence \( N \) is constant or zero-symmetric. By [2] and Ex.3.9 p.78 of [12] a constant near-ring is \( N_0 \)-simple iff it is cyclic of prime order. If \( N \) is zero-symmetric then \( nN = \{0\} \) for every \( n \in N \), and thus \( N \) is a zero-ring of prime order, or \( N \) is strongly monogenic and obviously \( A \)-simple. Conversely, if \( N \) is a zero-ring of prime order or a constant near-ring of prime order, then \( N \) is \( N_0 \)-simple. Let \( N \) be an \( A \)-simple and strongly monogenic near-ring. Let us suppose that \( M \) is a proper \( N_0 \)-subgroup of \( N \). Since \( N \) is an \( A \)-simple near-ring, then \( MN \neq \{0\} \) and since \( N \) is a strongly monogenic near-ring there is an element \( h \in M \) such that \( hN = N \). Since \( M \) is an \( N_0 \)-subgroup, \( hN \) is contained in \( M \), a contradiction. Thus \( N \) is \( N_0 \)-simple. \( \Diamond \)

We observe that a zero-symmetric near-ring which is \( A \)-simple and strongly monogenic is Blackett simple ([4]).

Definition 3. A near-ring \( N \) is strongly \( N_0 \)-simple if its subnear-rings belong to \( S_2 \).

We will call \( S_2^* \) the class of the strongly \( N_0 \)-simple near-rings.

Theorem 2. If \( N \) is an \( N_0 \)-simple near-ring and every subnear-ring \( M \) of \( N \) satisfies the d.c.c. on the \( M \)-subgroups, then \( N \) is strongly \( N_0 \)-simple.

Proof. By Th.1, if \( N \) is a zero-ring of prime order or a constant near-ring of prime order, then \( N \) is strongly \( N_0 \)-simple. Let \( N \) be an \( A \)-simple and strongly monogenic near-ring and let \( M \) be a subnear-ring of \( N \) with d.c.c. on the \( M \)-subgroups. Our aim is to show that \( M \) does not contain additive subgroups \( S \) so proving that \( SM \subseteq S \). Let us suppose \( S \) to be a proper \( M \)-subgroup of \( M \). Since \( N \) is \( A \)-simple then \( SN \neq \{0\} \), thus there is an element \( s \in S \) such that \( sN = N \), given that \( N \) is strongly monogenic. Firstly we observe that \( r(s) = \{0\} \) (where \( r(s) \) is the right annihilator of the element \( s \)). In fact \( r(s) \neq \{0\} \) implies \( r(s)N \neq \{0\} \), because \( r(s) \) is an \( N \)-subgroup of \( N \) and \( N \) is \( A \)-simple; thus \( r(s)N = N \) and \( N = sN = s[r(s)N] = \{0\}N = \{0\} \) and this is absurd. Moreover, since \( S \) is a proper \( M \)-subgroup of \( M \), \( sM \)
is strictly contained in \( M \). We set \( M_1 = sM \) and consider \( sM_1 \). It is an \( M \)-subgroup of \( M \) strictly contained in \( M_1 \), in fact if \( sM_1 = M_1 \), it would be \( ssM = sM \), that is \( s(sM - M) = \{0\} \). Since \( r(s) = \{0\} \), then \( sM = M \) and this was previously excluded. In this way we obtain a chain \( M_1 \supset sM_1 \supset s^2M_1 \supset \ldots \) which becomes stationary, due to d.c.c. on the \( M \)-subgroups. Since this is excluded, \( M \) is \( M \)-simple. 

**Proposition 1.** If \( N \) is a \( wJ \)-near-ring, then \( N \) is a \( (w - 1)J \)-near-ring.

**Proof.** We can easily prove that \( S_4 \subset S_3 \subset S_2 \subset S_1 \subset S_0 \) and consequently that a \( wJ \)-near-ring is a \( (w - 1)J \)-near-ring.

**Proposition 2.** The classes \( S_w (w \in \{0, 1, 2, 3, 4\}) \) are closed under homomorphisms and the classes \( S_w (w \in \{3, 4\}) \) are closed under substructures.

**Proof.** The near-rings belonging to \( S_3 \) and \( S_4 \) are without substructures and simple, so they do not have proper homomorphic images. Moreover, if \( N' = \varphi(N) \) is a homomorphic image of \( N \), each proper \( N_0 \)-subgroup (ideal) of \( N' \) derives from some proper \( N_0 \)-subgroup (ideal) of \( N \), thus \( N \in S_2 \) implies \( N' \in S_2 \) \((N \in S_0 \) implies \( N' \in S_0 \)). Moreover, if \( N \) is strongly monogenic and simple, then \( N' \) is strongly monogenic and simple, therefore \( N \in S_1 \) implies \( N' \in S_1 \).

Hence, by Prop.6 of [1]:

**Proposition 3.** The classes of the \( 3J \)-near-rings and of the \( 4J \)-near-rings are closed under substructures, homomorphic images and \( N_0 \)-subgroups.

We should observe that the classes \( S_w (w \in \{0, 1, 2\}) \) are not closed under substructures. In fact for example \( Q \in S_2 \) but \( \mathbb{Z} \notin S_0 \). Therefore we cannot apply Prop.6 of [1] and, in fact, even if we can prove that \( S_2 \) is closed under \( N_0 \)-subgroups, the class of the \( 2J \)-near-rings is not closed under \( N_0 \)-subgroups.

2-Jordan near-rings

The following Th.3, which provides a necessary and sufficient condition so that the class \( S_2 \) is closed w.r.t. substructures, uses the Th.1.33 of [11].

**Let** \( I \) be an ideal of a near-ring \( N \) and \( S \) a subnear-ring of \( N \). Then \( I \cap S \) is an ideal of \( S \), \( I \) is an ideal of \( I + S \) and \( I + S/I \) is isomorphic to \( S/I \cap S \).
Theorem 3. A near-ring $N$ has all its subnear-rings as $2J$-near-rings iff it contains an invariant series $N = N_1 \supset N_2 \supset \ldots \supset N_n =\{0\}$ whose principal factors $N'_i$ belong to $S_2^*$.  

**Proof.** Let $N$ be a near-ring whose subnear-rings are $2J$-near-rings. So $N$ is also a $2J$-near-ring. Hence let us consider an invariant series of $N$,

(α) \[ N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\} \]

whose principal factors belong to $S_2$. In order to show that the principal factors of (α) belong to $S_2^*$, we will show that every subnear-ring $M$ of $N'_i$ has the d.c.c. on the $M$-subgroups. Let $M$ be a subnear-ring of $N'_i$. Since $M$ is a homomorphic image of a subnear-ring of $N_i$ and consequently of $N$, by Proposition 2, it is a $2J$-near-ring. Therefore $M$ has an invariant series $M = M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$ whose factors belong to $S_2$. Hence these factors have the d.c.c. on the $(M'_i)_0$-subgroups. By Th.1 and Ex a) of [1] we can deduce that $M$ also has the d.c.c. on $M$-subgroups. Thus $N'_i$ belong to $S_2$ and every subnear-ring $M$ of $N'_i$ has the d.c.c. $M$. We apply Th.2 and $N'_i \in S_2^*$.

Conversely, let $N$ be a near-ring with an invariant series $N = N_1 \supset \ldots \supset N_n = \{0\}$ whose principal factors $N'_i$ belong to $S_2^*$. We can prove that the subnear-rings of $N$ are $2J$-near-rings. Let $M$ be a subnear-ring of $N$. We set $M_i = M \cap N_i$ and we obtain an invariant series of $M : M = M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$.

By the Theorem 1.33 of [11], $N_{i+1} \cap M_i/N_{i+1}$ is isomorphic to $M_i/N_{i+1} \cap M_i$ that coincides with $M_i/M_{i+1}$. Therefore $M'_i$ is isomorphic to $N_{i+1} + M_i/N_{i+1}$ and the latter is a subnear-ring of $N'_i$. Since $N'_i$ belongs to $S_2^*$, $M'_i$ belongs to $S_2$ and $M$ is a $2J$-near-ring. ◊

**Corollary 1.** The class of finite $2J$-near-rings is closed under substructures.

**Proof.** It follows from Th.2 and 3, given that, in the finite case, the d.c.c. hold. ◊

In the following $N$ will be a zero-symmetric near-ring.

**Theorem 4.** If $N$ is a near-ring with an $A$-simple and strongly monogenic ideal $I$ such that $N/I$ is a zero-ring of prime order, then $N = I \oplus J$ where $J = J_2(N)$.\(^{(2)}\)

\(^{(2)}\) $J_2(N)$ is the intersection of right annihilators of $N_0$-simple $N$-groups, see [12] p. 136.
Proof. Let $I$ be a proper ideal of $N$, otherwise the thesis is trivial. Since $N$ is zero-symmetric, $I$ is an $N$-subgroup of $N$, therefore $I J_2(N) = \{0\}$ and $J_2(N) \neq N$, $J_2(N) \neq I$ because $I$ is $A$-simple. Moreover $J_2(N) \neq \{0\}$. In fact: if $J_2(N) = \{0\}$, then $J_2(I) = \{0\}$ and $I$ is 2-semisimple with d.c.c. on the right annihilators. Hence $I$ has a left identity $e$ (see [2], [4], [12] p. 146) and by Pierce decomposition $N = r(e) + eN$. We observe that $r(e) \neq \{0\}$. In fact $r(e) = \{0\}$ implies $N = eN \subseteq I$ and this is excluded. Moreover $N/I$ is a zero-ring, therefore $[r(e)]^2 \subseteq I$ and hence $[r(e)]^2 = \{0\}$. In this way $r(e)$ is a non trivial nilpotent $N$-subgroup of $N$ and therefore $r(e) \subseteq J_2(N) = \{0\}$ (see [12] p. 153, [13]), a contradiction. Finally $I \cap J_2(N) = \{0\}$ because $I$ is simple and $N = I + J_2(N)$ because $N/I$ is of prime order. Hence $N = I \oplus J_2(N)$.

The following theorem shows that, given a zero-symmetric near-ring with an invariant series whose factors are in $S_2$, it is possible to construct another invariant series whose factors are in $S_2$ such that the $A$-simple and strongly monogenic factors precede the zero-ring factors.

Theorem 5. Let $N$ be a $2J$-near-ring and $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ an invariant series whose principal factors are in $S_2$. If $N'_i$ is a zero-ring and $N'_i$ is an $A$-simple and strongly monogenic near-ring then there is an ideal $M_i$ of $N$ such that $N_i \supset M_i \supset N_i+2$, $N_i/M_i$ is isomorphic to $N'_i$ and $M_i+2/N_i$ is isomorphic to $N'_i$.

Proof. Considering the near-ring $N''_i$, we set $I = f''_i (N_i)$ Given that $N''_i/N_i$ is isomorphic to $N'_i$ we have $N''_i/I$ isomorphic to $N'_i$. Therefore $N''_i/I$ is a zero-ring of prime order and $I$ is $A$-simple and strongly monogenic because it is isomorphic to $N''_i$. Hence, by Th.4, $N''_i = I \oplus J$ where $J \cong N'_i$ and therefore $N'_i \cong N''_i/J$. We set $M_{i+1} = (f''_i)^0 (J)$, that is $M_{i+1}/N_i$ is isomorphic to $N'_i$. Obviously $M_{i+1}$ is an ideal of $N_i$ and $N_i/M_{i+1} \cong (N_i/N_i)/M_{i+1}/N_i \cong \cong N''_i/J \cong I \cong N_i+1/N_i+2 = N'_i$. Hence $M_{i+1}$ is a maximal ideal of $N_i$.

Now we can show that $M_{i+1}$ is an ideal of $N$: the near-ring $N_i+1$ is an ideal of $N$, $M_{i+1}$ is an ideal of $N_i$, hence $N_i+1 M_{i+1} \subseteq \subseteq N_i+1 \cap M_{i+1}$. Moreover $N_i+1 \cap M_{i+1} = N_i+2$. In fact if $x \in N_i+1 \cap M_{i+1}$, then $x + N_i+2 \in N_i+1 \cap J = \{0\}$ and this implies that $x \in N_i+2$. Thus $N_i+1 \cap M_{i+1} \subseteq N_i+2$. Obviously $N_i+2 \subseteq N_i+1 \cap M_{i+1}$, therefore $N_i+1 \cap M_{i+1} = N_i+2$. We now set $(N_i+2 : N_i+1) = \{m \in N/N_i+1 m \subseteq \subseteq N_i+2\} = H$ which is an ideal of $N$ (see [11]). We obtain $M_{i+1} \subseteq$
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\( \subseteq H \cap N_i \) and \( H \cap N_i \) is strictly enclosed in \( N_i \); otherwise it would be \( N_i+1 \cap N_i \subseteq N_i+2 \) and hence \( N_i+1 \cap N_{i+1} \subseteq N_i+2 \), but \( N_i+1 \) is \( A \)-simple and this is excluded. Hence \( M_i+1 = H \cap N_i \). Thus \( M_i+1 \), as intersection of two ideals of \( N \), is an ideal of \( N \). ◊

**Theorem 6.** A non nilpotent 2J-near-ring \( N \), has the radical \( J_2(N) \) nilpotent and the factor \( N/J_2(N) \) is a direct sum of \( A \)-simple and strongly monogenic near-rings.

**Proof.** By Th.5, if \( N \) is a zero-symmetric 2J-near-ring, we can construct a new invariant series \( N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\} \) whose factors are in \( S_2 \), such that, if \( N'_j \) is \( A \)-simple and strongly monogenic and \( N'_j \) is a zero-ring, then \( i < j \). We set \( h \in I_n \), the smallest index such that \( N'_h \) is a zero-ring. Obviously \( N'_h \) is nilpotent. Therefore \( N_h \subseteq J_2(N) \). Moreover, if \( N_h \neq N \), the near-ring \( N/N_h \) contains an invariant series whose factors are \( N \)-simple and hence 2-semisimple. By Ex. f) of [1], \( N/N_h \) is 2-semisimple and therefore \( J_2(N) \subseteq N_h \). Hence \( J_2(N) = N_h \) and the radical \( J_2(N) \) is nilpotent. In this way \( N/J_2(N) \) has an invariant series satisfying the hypotheses of Th.4 of [1], thus \( N/J_2(N) \) is the direct sum of \( A \)-simple and strongly monogenic near-rings. ◊

The analogous, in ring-theory, brings us to the famous theorem of Artin-Noether. In fact, rings with an invariant series whose factors are in \( S_2 \), are rings with an invariant series whose factors are without right ideals\(^{(3)}\) and hence are either fields or zero-rings. Thus in a ring \( A \) satisfying the hypotheses of Th.6 the Jacobson radical \( J(A) \) is nilpotent and the factor \( A/J(A) \) is a direct sum of fields.

**Corollary 2.** Let \( N \) be a 2J-near-ring. Then \( \mathcal{P}(N) = \eta(N) = J_0(N) = J_1(N) = J_2(N) \).\(^{(4)}\)

**Proof.** It can be easily demonstrated, since \( N \) has the d.c.c. on the \( N \)-subgroups and \( J_2(N) \) is nilpotent (see 5.61 p. 162 of [12]). ◊

If \( N \) is a finite near-ring, we obtain:

**Corollary 3.** Let \( N \) be a finite near-ring such that \( N \neq J_2(N) \). Then:
1. If \( N \) is a 2J-near-ring and the \( A \)-simple factors present in a principal series are planar, then the additive group \( (N/J_2(N))^+ \) is nilpotent;
2. If \( N \) is a 3J-near-ring, the additive group \( (N/J_2(N))^+ \) is abelian.

**Proof.** The group \( (N/J_2(N))^+ \) is a direct sum of finite groups sup-

\(^{(3)}\) A ring having an invariant series whose factors are in \( S_2 \), is right artinian.

\(^{(4)}\) For the definitions of \( \mathcal{P}(N) \), \( \eta(N) \) and \( J_v(N) (v \in \{0,1,2\}) \) see [9], [11], [12].
porting planar near-rings. Therefore, as shown in [3], \((N/J_2(N))^+\) is nilpotent.

If \(N\) is a 3J-near-ring, the factors of the invariant series are without proper subnear-rings. Therefore, as proved in [6], (see also [7]) they are \(p\)-singular\(^{(5)}\) and therefore their additive group is elementary abelian, because they are simple. Thus \((N/J_2(N))^+\), being a direct sum of elementary abelian groups, is abelian. \(\diamondsuit\)

4-Jordan near-rings

In this section we will study the 4J-near-rings with particular reference to the nilpotent case. We recall that a near-ring \(N\) is nilpotent if there is an index \(n \in \mathbb{N}\) such that \(N^n = \{0\}\). We will call \(g(N)\) the least \(n \in \mathbb{N}\) such that \(N^n = \{0\}\) and \(\dim(N)\) the length of an invariant series whose factors are in \(S_4\).

**Theorem 7.** A near-ring \(N\) with an invariant series \(N = N_1 \supset N_2 \supset \cdots \supset N_n = \{0\}\) and whose factors are in \(S_4\) is nilpotent iff \(N^s \subseteq N_s\), for every \(s \in I_n\).

**Proof.** Let \(N\) be a nilpotent 4J-near-ring. We will show that, for every \(i \in I_n\), \(NN_i \subseteq N_{i+1}\). If \(NN_i \not\subseteq N_{i+1}\), there is an element \(a \in N\) such that \(aN_i \not\subseteq N_{i+1}\). Since \(aN_i\) is a subnear-ring of \(N_i\) and \(N_i/N_{i+1}\) is of prime order, \((aN_i + N_{i+1})/N_{i+1}\) is not a proper subnear-ring of \(N_i/N_{i+1}\). Therefore, either \(aN_i + N_{i+1} = N_{i+1}\) or \(aN_i + N_{i+1} = N_i\). Given that \(aN_i \not\subseteq N_{i+1}\), we have:

\[(\alpha)\quad aN_i + N_{i+1} = N_i\]

and \(a^hN_i = a^{h+1}N_i + a^hN_{i+1}\). Let \(h'\) be the smallest integer such that \(a^{h'}N_i \subseteq N_{i+1}\). This \(h'\) exists and it is \(h' > 1\) because otherwise, for every \(t \in N\), it would be \(a^tN_i + N_{i+1} = N_i\) and since \(N\) is nilpotent, it would be \(N_{i+1} = N_i\) and this is excluded. Therefore, by \((\alpha)\), we obtain \(a^{h'}N_i + a^{h'-1}N_{i+1} = a^{h'-1}N_i\), hence \(a^{h'-1}N_i \subseteq N_{i+1}\) in contrast to the hypothesis stating that \(h'\) is the smallest integer so that \(a^{h'}N_i \subseteq N_{i+1}\). Thus \(NN_i \subseteq N_{i+1}\) and consequently \(N^s \subseteq N_s\) for every \(s \in I_n\). The converse is trivial. \(\diamondsuit\)

\(^{(5)}\) For the definition of \(p\)-singular near-ring see [6].
Corollary 4. If $N$ is a nilpotent 4J-near-ring, $g(N) \leq \dim(N)$.

**Proof.** It is a consequence of Th.7. \hfill \checkmark

We can characterize the case in which $g(N) = \dim(N)$.

**Theorem 8.** Let $N$ be a nilpotent 4J-near-ring and let $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ a series whose factors are in $S_4$. The length of the chain and the nilpotence index of $N$ coincide iff $N_i = (N_{i+1} : N)_N$ for every $i \in I_{n-1}$.

**Proof.** We set $M_i = (N_{i+1} : N)_N = \{n \in N/Nn \subseteq N_{i+1}\}$. Let $g(N) = \dim(N) = n$. By Th.7, we have $NN_i \subseteq N_{i+1}$ and hence $N_i \subseteq M_i$. If $N_i$ is strictly contained in $M_i$, the series $N \supset M_i \supset N_i \supset \{0\}$ will be refinable (by Jordan-Hölder theorem) in a principal series where $M_i = \overline{N}_j$ with $j \leq i$. By Th.7, $N^j \subseteq \overline{N}_j$ and hence $N^j \subseteq M_i$. Therefore $N^{j+1} \subseteq NM_i \subseteq N_{i+1}$. Hence $N^{j+1+(n-i-1)} = N^{n-(i-j)} = \{0\}$. Given that $g(N) = n$, we obtain $i = j$, that is $M_i = \overline{N}_j = N_i$.

Conversely, let us suppose $N_i = M_i$ for every $i \in I_{n-1}$ and $g(N) = h$. Then $N^h = \{0\}$, therefore $N^{h-1} \subseteq (0 : N)_N = N_{n-1}$, in fact $N_{n-1}$ is the right annihilator of $N$ because $N_{n-1} = M_{n-1} = (N_n : N)_N$. Analogously $N^{h-2} \subseteq (N_{n-1} : N)_N = N_{n-2}$ and so on. After a finite number of steps we get $N \subseteq N_{n-h+1}$, thus $N = N_{n-h+1}$ and $n = h$. \hfill \checkmark

**Finally:**

**Theorem 9.** If $N$ is a nilpotent 4J-near-ring such that $g(N) = \dim(N)$, then $|N| = p^\alpha$, ($p$ prime).

**Proof.** We can prove this theorem by induction on $g(N)$. If $g(N) = 1$, $N = N_1 \supset N_2 = \{0\}$ is the principal series required and hence $|N| = p$.

Let us suppose the theorem proved for $g(N) = n - 1$ and let $N = N_1 \supset N_2 \supset \ldots \supset N_n = \{0\}$ be a series of $N$ whose factors are in $S_4$. Then $|N/N_{n-1}| = p$ and we can suppose $|N_{n-1}| = q$ ($q$ prime).

By Th.7, $N^{n-2} N = N^{n-1} \subseteq N_{n-1}$, therefore, for every $m \in N^{n-2}$, $mN \subseteq N_{n-1}$ and given that $N_{n-1}$ is of prime order, either $mN = \{0\}$ or $mN = N_{n-1}$. If $mN = \{0\}$, for every $m \in N^{n-2}$, then $N^{n-1} = \{0\}$ and this is excluded, thus $mN = N_{n-1}$ for some $m \in N$.

Considering now the left translation $\gamma_m : N \to mN$, we obtain an endomorphism of $N^+$ whose kernel is $r(m)$, the right annihilator of $m$ and whose image is $N_{n-1}$. Therefore $|\text{im } \gamma_m| = |N/\ker \gamma_m|$ that is $q = |N/\ker \gamma_m|$. Given that $\ker \gamma_m = r(m) \supseteq r(N) = N_{n-1}$, either $|\ker \gamma_m| = q$ or $|\ker \gamma_m| = q^\beta$. Thus: $q = q^\alpha / q^\beta$ and this implies $q^\beta = p^\alpha$, hence $p = q$. \hfill \checkmark
References


