THE CONGRUENCE LATTICE OF IMPLICATION ALGEBRAS

Sándor Radelecki

Mathematical Institute, University of Miskolc, 3515 Miskolc-
Egyetemváros, Hungary.

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Abstract: The variety of implication algebras is a minimal quasivariety.
It is 3-filtrial but not 2-filtrial. An implication algebra is tolerance-trivial
iff \((A, \leq)\) is a lattice, where the partial ordering "\(\leq\)" is defined as follows:
\[a \leq b \iff \exists x \in A \text{ such that } b = x \cdot a.\]

1. Introduction

Implication algebras are groupoids with a simple binary operation,
which yields a partially order. This derived order structure can be
considered as a generalization of Boolean lattices (see Prop.2).

Definition 1 ([1], [9]). A groupoid \((A, \cdot)\) is called an implication
algebra if the operation "\(\cdot\)" satisfies the following axioms:

\[ (a \cdot b) \cdot a = a \]
\[ (a \cdot b) \cdot b = (b \cdot a) \cdot a \]
\[ a \cdot (b \cdot c) = b \cdot (a \cdot c). \]

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Example. If \((B, \lor, \land, 0, 1, \lnot)\) is a Boolean algebra then \((B, \to)\) and 
\((B, /)\), where \(a \to b = a^\lor \lor b\) and \(a / b = a^\land \land b\) for all \(a, b \in B\), are both implication algebras.

Remark. If the algebra above is the Boolean algebra of propositional calculus then "\(\to\)" represents ordinary implication.

Implication algebras are examples of algebraic varieties which are 3-permutable, 3-congruence distributive and 3-congruence modular but are not either congruence permutable or 2-distributive or 2-modular: [9], [4].

In this paper we shall prove a new property of implication algebras, namely that they are 3-filtral but not 2-filtral (§2) and we shall characterize those implication algebras on which every compatible tolerance is a congruence (§3)

Let us first review a few concepts:

A variety \(V\) is congruence permutable (congruence 3-permutable) if \(\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1 (\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2)\) for any two congruences \(\Theta_1, \Theta_2 \in \text{Con } A\) and for any \(A \in V\) (where "\(\circ\)" is the relational product of congruences); 3-congruence modularity and 3-congruence distributivity mean that the systems of equations of H.P. Gumm and B. Johnson respectively for congruence modularity and congruence distributivity consist of at least \(3+1\) terms.

For example 3-distributivity means that the following system of equations (where \(n, i \in \mathbb{N}\); \(q_0, q_1, \ldots, q_n\) are 3-variable terms):

\[
\begin{align*}
q_0(x, y, z) &= x, & q_n(x, y, z) &= z \\
q_i(x, y, x) &= x, & 0 \leq i \leq n \\
q_i(x, x, y) &= q_{i+1}(x, x, y), & i \text{ even} \\
q_i(x, y, y) &= q_{i+1}(x, y, y), & i \text{ odd}
\end{align*}
\]

must contain at least \(3+1\) terms, i.e.: \(n = 3\).

For implication algebras these terms are:

\[
\begin{align*}
q_0(x, y, z) &= x, & q_3(x, y, z) &= z \\
q_1(x, y, z) &= [y \cdot (z \cdot x)] \cdot x, & q_2(x, y, z) &= (x \cdot y) \cdot z
\end{align*}
\]

for all \(x, y, z \in A\).

Filtral varieties can be defined using the notion of product congruence:
Let $A$ be the subdirect product of algebras $A_i$ ($i \in I$) and let $a_i$ denote the $i$-th component of $a \in A$ belonging to $A_i$. A congruence $\varphi \in \text{Con } A$, is called the product of the congruences $\varphi_i \in \text{Con } A_i$, $i \in I$ if $a \varphi b$ exactly when $a_i \varphi_i b_i$ for all $i \in I$. We write $\varphi = \prod_{i \in I} \varphi_i$.

Definition 2 ([7],[8]). A variety $\mathcal{V}$ is called an ideal variety iff for all $A \in \mathcal{V}$ every compact congruence on $A$ is a product congruence.

Definition 3 ([7],[8]). A variety $\mathcal{V}$ is called filtral if it is an ideal variety and it is semi-simple i.e. all its subdirect irreducible algebras are congruence-simple.

We shall denote the class of subdirect irreducible algebras of a variety $\mathcal{V}$ by $\text{SIV}_\mathcal{V}$, and the variety of implication algebras by $\mathcal{V}(1)$. E. Fried and E. Kiss [5] gave the following characterization of filtral varieties by term functions (see also [8]):

**Theorem** ([5],[8]). A variety $\mathcal{V}$ is filtral iff there is an $n \in \mathbb{N}$ and there are 3-variable terms $f_0, f_1, \ldots, f_n$ ($n > 1$) such that for any $x, y, z$ in any algebra of $\mathcal{V}$ we have:

\begin{align*}
\text{(a) } & f_0(x, y, z) = x, \quad f_n(x, y, z) = z, \\
\text{(b) } & f_i(x, y, x) = x, \quad \text{ (for all } i : 0 \leq i \leq n), \\
\text{(3) } & f_i(x, x, z) = f_{i+1}(x, x, z), \quad \text{for } i \text{ even,} \\
\text{(d) } & \text{for all } A \in \text{SIV}_\mathcal{V} \text{ and } x, y, z \in A, x \neq y: \quad f_i(x, y, z) = f_{i+1}(x, y, z), \quad \text{for } i \text{ odd.}
\end{align*}

Proceeding in the same way as in characterization of congruence modular and congruence distributive varieties by a system of term equation, we can use the following concept:

**Definition 4.** According to the theorem above, if the system (3) of equations for $\mathcal{V}$ needs at least $n + 1$ terms, then $\mathcal{V}$ is called $n$-filtral. Eg. $\mathcal{V}$ is 3-filtral if $n = 3$ and $f_0, f_1, f_2, f_3$ satisfy conditions (3).

Let us now list some properties of implication algebras:

**Property 1** ([1]). Let be $A$ an implication algebra. We can define an partially ordering relation "$\leq$" on $A$ as follows:

$$a \leq b \iff \exists x \in A : b = x \cdot a.$$  

J.C. Abbott has shown [1] that this relation is isotone on the left and antitone on the right with respect to "$\cdot$" (i.e. $\forall c \in A$, if $a \leq b : c \cdot a \leq c \cdot b$ and $a \cdot c \geq b \cdot c$); furthermore $(A, \leq)$ is a semilattice with identity, i.e. $\sup\{a, b\} = (a \cdot b) \cdot b$ exists for all $a, b \in A$ and there is an element
1 ∈ A such that x ≤ 1 for all x ∈ A. "≤" can be defined using 1, since
a ≤ b ⇔ a · b = 1.

**Property 2 ([1]).** If (A, ≤) is the semilattice corresponding to the
implication algebra (A, ·), then every principal filter (\(\{x|x ≥ a\}, ≤\))
is a Boolean lattice. Vice versa in every semilattice with the above
mentioned property one can define a binary operation "·" for which
(A, ·) is an implication algebra in the following way:

\[ a · b = (a \lor b)_b \]

where \((a \lor b)_b\) denotes the complement of \(a \lor b\) in the Boolean lattice
(\(\{x|x ≥ b\}, ≤\)).

**Property 3 ([1]).** For a pair \(a, b ∈ A\),\( \inf \{a, b\} \) exists exactly when
\(\{a, b\}\) has a common lower bound \(c ∈ A\). In that case \(\inf \{a, b\} =
\[= \[a \cdot (b · c)\] · c.\]

**Remark ([1]).** \((A, ≤)\) is a Boolean lattice iff it has a least element,
denoted by \(0\) \((0 ≤ x, \text{ for all } x ∈ A)\).

**Definition 5 ([1]).** If \((A, ·)\) is an implication algebra and if the derived
partially ordered set \((A, ≤)\) is a lattice (i.e. for all \(a, b ∈ A\) \(\inf \{a, b\} =
\[= a ∧ b\) exists), then \((A, ≤)\) (and \((A, ·, ≤)\) as well) is called an implication
lattice.

2. The variety and congruences of implication
algebras

One of the most notable properties of implication algebras is that
is a one-to-one correspondence between their congruences and their
filters.

A subset \(F ⊆ A\) of a partially ordered set \((A, ≤)\) is called a filter
if for all \(a ∈ F\) and \(x ∈ A\), \(x ≥ a ⇒ x ∈ F\) and if \(\inf \{x_1, x_2\} = x_1 ∧ x_2
exists for x_1, x_2 ∈ F\), then \(x_1 ∧ x_2 ∈ F\). E.g. \([a] = \{x ∈ A|x ≤ a\}\) is a
filter, called the principal filter belonging to \(a\). By Property 1 if \(a ≠ b
then \([a] ≠ [b]\).

One can easily show that the intersection of a given family \(\{F_i\}_{i ∈ I},
I ≠ \emptyset\) of filters of \((A, ≤)\) is also a filter; \(\prod_{i ∈ I} F_i\) can be defined as the
intersection of all filters containing the set \(\bigcup_{i ∈ I} F_i\). If \(F_A\) denotes the
set of all filters of an implication algebra \((A, ·)\), then \((F_A, \prod, \bigcap, A, \{1\})
is a distributive complete lattice with 1 and 0.
From now on let $\Theta[a]$ denote the congruence class of $\Theta$ belonging to $a \in A$, i.e.: $\Theta[a] = \{x \in A | x \Theta a\}$.

**Property 4 ([1]).** The mapping $i : \text{Con} A \to \mathcal{F}_A$, $i(\Theta) = \Theta[1]$ is an isomorphism between $(\text{Con} A, \wedge, \vee, 1_A, 0_A)$ and $(\mathcal{F}_A, \cap, \bigcup, A, \{1\})$. For any $F \in \mathcal{F}_A$, $i^{-1}(F) = \Theta_F$, where $a \Theta_F b \iff a \cdot b, b \cdot a \in F$ ($i^{-1}$ denotes the inverse of the mapping $i$).

**Proposition 1.** The variety of implication algebras is a minimal quasivariety.

**Proof.** We begin by showing that $\mathcal{V}(I)$ has only one subdirect irreducible algebra, namely the 2-element one.

Let $A \in \text{SI}(I)$, $\gamma$ its monolit, and $F_\gamma$ the filter belonging to $\gamma$. Since $\gamma \leq \Theta$ for all $\Theta \in \text{Con} A (\Theta \neq 0_A)$, therefore $F_\gamma \subseteq \bigcap_{x \in A} [x]$ and so there exists an $a \in F_\gamma$ such that $F_\gamma = \{a\} = \{1, a\}$ and

$$a \geq x \text{ for all } x \in A \setminus \{1\}.$$ (4)

Suppose now that there exists an $x \in A \setminus \{1\}$ such that $x \neq a$. Since $([x], \leq)$ is a Boolean lattice (see Prop.2) and $a \in [x]$, there exists an $a^- \in [x]$ such that $a^- \wedge a = x$, and $a^- \vee a = 1$.

Now (4) gives $a^- \leq a \neq 1$ - which is a contradiction. Thus $A = \{1, a\}$, i.e. $A$ has two elements.

Two element implication algebras are isomorphic to each other and so $\text{SI}(I)$ contains only one non-trivial algebra (and this one is congruence and subalgebra simple at the same time).

A locally finite variety $\mathcal{V}$ is a minimal quasivariety exactly when it has only one SI algebra and this can be embedded into every non-trivial $B \in \mathcal{V}$ (see [2], Cor.2).

By [1] the number of elements in any free implication algebra generated by $n$ elements is at most $2^n$. Therefore any finitely generated implication algebra is finite and so $\mathcal{V}(I)$ is locally finite.

On the other hand for every nontrivial $B \in \mathcal{V}(I)$ and $x \in B, x \neq 1, \{1, x\}$ is a two-element subalgebra of $B$ and thus $\mathcal{V}(I)$ satisfies all previous conditions. ◊

**Corollary 1.** Every implication algebra $(A, \cdot)$ is a subdirect power of two element implication algebra $(\{1, a\}, \cdot)$.

**Theorem 1.** The variety of implication algebras is 3-filtral but not 2-filtral.

**Proof.** Assuming that $\mathcal{V}(I)$ is 2-filtral means there are three 3-variable terms $f_0, f_1, f_2$ sufficient for $\mathcal{V}(I)$ in the system (3) of equations. But in
this case from (3) we get that \( \mathcal{V}(I) \) is 2-distributive, contradicting [8].

To prove that \( \mathcal{V}(I) \) is 3-filtral we shall use the terms \( q_0, q_1, q_2, q_3 \) from (2)—which were used first for distributivity. Let us check the identities of (3):

(a) is clear;
(b) \( q_i(x, y, z) = x, \quad 0 \leq i \leq 2 \) (by distributivity - (1));
(c) From (1) we have \( q_0(x, x, z) = q_1(x, x, z) \) and \( q_2(x, x, z) = q_3(x, x, z) \);
(d) Let \( x, y, z \) be elements of the subdirect irreducible algebra \( \{0, 1\}, \cdot \) and let \( x \neq y \):

\[
\begin{align*}
\text{If } x = 0 \text{ and } y = 1 \text{ then } q_1(0, 1, z) &= [1 \cdot (z \cdot 0)] \cdot 0 = (z \cdot 0) \cdot 0 = z, \\
q_2(0, 1, z) &= (0 \cdot 1) \cdot z = z; \\
\text{If } x = 1 \text{ and } y = 0 \text{ then } q_1(1, 0, z) &= [0 \cdot (z \cdot 1)] \cdot 1 = 1, \\
q_2(1, 0, z) &= (1 \cdot 0) \cdot z = 0 \cdot z. \text{ Since } 0 \cdot 0 = 1 \text{ and } 0 \cdot 1 = 1, \text{ we have } 0 \cdot z = 1.
\end{align*}
\]

To sum up: if \( x \neq y \) then \( q_1(x, y, z) = q_2(x, y, z) \) and so all the identities of (3) are satisfied. ♦

**Corollary 2.** Every compact \( \Theta \in \text{Con} A (A \in \mathcal{V}(I)) \) has a complement.

**Proof.** By [7] (and [8]) if \( \mathcal{V}(I) \) is filtral then every compact congruence on \( \mathcal{V} \) has a complement. ♦

Let \( \text{Con}^c A \) denote the lattice of compact congruences of \( A \); \( \text{Con}^c A \) is the same lattice together with the element "1\( A \)" and let \( B(\text{Con}^c A) \) be the Boolean lattice generated by \( \text{Con}^c A \). (This one always exists, see [6]). Denoting the complement of \( \Theta \in \text{Con} A \) by \( \Theta^\bot \), let us define the operation "*" on \( \text{Con} A \) as follows: \( \Theta \ast \varphi = \Theta^\bot \lor \varphi \). (This way we obtain from \( B(\text{Con}^c A) \) an implication algebra in which, by [1], \( (A, \cdot) \) can be dually embedded). Let \( \Theta_a \) denote the congruence belonging to the principal filter \( [a] (a \in A) \), (and at the same time to the element \( a \in A \) as well).

**Proposition 2.** Let \( (A, \cdot) \) be an implication algebra and \( (A, \leq) \) the derived partially ordered set. The following statements are equivalent:

(i) \( (A, \leq) \) is a Boolean lattice;

(ii) \( (A, \leq) \) and \( \text{Con}^c A, \leq \) are dually order-isomorphic;

(iii) \( (A, \cdot) \) and \( B(\text{Con}^c A), \ast \) are dually isomorphic implication algebras.

**Proof.** (i) \( \Rightarrow \) (ii) by [11]. (For a more general construction see [6]).

(ii) \( \Rightarrow \) (i) and (iii) \( \Rightarrow \) (ii): Since \( \text{Con}^c A \) and \( B(\text{Con}^c A) \) both have a greatest element, \( (A, \leq) \) has a least element and therefore by [1] it is a Boolean algebra.
(i) $\Rightarrow$ (iii): If $\Theta \in \text{Con}^* A$, then $\Theta$ can be written as a finite union of principal filters $[a_1], \ldots, [a_n] = \{a_1, \ldots, a_n \in A, n \in \mathbb{N}\}$. Since $(A, \leq)$ is a lattice, $[a_1] \cap \ldots \cap [a_n] = [a_1 \land \ldots \land a_n]$ and therefore $\Theta[1]$ is a principal filter, i.e. there is an $a_\Theta \in A$ such that $[a_\Theta] = \Theta[1]$.

If $\bar{a}$ denotes the complement of $a$ and $\Theta_{\bar{a}}$ the corresponding congruence then $[a] \cap [\bar{a}] = \{x | x \geq a \text{ and } x \leq \bar{a}\} = \{x | x \geq 1\} = \{1\}$, so $\Theta_a \cap \Theta_{\bar{a}} = 0_A$ and $[a] \cup [\bar{a}] = [a \land \bar{a}] = [0] = A$, i.e. $\Theta_a \lor \Theta_{\bar{a}} = 1_A$. Hence $\Theta_a$ and $\Theta_{\bar{a}}$ are complements of each other; furthermore since for all $\Theta \in \text{Con}^c A$ there is an $a \in A$ such that $\Theta_a = \Theta$, $\Theta \in \text{Con}^c A$ holds as well (for all $\Theta \in \text{Con}^c A$). However, this means that $\text{Con}^* A = \text{B(Con}^* A)$ and by (i)$\Leftrightarrow$ (ii) $(A, \leq)$ and $(\text{B(Con}^* A), \leq)$ are dually order isomorphic Boolean algebras. But in that case, by [1] again, they are dually isomorphic as implication algebras. $\diamond$

3. Reflexive, compatible relations on implication algebras

A compatible relation $\rho \leq A \times A$ on $(A, \cdot)$ is called a compatible tolerance if $\rho$ is reflexive and symmetric ([3]).

Definition 6 ([3]). An algebra $A \in \mathcal{V}$ is called tolerance-trivial (T-trivial) if every compatible tolerance on $A$ is a congruence (i.e. transitive as well).

Theorem 2. Let $(A, \cdot)$ be an implication algebra. Then the following statements are equivalent:

(i) Every reflexive compatible relation on $(A, \cdot)$ is a congruence;
(ii) $(A, \cdot)$ is tolerance-trivial;
(iii) $(A, \leq)$ is an implication lattice.

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Let us define a relation $\rho$ as follows: $a \rho b$ $\iff$ there is a $k \in A$ such that $a \geq k$ and $b \geq k$. By definition $\rho$ is reflexive and symmetric. Let us show that $\rho$ is compatible as well. Consider $c \rho d$ ($c, d \in A$). This means that there is an $l \in A$ such that $c \geq l$ and $d \geq l$. Then $ca \geq a \geq k$ and $db \geq b \geq k$, while $ac \geq c \geq l$ and $bd \geq d \geq l$, thus $ca \rho db$ and $ac \rho bd$, i.e. $\rho$ is compatible. By (ii) $\rho$ is a congruence and $1 \rho a$ for any $a \in A$. Therefore $\rho = 1_A$. However, this means that for any $a, b \in A, \{a, b\}$ has a lower bound $m \in A$. By Prop.3 of [1] $\inf \{a, b\}$ exists for all $a, b \in A$ and hence $(A, \leq)$ is an implication lattice.
(iii) \(\Rightarrow\) (i): Let us now assume that \((A, \cdot)\) is an implication lattice. Using the idea of [4] (Th.8) first we show that if \((A, \leq)\) is a Boolean lattice then it satisfies (i). Indeed in that case there is a \(0 \in A\) such that \(0 \leq x\) for all \(x \in A\) and by [1] again the complement of \(a\), denoted by \(\overline{a}\), can be obtained as \(\overline{a} = a \cdot 0\). Since \(a \lor b = (a \cdot b) \cdot b, a \land b = [a \cdot (b \cdot 0)] \cdot 0\), every compatible relation on \((A, \cdot)\) is also a compatible relation on \((A, \land, \lor, 1, 0, \overline{\cdot})\). But since this algebra belongs to a Mal’cev variety all its reflexive compatible relations are congruences [3].

Now let \((A, \cdot)\) be an implication lattice and \(\rho\) a compatible reflexive relation on \(A\). Let \(a \rho b, b \rho c\) (for \(a, b, c \in A\)). Then \((a \land b) \land c = d\) exists and it is the greatest lower bound of \(\{a, b, c\}\). The restriction of \(\cdot\) to the principal filter \([d]\) is a Boolean algebra (with "0" element \(d\)) and \(a, b, c \in [d]\).

On the other hand the restriction of \(\rho\) to \([d]\) is also compatible and reflexive and thus it is also a congruence on \([d, \cdot\) as well. \(\checkmark\)

**Corollary 3.** Let \((A, \cdot)\) be an implication algebra. If the derived structure \((A, \leq)\) is an implication lattice, then the congruences of \((A, \cdot)\) permute.

**Proof.** In this case \((A, \cdot)\) is tolerance-trivial by Th.2. According to [10] every tolerance-trivial algebra has permutable congruences. \(\checkmark\)

**Corollary 4.** For a finite implication algebra \((A, \cdot)\) the following statements are equivalent:

(i) The derived partially ordered set \((A, \leq)\) is a Boolean lattice;

(ii) \((A, \cdot)\) is tolerance-trivial;

(iii) \((A, \cdot)\) and \((\text{Con} A, \ast)\) are dually isomorphic;

(iv) \((A, \leq)\) and \((\text{Con} A, \leq)\) are dually order isomorphic.

**Proof.** The proof is based on the fact that if \(A\) is finite then all its congruences are compact and so \(\text{Con} A = \text{Con}^c A = \text{Con}^{\ast c} A = B(\text{Con}^{\ast c} A)\). Applying Prop.2 we get Cor.4.

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