ON AN INTEGRAL EQUATION

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Abstract: A class $G$ of functions $f$ is introduced such that the integral equation $x(t) = \int_0^t f(s, x(s)) \, ds$ has solutions. This class is more general than the class of functions $f$ satisfying the classical Carathéodory's conditions.

Let $\mathbb{R}$ be the set of all reals, $I = [t_0, t_0 + a]$, $J = \{x \in \mathbb{R}^n : |x - x_0| \leq b\}$. A function $f : I \times J \to \mathbb{R}^n$ satisfies the Carathéodory's conditions (C) if

(i) almost all sections $f_t(x) = f(t, x)$ ($t \in I$, $x \in J$) are continuous,
(ii) all sections $f^x(t) = f(t, x)$ ($t \in I$, $x \in J$) are measurable (in the sense of Lebesgue), and
(iii) there is an integrable function (in the sense of Lebesgue) $m : I \to \mathbb{R}$ such that $|f(t, x)| \leq m(t)$ for every $(t, x) \in I \times J$.

It is well known the following theorem:

Theorem 0. ([2], p.7–8, Th. 1). Suppose that $f : I \times J \to \mathbb{R}^n$ is a function satisfying the conditions (C) and $d$ is a number such that $0 < d \leq a$, $\bar{g}(t_0 + d) = \int_{t_0}^{t_0+d} m(s) \, ds \leq b$. Then there is an absolutely

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continuous function \( h : [t_0, t_0 + d] \to J \) such that \( h(t_0) = x_0 \), and \( h(t) = x_0 + \int_{t_0}^{t} f(s, h(s)) ds \).

In this paper we prove that Th. 0 remains true if the conditions (C) will be replaced by more general conditions (G).

We say that a function \( f : I \times J \to \mathbb{R}^n \) satisfies the conditions (G) if

(i) for every continuous function \( h : I \to J \) the superposition \( t \to f(t, h(t)) \) is measurable,

(ii) there exists an integrable function \( m : I \to \mathbb{R} \) such that \( |f(t, x)| \leq m(t) \) for every \((t, x) \in I \times J\), and

(iii) there is a sequence of functions \( f_k : I \times J \to \mathbb{R}^n \) satisfying the conditions (C) with \( |f_k(t, x)| \leq m(t) \) for \((t, x) \in I \times J\), \( k = 1, 2, \ldots \), and such that for every subsequence \( (f_{k_i}) \), for every sequence of continuous functions \( g_n : I \to J \) which converges uniformly on \( I \) to a function \( g \) and for every \( t \in I \) there is a strictly increasing sequence \( (n_i) \) of positive integers such that

\[
\lim_{i \to \infty} \int_{t_0}^{t} f_{k_{n_i}}(s, g_{n_i}(s)) ds = \int_{t_0}^{t} f(s, g(s)) ds.
\]

**Theorem 1.** Suppose that \( f : I \times J \to \mathbb{R}^n \) satisfies the conditions (G) and \( d \) is a number such that \( 0 < d \leq \alpha \), and \( \overline{g}(t_0 + d) = \int_{t_0}^{t_0 + d} m(s) ds \leq b \). Then there is an absolutely continuous function \( h : [t_0, t_0 + d] \to J \) such that

\[
h(t) = x_0 + \int_{t_0}^{t} f(s, h(s)) ds, \quad t \in [t_0, t_0 + d].
\]

**Proof.** Since \( f \) satisfies the conditions (G), there is a sequence of functions \( f_k \) satisfying the condition (iii). Without loss of generality we may assume that \( |f_k(t, x)| \leq m(t) \) for \((t, x) \in I \times J\), \( k = 1, 2, \ldots \). Since each \( f_k \) \( (k = 1, 2, \ldots) \) satisfies the conditions (C), by Th. 0 there are absolutely continuous functions \( h_k : [t_0, t_0 + d] \to J \) which satisfy the integral equations...
\[ h_k(t) = x_0 + \int_{t_0}^{t} f_k(s, h_k(s)) \, ds \quad \text{for} \quad t \in [t_0, t_0 + d]. \]

Remark that the functions \( h_k \) (\( k = 1, 2, \ldots \)) are uniformly bounded and equicontinuous on \([t_0, t_0 + d]\). By the Ascoli-Arzelà Theorem, there is a subsequence \((h_{k_i})_i\) which converges uniformly on \([t_0, t_0 + d]\) to a continuous function \( h : [t_0, t_0 + d] \to \mathbb{R}^n \). We shall prove that

\[ h(t) = x_0 + \int_{t_0}^{t} f(s, h(s)) \, ds \quad \text{for} \quad t \in [t_0, t_0 + d]. \]

Evidently, \( h_k(t_0) = x_0 \) (\( k = 1, 2, \ldots \)). So \( h(t_0) = \lim_{i \to \infty} h_{k_i}(t_0) = x_0 \).

Fix \( t \in [t_0, t_0 + d] \). There exists a subsequence \((m_j)_j\) of the sequence \((k_i)_i\) such that

\[ \lim_{j \to \infty} \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s)) \, ds = \int_{t_0}^{t} f(s, h(s)) \, ds. \]

Since

\[ h_{m_j}(t) = x_0 + \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s)) \, ds, \]

and

\[ \lim_{j \to \infty} h_{m_j}(t) = h(t), \]

we obtain by (jjj) the relation

\[ h(t) = \lim_{j \to \infty} h_{m_j}(t) = \lim_{j \to \infty} \left( x_0 + \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s)) \, ds \right) = \]

\[ = x_0 + \lim_{j \to \infty} \int_{t_0}^{t} f_{m_j}(s, h_{m_j}(s)) \, ds = x_0 + \int_{t_0}^{t} f(s, h(s)) \, ds. \]

From Th. 1 it follows immediately

**Corollary 1.** If a function \( f : I \times J \to \mathbb{R}^n \) satisfying the conditions (G) is such that for every continuous function \( h : [t_0, t_0 + d] \to J \) the superposition \( t \to f(t, h(t)) \) is a derivative then there exists a solution of the Cauchy's problem \( y'(t) = f(t, y(t)), y(t_0) = x_0 \), defined on \([t_0, t_0 + d]\).
Recollect that \( g : [t_0, t_0 + d] \to \mathbb{R}^n \) is a derivative at a point \( t \) if
\[
\lim_{r \to t} \int_r^t g(s)ds/(r - t) = g(t) \quad ([1 \text{ or } [3]).
\]

**Theorem 2.** If \( f, g : I \times J \to \mathbb{R}^n \) are functions satisfying the conditions (G) then the sum \( f + g \) satisfies the conditions (G).

**Proof.** Evidently, the sum \( f + g \) satisfies the conditions (j), (jj). Let 
\( (f_k), (g_k) \) be sequences of functions satisfying the condition (jjj) for \( f \) and \( g \), respectively. Obviously, the sums \( f_k + g_k \ (k = 1, 2, \ldots) \) satisfy the conditions (C). Suppose that a sequence of continuous functions \( h_k : I \to J \) converges uniformly on \( I \) to a function \( h \). Fix \( t \in I \). Let \( (k_n) \) be a strictly increasing sequence of positive integers. By (jjj) there are a subsequence \( (n_i) \) of the sequence \( (1, 2, \ldots) \) and a subsequence \( (i_j) \) of \( (n_i) \) such that
\[
\lim_{i \to \infty} \int_{t_0}^t f_{k_{n_i}}(s, h_{n_i}(s)) = \int_{t_0}^t f(s, h(s))ds, \quad \text{and}
\]
\[
\lim_{j \to \infty} \int_{t_0}^t g_{k_{i_j}}(s, h_{i_j}(s)) = \int_{t_0}^t g(s, h(s))ds.
\]

Consequently,
\[
\lim_{j \to \infty} \int_{t_0}^t \left( f_{k_{i_j}}(s, h_{i_j}(s)) + g_{k_{i_j}}(s, h_{i_j}(s)) \right) ds =
\]
\[
= \int_{t_0}^t (f(s, h(s)))ds + \int_{t_0}^t g(s, h(s))ds. \quad \diamond
\]

**Remark 1.** Analogously as above we may prove that the product \( kf \) satisfies the conditions (G) whenever \( k \in \mathbb{R} \) is a constant and the function \( f : I \times J \to \mathbb{R}^n \) satisfies the conditions (G).

**Remark 2.** From Remark 1 and Th. 2 it follows that the space \( G = \{ f : I \times J \to \mathbb{R}^n : f \text{ satisfies (G)} \} \) with the metric \( \rho(f, g) = \min(1, \sup \{|f(t, x) - g(t, x)| : (t, x) \in I \times J \}) \) is a linear metric space.

**Theorem 3.** Assume the Continuum Hypothesis. Then the set \( C = \{ f : I \times J \to \mathbb{R}^n : f \text{ satisfies (C) } \} \) is closed and nowhere dense in \( G \).
Proof. Of course, if \( f \in C \) then \( f \) satisfies the conditions (i), (ii) and the functions \( f_k = f \ (k = 1, 2, \ldots) \) satisfy all requirements of the condition (iii). So \( C \subset G \). Moreover, if a sequence of functions \( f_k : I \times J \to \mathbb{R}^n \) satisfying the conditions (C) converges uniformly (with respect to the metric \( \rho \) from Remark 2) to a function \( f \) then \( f \) satisfies also the conditions (C). So \( C \) is a closed set in \( G \) with respect to the metric \( \rho \). Fix \( f \in C \) and \( \varepsilon > 0 \ (\varepsilon < 1) \). Denote by \( \omega \) the first ordinal number of the continuum power. Let \( (h_\alpha)_{\alpha < \omega_1} \) be a transfinite sequence of all continuous functions \( (h_\alpha : I \to J) \) and let \( (F_\alpha)_{\alpha < \omega_1} \) be a transfinite sequence of all closed subsets of \( I \times J \) which are of positive (Lebesgue) measure and all sets \( E_t = \{(t, x) : x \in J\} \), \( t \in I \). Denote by \( G(h_\alpha) \) the graph of the function \( h_\alpha \) \( (\alpha < \omega_1) \). By transfinite induction, there is a set

\[
B = \{(t_\alpha, x_\alpha) \in I \times J : \alpha < \omega_1\}
\]

such that

\[
(t_\alpha, x_\alpha) \in F_\alpha - \bigcup_{\beta < \alpha} G(h_\beta), \quad \text{and} \quad x_\alpha \neq x_0 \quad \text{for} \quad \alpha < \omega_1;
\]

and for each \( t \in I \) the intersection \( B \cap E_t \) contains a sequence \( ((t, x_k))_k \) such that \( \lim_{k \to \infty} x_k = x_0 \). Let \( u \in \mathbb{R}^n \) be a point such that \( |u| = 1 \). Let us put

\[
g(t, x) = \begin{cases} 
\varepsilon u & \text{for} \ (t, x) \in B \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
h = f + g.
\]

Evidently, \( \rho(f, h) = \varepsilon \). To prove that \( h \in G - C \) it suffices to show that \( g \in G - C \). Since for each \( \alpha < \omega_1 \) the set \( \{t \in I : g(t, h_\alpha(t)) \neq 0\} \) is countable, \( g \) satisfies the conditions (j), (jj) and

\[
\int_I g(s, h_\alpha(s)) \, ds = 0 \quad \text{for} \quad \alpha < \omega_1.
\]

Consequently, \( g \) satisfies the condition (jjj), and \( g \in .G \). Fix \( t \in I \). Since \( g(t, x_0) = 0 \) and \( x_0 \) is an accumulation point of the set \( B \cap E_t \), the section \( g_t \) is not continuous at \( x_0 \). So \( g \notin C \), and the proof is completed.

\[\Diamond\]

Remark 3. In Th. 4 the Continuum Hypothesis can be replaced by the Martin's Axiom.

Example 1. Let \( I = [0, 1] \), \( J = [-1, 1] \), and let
\[ f(t, x) = \begin{cases} 
1 & \text{if } x = 0 \text{ and } 1/(2n + 1) < t < 1/2n, \ n = 1, 2, \ldots \\
0 & \text{otherwise.} 
\end{cases} \]

The function \( f \) is of Baire class 1. If the function \( x : [0, d] \to J \ (d \leq 1) \) satisfies the equation

\[(*) \quad x(t) = \int_0^t f(s, x(s)) ds \]

then \( x(0) = 0 \), \( x \) is nondecreasing, and \( x(t) > 0 \) for \( t > 0 \ (t \leq d) \). But, in this case \( f(s, x(s)) = 0 \) for \( s > 0 \) and \( x(t) = 0 \) for \( t \in [0, d] \). This contradiction proves that the integral equation (*) has not an absolutely continuous solution, and consequently \( f \notin G \).

**Example 2.** Let \( I = [0, 1] \) and \( J = [-1, 1] \). Denote by \( T_e \) and \( T_d \), respectively the euclidean and the density topologies in \( R^2 \) (for the definition of the density topology see [1]). There is an approximately continuous (i.e. \( (T_d, T_e) \) continuous) function \( g : J \to [0, 1] \) is such that \( g[1/k] = 1 \) for \( k = 1, 2, \ldots \), and \( g(0) = 0 \) (see [1]). Consequently, the function \( f(t, x) = g(x) \) is a \( (T_e \times T_d, T_e) \) continuous mapping. Assume that \( f \in G \), let \( (f_k) \) be a sequence of functions from \( G \) corresponding to \( f \) by the condition (jjj). For \( k = 1, 2, \ldots \) there are an index \( n_k \) and a number \( y_k \) such that \( n_{k+1} > n_k \), \( |y_k - 1/k| < 1/k(k + 1) \), and

\[(i) \quad \left| \int_0^1 f_{n_k}(s, y_k) ds - \int_0^1 f(s, 1/k) ds \right| < 1/2. \]

Since

\[ \int_0^1 f(s, 1/k) ds = 1, \]

it follows from (i) that

\[ \int_0^1 f_{n_k}(s, y_k) ds > 1/2. \]

Then the sequence \( (y_k) \) converges uniformly to 0 and there is not a strictly increasing sequence \( (k_i) \) of positive integers such that
\[
\lim_{i \to \infty} \int_0^1 f_{n_{k_i}}(s, y_{k_i}) ds = \int_0^1 f(s, 0) ds = 0.
\]

So, \( f \notin G \). Observe that the integral equation (*) has a solution \( x(t) = 0 \) for \( t \in I \).

References

