COMMUTATIVITY THEOREM FOR s-UNITAL RINGS

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Abstract: The main result of this paper is a commutativity theorem for associative rings satisfying the polynomial identity $x'[x^n, y]y^s = \pm [x, y^m]$ (see Th. 1).

1. Introduction

Throughout the present paper $R$ will represent an associative ring (with or without unity 1), $Z(R)$ the center of $R$, $N(R)$ the set of all nilpotent elements of $R$, $N'(R)$ the set of all zero divisors of $R$, and $C(R)$ the commutator ideal of $R$. A ring $R$ is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x$ in $R$. Further, $R$ is called s-unital if $R$ is both left and right s-unital, that is $x \in xR \cap Rx$, for every $x$ in $R$. If $R$ is s-unital (resp. left or right s-unital), then for any finite subset $F$ of $R$, there exists an element $e$ in $R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$), for every $x$ in $F$.

For any $x, y$ in $R$, we write as usual $[x, y] = xy - yx$. For a positive integer $n$, we consider the following property of a ring $R$
Q(n): For all \( x, y \) in \( R \), \( n[x, y] = 0 \) implies \([x, y] = 0\).

Obviously, every \( n \)-torsion free ring \( R \) has the property \( Q(n) \) and every ring \( R \) has the property \( Q(1) \). If a ring \( R \) has the property \( Q(n) \), then \( R \) has the property \( Q(m) \) for any factor \( m \) of \( n \).

In a recent paper [2] we considered (one-sided) \( s \)-unital rings \( R \) satisfying

\( (P) \): There exists non-negative integers \( m, n, s \) and \( t \), \( m > 0 \) or \( n > 0 \), and \( s \neq t \) for \( m = n = 1 \) such that \( x^t[x^n, y] = \pm x^s[x, y^m] \), for all \( x, y \) in \( R \), or \( x^t[x^n, y] = \pm [x, y^m] x^s \), for all \( x, y \) in \( R \).

Now, our objective is to investigate the commutativity of a ring \( R \) which satisfies the polynomial identity

\( (1) \)

\[ x^t[x^n, y]y^s = \pm [x, y^m], \]

for some given non-negative integers \( m, n, s \) and \( t \). Since we, as in the case that \( R \) has a unity 1, under \( x^ty \), resp. \( xy^s \), for \( t = 0 \), resp. \( s = 0 \), understand \( y \), resp. \( x \), the above identity take sense also when some of the exponents becomes zero. For \( m = n = 0 \), or \( m = n = 1 \) and \( s = t = 0 \), any ring \( R \) satisfies the identity \((1)\), and thus, in this case, she cannot contribute to the commutativity of a ring. Hence, we can exclude the above mentioned values of non-negative integers \( m, n, s \) and \( t \). For the remained values we will prove here three theorems. The main result of the present paper is the following

**Theorem 1.** Let \( m, n, s \) and \( t \) be fixed non-negative integers such that \( m > 0 \) or \( n > 0 \), and \( s > 0 \) or \( t > 0 \) if \( m = 1 \), \( n = 1 \). If \( R \) is a ring which satisfies the polynomial identity \((1)\), then \( R \) is commutative provided that one of the following additional conditions is fulfilled:

(a) \( m = 0 \), and \( R \) is an \( s \)-unital (resp. a left \( s \)-unital for \( s = 0 \), or a right \( s \)-unital for \( t = 0 \)) ring with property \( Q(n) \);

(b) \( n = 0 \), and \( R \) is a left or right \( s \)-unital ring with the property \( Q(m) \);

(c) \( m = 1, n \geq 1 \), or \( m > 1, n = 1 \) and \( s = t = 0 \);

(d) \( m > 1, n > 1 \), and \( R \) is a left or right \( s \)-unital ring with the property \( Q(m) \);

(e) \( m > 1, n = 1, s + t > 0 \), and \( R \) is a left or right \( s \)-unital ring (with the property \( Q(m + 1) \) for \( t = 0 \)).

2. Preparation for the proof

In the preparation for the proof of the above theorem, we start by stating without proof the following well-known lemmas.
Lemma 1. ([4, Lemma]). Let $R$ be a ring with $1$, and let $f$ be a polynomial function of two variables such that $f(x + 1, y) = f(x, y)$ for all $x, y \in R$. If there exists a positive integer $n$ such that $x^n f(x, y) = 0$ for all $x, y \in R$, then $f(x, y) = 0$ for all $x, y \in R$.

Lemma 2. ([9, Lemma 3]). Let $x$ and $y$ be elements in a ring $R$. If $[x, [x, y]] = 0$, then $[x^k y] = k x^{k-1} [x, y]$ for all integers $k \geq 1$.

Lemma 3. ([14, Lemma]). Let $R$ be a left (resp. right) s-unital ring. If for each pair of elements $x$ and $y$ in $R$, there exists a positive integer $k = k(x, y)$ and an element $e = e(x, y)$ of $R$ such that $x^k e = x^k$ and $y^k e = y^k$ (resp. $e x^k = x^k$ and $e y^k = y^k$), then $R$ is an s-unital ring.

An especially important role in proving all results of this paper play the following two results. The first is due to T. P. Kezlan [7, Th.] and H. E. Bell [3, Th. 1] (also see [12, Prop. 2]), and the second was proved by W. Streb [13, Hauptsatz 3].

Theorem KB. Let $f$ be a polynomial in non-commuting indeterminates $x_1, \ldots, x_n$ with (relatively prime) integral coefficients. Then the following are equivalent:

1) For any ring $R$ satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal;
2) every semi-prime ring $R$ satisfying $f = 0$ is commutative;
3) for every prime $p$, $(GF(p))_2$ fails to satisfy $f = 0$.

Theorem S. Let $R$ satisfy a polynomial identity of the form $[x, y] = p(x, y)$, where $p(X, Y) \in \mathbb{Z}[X, Y]$, the ring of polynomials in two non-commuting indeterminates over the ring $\mathbb{Z}$ of integers, has the following properties:

(i) $p(X, Y)$ is the kernel of the natural homomorphism from $\mathbb{Z}(X, Y)$ to $\mathbb{Z}[X, Y]$, the ring of polynomial in two commuting indeterminates;
(ii) each monomial of $p(X, Y)$ has total degree at least 3;
(iii) each monomial of $p(X, Y)$ has $X$-degree at least 2, or each monomial of $p(X, Y)$ has $Y$-degree at least 2.

Then $R$ is commutative.

Now, we need the following Lemma which enables us to reduce the proof of Th. 1 to ring $R$ with unity 1 (if $R$ is left or right s-unital).

Lemma 4. Let $m, n, s$ and $t$ be fixed non-negative integers such that $m > 0$ or $n > 0$, and $s > 0$ or $t > 0$ if $m = n = 1$. If a ring $R$ satisfies (1), then $R$ is s-unital in all of the following cases:

(a') $m = 0$, and $R$ is a left s-unital ring for $s = 0$, or a right s-unital ring for $t = 0$;
(b') \( n = 0 \), and \( R \) is a left or right \( s \)-unital ring;
(c') \( m > 1, n > 1 \) (or \( n = 1, s + t > 0 \)) and \( R \) is a left or right \( s \)-unital ring.

**Proof.** Let \( x \) and \( y \) be arbitrary elements in \( R \). If \( R \) is a left (resp. right) \( s \)-unital ring, then we can choose an element \( e \) (resp. \( f \)) in \( R \) such that \( ex = x \) and \( ey = y \) (resp. \( xf = x \) and \( yf = y \)).

Case (a'): For \( m = 0 \) the identity (1) reduces to

\[
(2) \quad x^t[x^n, y]^s = 0 \quad \text{for all} \quad x, y \in R.
\]

If \( s = 0 \) and \( R \) is left \( s \)-unital, then by (2), for \( x = e \), we get \( y = ye^n \), and thus, \( R \) is \( s \)-unital. For \( t = 0 \), and \( R \) a right \( s \)-unital ring, from (3) we derive \( x^n = fx^n \) and \( y^n = fy^n \), which by Lemma 3, means that \( R \) is also left \( s \)-unital.

Case (b'): For \( n = 0 \), the identity (1) becomes

\[
(3) \quad [x, y^m] = 0 \quad \text{for all} \quad x, y \in R.
\]

Hence, by (3), \( x = x^m \) (resp. \( x = f^m x \)) if \( R \) is left (resp. right) \( s \)-unital and thus, \( R \) is \( s \)-unital.

Case (c'): If \( R \) is left \( s \)-unital, then by (1), \( x = xe^m - x^{n+t}exe^s + x^{n+s+t} \in xR \), since \( m > 1 \) and \( n > 1 \) (or \( n = 1 \) and \( s + t > 0 \)). Hence, \( R \) is \( s \)-unital.

Similarly, one can see that \( R \) is \( s \)-unital if \( R \) is right \( s \)-unital. \( \diamond \)

Further, we prove that, for the ring in Th. 1, \( C(R) \subseteq N(R) \). In fact, we prove the following lemma:

**Lemma 5.** Let \( m, n, s \) and \( t \) be fixed non-negative integers such that \( m > 0 \) or \( n > 0 \), and \( s > 0 \) or \( t > 0 \) if \( m = n = 1 \). If \( R \) satisfies the polynomial identity (1), then the commutator ideal \( C(R) \) of \( R \) is a nil ideal, i.e. \( C(R) \subseteq N(R) \).

**Proof.** In view of Th. KB, it suffices to prove that, for every prime \( p \), there exist \( x, y \) in the full ring \( (GF(p))^2 \) of \( 2 \times 2 \) matrices over Galois field \( GF(p) \) which fail to satisfy the identity (1). Actually, we can take

\[
\begin{align*}
x &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{for } m = 0, \\
x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & y &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \text{for other cases}. \end{align*}
\]

For \( s = t = 0 \), and \( m > 1, n = 1 \), the ring \( R \) in Th. 1 is commutative by Herstein's criterion [5, Th. 18] and also by Th. S, and for \( m = 1 \)}
and \( n \geq 1 \), by Th. S, \( R \) is commutative for arbitrary non-negative integers \( s \) and \( t \) (such that \( s > 0 \) or \( t > 0 \) if \( n = 1 \)). In all remaining cases, the ring \( R \) in Th. 1 is \( s \)-unital by Lemma 4. Hence, for these cases, in view of [6, Prop. 1], we can and will assume that \( R \) has unity 1. Under this assumption, as the next step in the proof of Th. 1, we have

**Lemma 6.** For the ring \( R \) in Th. 1, all nilpotent elements are central, i.e. \( N(R) \subseteq Z(R) \).

**Proof.** Take an arbitrary element \( a \) in \( N(R) \). Then there exists a positive integer \( p \) such that

\[
(4) \quad a^k \in Z(R) \quad \text{for all integers } k \geq p, \quad p \text{ minimal.}
\]

If \( p = 1 \), then \( a \in Z(R) \). Suppose that \( p > 1 \), and set \( b = a^{p-1} \). By (4), we have

\[
(5) \quad b^k \in Z(R), \quad \text{and } b^k[x, b] = [x, b]b^k = 0 \quad \text{for all }
\]
\[
x \in R \quad \text{and all integers } k > 1.
\]

1) Let \( m = 0 \) and suppose that \( R \) has the property \( Q(n) \). Set \( 1 + b \) for \( x \) in (2). In view of (5) and the invertibility of \( (1 + b)^t \), we get \( n[b, y]y^s = 0 \) for all \( y \in R \), hence, by Lemma 1, \( n[b, y] = 0 \) for all \( y \in R \). In view of the property \( Q(n) \), this yields \([b, y] = 0 \) for all \( y \in R \), i.e. \( a^{p-1} \in Z(R) \), which contradicts to the minimality of \( p \) in (4).

2) Let now \( n = 0 \) and suppose that \( R \) has the property \( Q(m) \). Set \( 1 + b \) for \( y \) in (3). Then, in view of (5), \( m[x, b] = 0 \), for all \( x \in R \), hence by \( Q(m) \), \([x, b] = 0 \) for all \( x \in R \), i.e. \( a^{p-1} \in Z(R) \), which is a contradiction.

3) Let \( m > 1, n > 1 \) and suppose that \( R \) has the property \( Q(n) \). Then by (5), for \( x = b \), the identity (1) gives \([b, y^m] = +b^t[b^t, y]y^s = 0 \) for all \( y \in R \). Therefore, setting \( 1 + b \) for \( x \) in (1), we get \((1 + b)^t[(1 + + b)^t, y]y^s = 0 \) for all \( y \in R \). In view of (5) and the invertibility of \( (1 + b)^t \), this implies \( n[b, y]y^s = 0 \) for all \( y \in R \), hence, by Lemma 1 and the property \( Q(n) \), \([b, y] = 0 \) for all \( y \in R \), i.e. \( a^{p-1} \in Z(R) \), and this is a contradiction.

4) Let, finally, \( m > 1, n = 1 \) and \( s + t > 0 \). If \( t > 0 \), then setting \( 1 + b \) for \( x \) in (1), we get, in account of (5), \([b, y]y^s = -tb[b, y]y^s + [b, y^m] = -tb[b, y]y^s + b^t[b, y]y^s \), i.e. \([b, y]y^s = -tb^2[b, y]y^s + b^{t+1}[b, y]y^s = 0 \). Hence, \([b, y]y^s = -tb[b, y]y^s + b^t[b, y]y^s = 0 \) for all \( y \in R \). According to Lemma 1, this yields \([b, y] = 0 \) for all \( y \in R \), i.e. \( a^{p-1} \in Z(R) \). If \( t = 0 \), then from (1), for \( y = 1 + b \), we get \([x, b](1 + sb) = \pm m[x, b], \) i.e. \((m \mp 1)[x, b] = \pm s[x, b]b\), or, by (5), \((m \mp 1)[x, b]b = 0\), i.e. \((m \mp 1)[x, b]b = 0\). In view of \( Q(m \mp 1) \), this yields \([x, b]b = 0\), i.e. \((m \mp 1)[x, b]b = 0\).
for all \( x \in R \), and thus, \([x, b] = 0\) for all \( x \in R \), i.e. \( a^{p-1} \in Z(R) \), a contradiction. \( \diamond \)

By Lemmas 5 and 6, for the ring \( R \) in Th. 1, we have
\[
C(R) \subseteq N(R) \subseteq Z(R),
\]
hence, especially,
\[
[x, [x, y]] = 0 \quad \text{for all} \quad x, y \in R. \tag{7}
\]
In view of (7) and Lemma 2, the identity (1) can be rewritten in the form
\[
x^{n+t-1}[x, y]y^s = \pm m[x, y]y^{m-1} \quad \text{for all} \quad x, y \in R. \tag{1'}
\]

By an argument similar to Lemma 1, it is easily to see, that for a ring \( R \) with unity 1 satisfying the identity (1'), and any \( x, y \in R \),
\[
m[x, y] = 0 \quad \text{if and only if} \quad n[x, y] = 0. \tag{8}
\]
Especially, for such a ring \( R \), the properties \( Q(m) \) and \( Q(n) \) are equivalent.

3. Proof of main result and some comments and supplements

Proof of Th. 1. Case (a): Let \( m = 0 \) and suppose that \( R \) has the property \( Q(n) \). Then (1'), in view of Lemma 1 and the property \( Q(n) \), implies
\[
[x, y] = 0 \quad \text{for all} \quad x, y \in R.
\]

Case (b): If \( n = 0 \), and \( R \) has the property \( Q(m) \), then (1'), Lemma 1 and the property \( Q(m) \) yield
\[
[x, y] = 0 \quad \text{for all} \quad x, y \in R.
\]

Case (c): The commutativity of \( R \) in this case, was established earlier.

Case (d): Let \( m > 1, n > 1 \) and \( R \) be a ring with unity having the property \( Q(m) \). Since \( R \) also satisfies (1'), \( R \) has the property \( Q(n) \) too. Now, set \( 1 + x \) for \( x \) in (1), and combine the identity (1) with obtained one. Then we get \((1 + x)^t[(1 + x)^n, y]y^s = x^t[x^n, y]y^s\) for all \( x, y \in R \), hence, by Lemma 1, \((1 + x)^t[(1 + x)^n, y] = x^t[x^n, y]\) for all \( x, y \in R \). The last identity implies
\[
n[x, y] = f(x, y) \quad \text{for all} \quad x, y \in R, \tag{9}
\]
where \( f(X, Y) \) is a polynomial satisfying conditions of Th. S. But, the ring \( R \) satisfies the identity
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(10) \( k[x, y] = 0 \) for all \( x, y \in R \), and \( k = (2^{n+t} - 2)m \).

Namely, setting in (1'), \( 2x \) for \( x \) and combining the identity (1') with obtained one, we get \( k[x, y]y^{m-1} = 0 \) for all \( x, y \in R \), and \( k = (2^{n+t} - 2)m \), or, by Lemma 1, the identity (10). Now, by (10), there exists a minimal positive integer \( p \) such that

(11) \( p[x, y] = 0 \) for all \( x, y \in R \).

If \( p = 1 \), then \( R \) is commutative. Otherwise, by \( Q(n) \), \( n \) is relatively prime to \( p \), hence, there exist integers \( n' \) and \( p' \) such that \( 1 = nn' + pp' \), and thus, in view of (9) and (11),

\[ [x, y] = n'f(x, y) \text{ for all } x, y \in R. \]

Hence, \( R \) is commutative by Th. S.

Case (e): Let \( m > 1 \), \( n = 1 \), \( s + t > 0 \), and let \( R \) be a ring with unity 1.

For \( t > 0 \), we can derive (9) as in the case (d). Since now \( n = 1 \), this means that \( R \) is commutative (see Th. S).

If \( t = 0 \), then \( R \) has the property \( Q(m \mp 1) \). In this case, the identity (1'), for \( s = m - 1 \), in view of Lemma 1 and the property \( Q(m \mp 1) \), gives

\[ [x, y] = 0 \text{ for all } x, y \in R. \]

For \( s \neq m - 1 \), setting \( 1 + y \) for \( y \) in (1'), we get

(12) \( (m \mp 1)[x, y] = g(x, y) \) for all \( x, y \in R \),

where \( g(x, y) \) is a polynomial satisfying the conditions of Th. S. Since now, \( s + 1 \neq m \), from (1') we can easily derive

(13) \( k[x, y] = 0 \) for all \( x, y \in R \), and \( k = |2^{s+1} - 2^m|n \).

Thus, there exists again a minimal positive integer \( p \) for which (11) is satisfied. But then, from (12) and (13) we get, similarly as in the foregoing case,

\[ [x, y] = m'g(x, y) \text{ for all } x, y \in R, \]

and this, in view of Th. S, yields the commutativity of \( R \).

The following results are immediate consequences of Th. 1.

**Corollary 1 ([8, Th.]).** Let \( m, t \) be fixed non-negative integers. Suppose that \( R \) satisfies the polynomial identity \( x^t[x, y] = [x, y^m] \). Then

a) if \( R \) is left \( s \)-unital, then \( R \) is commutative except for \((m, t) = (1, 0)\);
b) if $R$ is right $s$-unital, then $R$ is commutative except for $m = 1$, $t = 0$; and also $m = 0$, $t > 0$.

**Corollary 2** ([11, Th. 2.]). Let $m \geq n \geq 1$ be fixed integers with $mn > 1$, and let $R$ be an $s$-unital ring. Suppose that every commutator $[x, y]$ in $R$ is $m!$-torsion free. If further, $R$ satisfies the polynomial identity $[x^n, y] = [x, y^m]$, then $R$ is commutative.

**Corollary 3** ([1, Lemma 2(2)]). Let $R$ be a ring with unity and $n > 1$ a fixed positive integer. If $R$ is $n$-torsion free and satisfies the identity $[x^n, y] = [x, y^n]$, then $R$ is commutative.

Finally, as complements to Th. 1, we prove the following two theorems, which are similar to Th. 3, resp. Th. 4 in [2].

**Theorem 2.** Let $R$ be a left or right $s$-unital ring which satisfies (1) and has the property $Q(2)$. Suppose that one of the integers $m - s - 1$ and $n + t - 1$ is odd. If, moreover, $R$ has one of the properties $Q(m)$, $Q(n)$, or especially, if $(m, n) = 2^r$ for some non-negative integer $r$, then $R$ is commutative.

**Proof.** If $m - s - 1$, resp. $n + t - 1$ is an odd integer, then from (1), for $-y$ instead of $y$, resp. for $-x$ instead of $x$, one gets $x^t[x^n, y]y^s = \pm[x, y^m]$. This, combined with (1), yields, in view of $Q(2)$,

$$x^t[x^n, y]y^s = 0, \quad [x, y^m] = 0 \quad \text{for all} \quad x, y \in R. \tag{14}$$

In view of the second part of (14), we see as in the proof of case (b) in Th. 1, that $R$ is $s$-unital, $R$ has the property $C(R) \subseteq N(R)$ and that

$$m[x, b] = 0 \quad \text{for all} \quad x \in R, \tag{15}$$

where $b$ is defined as in the proof of Lemma 6. Now, by Lemma 1, from the first part of (14), one gets

$$x^t[x^n, y] = 0 \quad \text{for all} \quad x, y \in R. \tag{16}$$

Setting $1 + b$ for $x$ in (16), we arrive, in view of (5) and the invertibility of $(1 + b)^t$, at the identity

$$n[b, y] = 0 \quad \text{for all} \quad y \in R. \tag{17}$$

If $R$ has one of the properties $Q(m)$ and $Q(n)$, or, especially, if $(m, n) = 2^r$ for some non-negative integer $r$, then from (15) and (17) one can easily derive

$$[b, y] = 0 \quad \text{for all} \quad y \in R, \quad \text{i.e.} \quad \alpha^{p - 1} \in R.$$

This contradiction shows that $N(R) \subseteq Z(R)$, and thus $R$ satisfies (6), hence also (7). Therefore, by Lemma 2, the identities in (14) can be
rewritten in the form
\[ nx^{n+t-1} [x, y] y^s = 0, \quad m[x, y] y^{m-1} = 0 \quad \text{for all} \quad x, y \in R, \]
hence, in view of Lemma 1,
\[ n[x, y] = 0, \quad m[x, y] = 0 \quad \text{for all} \quad x, y \in R. \]
From (18), in view of \( Q(2) \), follows the commutativity of \( R \), since \( R \) has one of the properties \( Q(m) \) and \( Q(n) \), or especially, \( (m, n) = 2^r \) for some non-negative integer \( r \). \( \Diamond \)

**Theorem 3.** Let \( R \) be a left or right \( s \)-unital ring which satisfies (1). Suppose that \( s \neq m \), resp. \( n + t > 1 \), and \( R \) has the property \( Q(k) \), where \( k = |2^m - 2^s| \), resp. \( k = 2^{n+t} - 2 \). Then \( R \) is commutative, provided that \( R \) has one of the properties \( Q(m) \) and \( Q(n) \), or, especially, \( (m, n) = 2^r \cdot r' \) for some non-negative integer \( r \) and some odd divisor \( r' \) of \( k \).

**Proof.** If \( s \neq m \), resp. \( n + t > 1 \), and \( R \) has the property \( Q(k) \) for \( k = |2^m - 2^s| \), resp. \( k = 2^{n+t} - 2 \), then from (1), for \( 2y \) instead of \( y \), resp. for \( 2x \) instead of \( x \), in view of \( Q(k) \), one derives (14). Since \( k \) is even, and \( R \) has the property \( Q(k) \), then \( R \) has also the property \( Q(2^r r') \) for every non-negative integer \( r \) and every odd divisor \( r' \) of \( k \). Now, the proof is similar to the proof of Th. 2, and can be omitted. \( \Diamond \)

**Remark 1.** If \( R \) is a right (resp. left) \( s \)-unital ring which satisfies the identity
\[ y^s [x^n, y] x^t = \pm [x, y^m], \]
then the opposite ring \( R' \) of \( R \) is left (resp. right) \( s \)-unital and satisfies the identity (1). Thus all previous results still true if one replaces “left (resp. right) \( s \)-unital” by “right (resp. left) \( s \)-unital” and the identity (1) by the identity (19).

**References**


