ROBOT-MANIPULATORS AS SUBMANIFOLDS

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Abstract: Robot-manipulators with less than six degrees of freedom are considered as submanifolds of the pseudo-Riemannian Lie group \( C_6 \) of all orientation preserving congruences of the Euclidean space. They are generalisations of quadratical ruled surfaces in Euclidean geometry to the geometry of \( C_6 \). In the paper we discuss the problem of existence of one more “straight” line of such a submanifold and describe relations of this problem to the geometry of the motion of robot-manipulators.

The paper is a straightforward continuation of [1] and [2] and therefore we shall use results and denotations from [1] and [2] without special reference. In the presented paper we shall limit ourselves to robot-manipulators with less then 6 degrees of freedom and we shall treat them as submanifolds of the pseudo-Riemannian homogeneous space \( C_6 \).

Let us consider a \( p \)-parametric robot-manipulator \( g \) as the mapping

\[
g : \mathbb{R}^p \rightarrow C_6 : [u_1, \ldots , u_p] \rightarrow g(u_1 , \ldots , u_p) = g_1(u_1) \cdots g_p(u_p),
\]

where \( g_i(u_i) = \exp(u_i X_i) \) and \( X_1, \ldots , X_p \) are linearly independent vectors from \( L \). \( g \) determines an imbedded submanifold of \( C_6 \) on some
neighbourhood \( U(0) \) of \( 0 \in \mathbb{R}^p \); let us denote \( g(U) = M \).

Robot manipulators with \( p < 6 \) were locally characterized as submanifolds of \( C_6 \) in [1]: Let us notice that geodesic lines in \( C_6 \) are left or right translates of screw-motions (including rotations and translations as special cases). From the definition of the robot manipulator we see that \( M \) is a submanifold such that it has \( p \) independent geodesic lines passing through each of its points. It was proved in [1], Th. 1, that this property is also a local characterisation of robot-manipulators in the general case – if a submanifold of \( C_6 \) has the above mentioned property, it is locally a robot-manipulator (if we consider only rotations and translations, we have to take isotropic geodesic lines only). This shows that a robot-manipulator is a generalisation of a quadratical ruled surface in the Euclidean space to the geometry of the pseudo-Riemannian space \( C_6 \) – a \( p \)-parametric robot-manipulator is characterized as a submanifold of \( C_6 \) with \( p \) systems of straight lines (geodesics) on it similarly as quadratical ruled surfaces in \( E_3 \) are the only 2-dimensional submanifolds with two systems of straight lines on them.

Similarly as in the \( 6 \)-parametric case we have the coordinate system \( u = [u_1, \ldots, u_p] \) given on \( g \) in a neighbourhood \( U(0) \) of \( 0 \in \mathbb{R}^p \). The induced pseudo-Riemannian metric is given by the same formula as in the \( 6 \)-parametric case, \( h_{ij} = \langle Y_i, Y_j \rangle \), where \( Y_i = Adg_p^{-1} \ldots Adg_{i+1}^{-1} X_i \). The affine connection induced by \( h_{ij} \) is the Levi-Civita connection and therefore it is given by the same formula as in [2],

\[
\Gamma_{ij,m} = \frac{1}{2} \varepsilon_{ij} \langle [Y_i, Y_j], Y_m \rangle \quad i, j, m = 1, \ldots, p.
\]

The same is true for equations of geodesic lines, which are

\[
u''_i + \Gamma_{ijk} u'_j u'_k = 0 \quad i, j, k = 1, \ldots, p,
\]

as in any submanifold of the pseudo-Riemannian space.

From now on we shall consider robot-manipulators with rotational axes only. In this case we have \( h_{ii} = 0 \). As the values of \( g_{ij}, h_{ij}, \Gamma_{ij,k} \) depend only on the instantaneous position of axes of the robot-manipulator, it is not difficult to find the geometrical meaning of \( g_{ij}, h_{ij}, \Gamma_{ij,k} \) for a given robot-manipulator.

Let \( Y_g, Y_r, Y_s \) be three pairwise different axes of the \( p \)-parametric robot-manipulator \( g \), \((i, j, k)\) be an even (cyclic) permutation of \((q, r, s)\). Let us denote \( \alpha_k \) the angle of \( Y_i, Y_j \), \( a_k \) the distance of \( Y_i \) and \( Y_j \), \( C_k = \cos \alpha_k, S_k = \sin \alpha_k \). Let \( Y_i = (x_i; y_i), Y_j = (x_j; y_j), Y_k = (x_k; y_k), \delta = \)
= |x_i, x_j, x_k| > 0, where x_i^2 = x_j^2 = x_k^2 = 1, (x_i, y_i) = (x_j, y_j) =
= (x_k, y_k) = 0.

**Lemma 1.** \( g_{ij} = C_k, h_{ij} = \frac{1}{2} S_k a_k, \Gamma_{ij,k} = \frac{1}{2s} \varepsilon_{ij} S(a_i S_i (C_j C_k - C_i)), \)
where \( S \) denotes the cyclic sum over \((i, j, k)\).

**Proof.** Let us write \( y_i = m_i^a x_a \). We have
\( (x_i, x_j) = C_k, (x_i, y_j) +
+ (x_j, y_i) = a_k S_k, <[Y_i, Y_j], Y_k> = |x_i, x_j, y_k| + |x_i, y_j, x_k| + |y_i, x_j, x_k|\)
\( = \delta \text{Tr}(m). a_i S_i = (x_j, y_k) + (x_k, y_j) = (x_j, m_i^a x_i + m_j^b x_j + m_k^c x_k) +
+ (x_k, m_j^b x_i + m_j^a x_j + m_k^b x_k) = m_k^a C_k + m_j^b C_j + m_k^c C_i + m_j^b C_j + m_j^a C_i + m_k^b C_j.\)
\( S[a_i S_i (C_j C_k - C_i)] = S[m_i^a C_j C_k^2 + m_j^a C_j C_k + (m_k^a + m_j^a) C_j C_k + (m_k^a + m_j^a) C_i C_j - m_j^a C_i C_k - m_k^a C_i C_k - (m_k^a + m_j^a) C_i - (m_k^a + m_j^a) C_k^2]] =
= S[-m_k^a C_j S_k^2 - m_j^a C_k S_j^2 + (m_k^a + m_j^a)(C_i C_j C_k - C_i^2)] = S[m_j^a S_j^2 +
+ (m_k^a + m_j^a)(C_i C_j C_k - C_i^2)] = \delta^2 \text{Tr}(m), \) because \( (x_j, y_j) = (x_j, m_j^a x_i +
+ m_j^b x_j + m_j^a x_k) = m_j^a C_j + m_j^b C_k + m_j^a C_i = 0, \delta^2 = 2C_i C_j C_k + S_j^2 - C_i^2 - C_k^2\)
and the formula follows. \( \diamond \)

To give more insight into the formula for \( \Gamma_{ij,k} \), let us consider a
3-parametric robot-manipulator with axes \( Y_1 = (x_1, y_1), Y_2 = (x_2, y_2)\)
and \( Y_3 = (x_3, y_3) \). Let \( Z_3 \) be the axis of \( Y_1 \) and \( Y_2 \), \( Z_1 \) be the axis of \( Y_2 \)
and \( Z_3 \) \( Z_1 \) be the angle between \( Z_3 \) and \( Z_1 \), \( d_2 \) be the distance between
\( Z_3 \) and \( Z_1 \). \( a_1 \) be the distance of \( Y_2 \) and \( Y_3 \), \( a_3 \) be the distance of \( Y_1 \)
and \( Y_2 \). We have the following \( (c_2 = \cos u_2, s_2 = \sin u_2)\)

**Lemma 2.** \( 4\Gamma_{12,3} = -s_2(a_1 C_1 S_3 + a_3 S_1 C_3) - d_2 c_2 S_1 S_3. \)

**Proof.** We have \( Z_3 = S_3^{-1}(x_1 x_2; x_2 x_1 + y_1 x_2 + C_3 a_3 S_3^{-1} x_1 x_2),\)
\( Z_1 = S_1^{-1}(x_2 x_3; x_2 x_3 + y_2 x_3 + C_1 a_1 S_1^{-1} x_2 x_3), (x_1 x_2, x_2 x_3 \times x_3) = c_2 S_1 S_3 = C_1 C_3 - C_2, d_2 s_2 = S_1^{-1} S_3^{-1}(C_3 a_1 S_1 + C_1 a_3 S_3 -
- a_2 S_2 + C_1 a_1 S_3 + C_3 a_3 c_2 S_1) \) and hence
\( a_2 S_2 = C_3 a_1 S_1 + C_1 a_3 S_3 + C_1 a_1 S_3 + C_3 a_3 c_2 S_1 - d_2 s_2 S_1 S_3. \)

This yields
\( 4\Gamma_{12,3} = \frac{\partial}{\partial u_2} (h_{13}) = \frac{\partial}{\partial u_2} (a_2 S_2) = -s_2(a_1 C_1 S_3 + a_3 S_1 C_3) - d_2 c_2 S_1 S_3. \) \( \diamond \)

Now we are going to study the relation between the Levi–Civita
connection on \( M \), which is determined by the induced scalar product
on \( M \) and the Cartan connection on \( C_6 \). For the Cartan connection \( \nabla \)
we have the following splitting:

\( \nabla Y_j Y_j = \frac{1}{2} \varepsilon_{ij} ([Y_i, Y_j]_1 + [Y_i, Y_j]_2), \)
where \( [Y_i, Y_j]_1 \) denotes the component into the tangent space of the
submanifold \( M \) into the space \( Y = \{Y_1, \ldots, Y_p\} \) generated by vectors
\[ Y_1, \ldots, Y_p. \quad [Y_i, Y_j]_2 \text{ denotes the component into the orthogonal complement } Z = Y^\perp \text{ of } Y \text{ in } L \text{ with respect to the Klein form.} \]

**Remark.** It is convenient to translate the tangent space of \( C_6 \) at the point \( g \) to the Lie algebra \( L \) by left translations, \( Y_i = L_{g^{-1}}(\frac{\partial}{\partial u_i})_g \). The splitting (1) is invariant, because the Klein form is invariant. We have to suppose that \( M \) is a submanifold of the pseudo-Riemannian manifold \( C_6 \), which requires the tangent space of \( M \) to be non-degenerated in the induced metric. In this case \( L \) is the direct sum of \( Y \) and \( Z = Y^\perp \).

Let \( Z_a, a = p + 1, \ldots, 6, \) be a basis in \( Z = Y^\perp, Y = \{Y_1, \ldots, Y_p\} \). We have

\[
\frac{1}{2} \varepsilon_{ij} = [Y_i, Y_j]_1 = \tilde{\Gamma}^k_{ij} Y_k, \quad \frac{1}{2} \varepsilon_{ij} [Y_i, Y_j]_2 = H^a_{ij} Z_a
\]

\[ i, j, k = 1, \ldots, p; a = p + 1, \ldots, 6. \]

(1) now reads as

(2)

\[
\frac{1}{2} \varepsilon_{ij} [Y_i, Y_j] = \tilde{\Gamma}^k_{ij} Y_k + H^a_{ij} Z_a.
\]

Scalar multiplication of (2) by \( Y_m \) yields \( \Gamma_{ij,m} = \tilde{\Gamma}_{ij,m}, i, j, m = 1, \ldots, p \), which gives the relation between the Cartan connection on \( C_6 \) and the Levi–Civita connection on \( M \). \( \Gamma_{ij,m} \) are Christoffel symbols of the Cartan connection for any 6-parametric robot-manipulator, which has axes of the given \( p \)-parametric robot-manipulator as its first \( p \) axes. Scalar multiplication of (2) by \( Z_b \) yields

\[
\frac{1}{2} \varepsilon_{ij} <[Y_i, Y_j], Z_b> = H^a_{ij} <Z_a, Z_b>,
\]

which determines \( H^a_{ij} \), because the matrix \( <Z_a, Z_b> \) is nonsingular. Coefficients \( H^a_{ij}, i, j = 1, \ldots, p; a = p + 1, \ldots, 6 \) determine the so-called *second metric tensor* of the submanifold \( M \). The second metric tensor is a bilinear form on \( T(M) \) with values in \( T(M)^\perp \). With respect to the second metric tensor there is a fundamental difference between the classical geometry of submanifolds of multi-dimensional Euclidean spaces and geometry of submanifolds of \( C_6 \):

The geometry of \( C_6 \) gives the possibility to define a canonical orthonormal basis in each space \( T(M)^\perp \) for \( p = 2, \ldots, 5 \) in a way, which is independent of the geometry of the submanifold \( M \). Such a construction is in the Euclidean geometry possible only for codimension 1.

To construct the canonical basis in \( T(M)^\perp \), we have to know whether the tangent space \( T_m(M) \) at the point \( m \in M \) is degenerated or not. Let us discuss this problem at first. The tangent space \( T_m(M) \)
is non-degenerated iff the matrix \( h_{ij} \) of the fundamental metric tensor is non-singular, \( \det h_{ij} \neq 0 \).

Let an instantaneous position of a \( p \)-parametric robot-manipulator be determined by axes \( Y_1, \ldots, Y_p \in L \). Then \( h_{ij} = \langle Y_i, Y_j \rangle ; i, j = 1, \ldots, p \). \( h_{ij} = \frac{1}{2} a_{ij} \sin \alpha_{ij} \), where \( a_{ij} \) is the distance of \( Y_i \) and \( Y_j \), \( \alpha_{ij} \) is the angle of \( Y_i \) and \( Y_j \). For instance for \( p = 2 \) we have

\[
\det h_{ij} = \begin{vmatrix} 0 & h_{12} \\ h_{12} & 0 \end{vmatrix} = -h_{12}^2,
\]

for \( p = 3 \) we have

\[
\det h_{ij} = \begin{vmatrix} 0 & h_{12} & h_{13} \\ h_{12} & 0 & h_{23} \\ h_{13} & h_{23} & 0 \end{vmatrix} = 2h_{12}h_{23}h_{13}.
\]

For the description of a \( p \)-parametric robot-manipulator we shall use the so called Denavit-Hartenberg parameters: Let \( X_1, \ldots, X_p \) be axes defining the robot-manipulator, \( Y_1, \ldots, Y_p \) be any instantaneous position of \( X_1, \ldots, X_p \). Let \( a_i \) be the distance of \( X_i, X_{i+1} \), \( \alpha_i \) be the angle of \( X_i, X_{i+1} \), \( d_{i+1} \) be the distance of the axis of lines \( X_i, X_{i+1} \) from the axis of lines \( X_{i+1}, X_{i+2} \) (the offset), \( u_i \) be the angle of those axes. We write \( S_i = \sin \alpha_i, C_i = \cos \alpha_i, s_i = \sin u_i, c_i = \cos u_i \).

**Lemma 3.** A 2-parametric robot-manipulator is a submanifold of \( C_6 \) iff \( a_1 S_1 \neq 0 \) and it has index 1. A 3-parametric robot-manipulator is a submanifold of \( C_6 \) iff

\[
a_1 a_2 S_1 S_2 \neq 0, d_2^2 + (a_1^2 - a_2^2)^2 + (\cot^2 \alpha_1 - \cot^2 \alpha_2)^2 \neq 0.
\]

It has index 1 or 2 according to the sign of \( h_{12}h_{23}h_{13} \).

**Proof.** We use the expression for \( h_{13} \) in the proof of Lemma 2. \( \Diamond \)

**Remark.** Any \( p \)-parametric robot-manipulator has nonzero index because it has isotropic lines (rotations).

For \( p > 3 \) we have more complicated situation, but similarly as in Lemma 3 we can see that the equation \( \det h_{ij} = 0 \) is algebraic in \( \cos u_i \) and \( \sin u_i \). Such an equation can be changed into an algebraic equation by a suitable substitution. This means that either the equation \( \det h_{ij} = 0 \) is identically satisfied or the set of positions, for which \( \det h_{ij} \neq 0 \) is dense and open. This justifies our assumption that the induced metric is defined on \( M \) and we can write (1).

Now we shall construct the canonical basis in \( T_m(M) \). Let \( p = 3 \) and let \( Z = T_m(M) = \{X_1, X_2, X_3\} \) be a nondegenerated 3-dimensional subspace in \( L \), let us write \( X_i = (y_i, z_i) \). Let us consider
only the general case, for which vectors $y_i$ are linearly independent. We
can choose the basis \{X_i\} in such a way that $y_i$ are orthonormal. Then
$z_i = m_i y_j$. Let \{y'_i\} = $\gamma \{y_i\}$ be another orthonormal triple, $\gamma \in O(3)$. Then
$$\{z'_i\} = \gamma \{z_i\} = \gamma m \{y_i\} = \gamma m \gamma^T \{y'_i\}, \quad m = (m_i^2).$$
We obtain a new matrix $m' = \gamma m \gamma^T$. This means that we can choose
the basis in the space $Z$ in such a way that the symmetric part of $m$ is
diagonal (the symmetric and skew-symmetric parts of $m$ transform sepa-
ately). This procedure means geometrically the transformation of the
ruled hyperboloid determined by $Z$ to the main axes. We also see that
the canonical system of coordinates in $T_m(M) \perp$ yields immediately a
canonical basis in $T_m(M)$, because $T_m(M)$ determines the other system
of straight lines on the same hyperboloid.

For $p = 4$ we obtain a similar situation. The space $T_m(M)$ de-
termines a linear congruence and $T_m(M) \perp$ determines axes of this con-
gruence. These axes determine the canonical basis – see for instance
[1].

The 5-dimensional case is obvious – the orthogonal complement
is one-dimensional. Geometrically it means that we have to find the
axis of a linear complex. The degeneration of the induced metric has
obvious geometrical meaning in this case: the induced metric in $T_m(M)$
is degenerated iff all axes of the robot-manipulator intersect one straight
line. The linear complex determined by $T_m(M)$ is special in this case
and the orthogonal complement $Z = T_m(M) \perp$ is a one-dimensional
isotropic subspace and the induced metric degenerates.

The most interesting case is the case $p = 2$. Let $X_1, X_2$ be two
straight lines such that $<X_1, X_2> \neq 0$. Let $Y$ be the 3-dimensional
subspace in $L$ generated by vectors $X_1, X_2, X_3 = [X_1, X_2]$. Let $X =
= m_1 X_1 \in Y$ be an arbitrary vector. Then $<X, X> = a_1 S_2 (m_1 m_2
- m_3^2 C_1)$, because $<[X_1, X_2], [X_1, X_2]> = -a_1 C_1 S_2$. This shows that $Y$ is
nondegenerated iff $a_1 C_1 S_2 \neq 0$. In this case we obtain a 3-dimensional
complement $Y \perp$ in which we construct the canonical basis in the same
way as for the case $p = 3$. Because $<X_1, X_3> = <X_2, X_3> = 0$ we have
$X_3 \in \{X_1, X_2\} \perp$ and we have a canonical basis in $T_m(M) \perp$.

A submanifold $M$ of a pseudo-Riemannian manifold is called flat
at the point $m \in M$ iff the second metric tensor $H = 0$ at $m$. $M$ is
called totally geodesic iff it is flat at all its points.

**Theorem 1.** Let $g: \mathbb{R}^p \rightarrow C_6$ be a $p$-parametric robot-manipulator
with only rotational axes. If $g$ is flat at one of its points, it is totally geodesic. Then $S_i = 0$ for $i = 1, \ldots, p - 1$ or $a_i = 0$ for $i = 1, \ldots, p - 1$ and $d_j = 0$ for $j = 2, \ldots, p - 1$; $p = 3, 4, 5$. A 2-parametric robot manipulator has no flat points.

**Proof.** Let $g$ be determined by vectors $X_1, \ldots, X_p \in L.H = 0$ at the point $0 \in \mathbb{R}^p$ shows that the component of $[X_i, X_j]$ into the orthogonal complement $\{X_1, \ldots, X_p\}^\perp$ is equal to zero. This means that $[X_i, X_j]$ must be a linear combination of $X_1, \ldots, X_p$. This shows that $\{X_1, \ldots, X_p\}$ must be a subalgebra generated by rotations. There are only two such subalgebras, the algebra $SO(3)$ of the group of all spherical motions and the Lie algebra of the group of all congruences of the plane $E_2$. This follows that all axes of the robot-manipulator pass through one point or all of them are parallel. ◊

**Remarks.** 1. Strictly speaking we have to suppose $p = 3$, because for $p = 4$ and $p = 5$ vectors $X_1, \ldots, X_p$ are not linearly independent. In that case we consider the manifold $M = g(\mathbb{R}^p)$ as a subset of $C_6$.

2. If we admit robot-manipulators with translational (prismatic) and screw joints, we obtain all connected subgroups of $C_6$ as totally geodesic submanifolds generated by robot-manipulators. Their list can be found for instance in [3].

3. Th. 1 shows that robot-manipulators have one more property of ruled quadratical surfaces in $E_3$—if such a quadratical surface has one flat point, it splits into planes. Let us remark that robot-manipulators with only rotational axes are generalisations of the one-sheet hyperboloid, robot-manipulators with prismatic joints are generalisations of the hyperbolic paraboloid.

Robot-manipulators with $a_i = 0$, $i = 1, \ldots, p - 1$ will be called *spherical* robot-manipulators, robot-manipulators with $S_i = 0$, $i = 1, \ldots, p - 1$ will be called *planar* robot-manipulators. A curve $c(t)$ on a pseudo-Riemannian submanifold $M$ is called *asymptotic* iff $H(c'(t)) = 0$, where $c'(t)$ is the tangent vector of $c(t)$. This shows that a geodetic curve on $M$ is asymptotic iff it is a geodetic curve of the enveloping space of $M$. This means that a $p$-parametric robot-manipulator is characterized as a submanifold with $p$ independent asymptotic geodesic curves passing through each of its points. (Independent means independent tangent vectors and the statement is true only for regular points.)

In the next part of the paper we shall consider some special properties of robot-manipulators. For instance we may ask if there exist robot-manipulators which have one more asymptotic geodesic line
(apart from those given above). The answer is positive because totally geodesic robot-manipulators have all geodesic lines asymptotic. Interesting question is whether there are some other solutions. In the Euclidean geometry of $E_3$ the answer is negative – if a ruled quadratic surface contains one more straight line it must be flat. For robot-manipulators we have nontrivial solutions of this problem – for instance the so called Bennets mechanism is one of them and some other cases are known. The general solution of this problem is not known. The above mentioned problem is not uninteresting from the practical point of view, because an asymptotic geodesic curve on a robot-manipulator with less than six degrees of freedom means a translation, rotation or a screw-motion. This means that we ask whether the end-effector of such a robot-manipulator can perform a rotation different from the rotation around one of its axes or if it can perform a translation or a screw motion. We shall see that this problem is closely connected with some other problems concerning robot-manipulators. To simplify our language we shall introduce some definitions:

**Definition 1.** A $p$-parametric robot-manipulator is called *singular* iff $\dim\{Y_1, \ldots, Y_p\} < p$ for all positions of the robot-manipulator. We say that a $p$-parametric robot-manipulator has an *additional degree of freedom* iff there exists such a location of the end-effector (uncertainty position) that its joints can move with the end-effector fixed.

**Remark.** For instance the spherical and planar robot-manipulators are singular for $p > 3$. Robot-manipulators with $p > 6$ have additional degree of freedom at most of their positions; therefore we shall suppose $p \leq 6$.

Because we can identify the coordinate system in the moving space with the one in the fixed space for any fixed location of the end-effector, we can write the equation for the additional degree of freedom in the form

$$g_1(u_1(t)), \ldots, g_p(u_p(t)) = e,$$

where $u_i(t)$ are functions of one parameter $t$ and at least one of these functions is not constant.

Let us denote by $S_p$ the subgroup of all permutations of numbers $(1, \ldots, p)$, which is generated by the cyclic permutation $(1, \ldots, p) \rightarrow (p,1,\ldots,p-1)$ and by the permutation $(1,\ldots,p) \rightarrow (p,p-1,\ldots,1)$. The group $S_p$ operates on vectors $X_1, \ldots, X_p$ in a natural manner.

**Lemma 4.** The group $S_p$ preserves the property of additional degree of freedom.
Proof. Let \( g_1 \ldots g_p = e \). Then \( g_1 \ldots g_{p-1} = g_p^{-1} \) and \( g_p g_1 \ldots g_{p-1} = e \). Similarly \( g_p^{-1}(u_p) \ldots g_1^{-1}(u_1) = e \) and \( g_p(-u_p) \ldots g_1(-u_1) = e \).  

Lemma 5. Let the \( p \)-parametric robot-manipulator \( g \) have an additional degree of freedom. Then there exists a robot-manipulator \( g' \) equivalent with \( g \), such that the \( (p-1) \)-parametric robot-manipulator obtained from \( g' \) by leaving out the last axis has an additional asymptotic geodesic curve.  

Proof. Let \( g_1(u_1(t)) \ldots g_p(u_p(t)) = e \) for functions \( u_1(t), \ldots, u_p(t) \). Then there exists \( a, 1 \leq a \leq p \), such that \( u_a(t) \) is not constant. This yields 
\[
g_a(u_a(t)) g_{a+1}(u_{a+1}(t)) \ldots g_p(u_p(t)) g_1(u_1(t)) \ldots g_{a-1}(u_{a-1}(t)) = e
\]
and therefore 
\[
g_{a+1}(u_{a+1}(t)) \ldots g_p(u_p(t)) g_1(u_1(t)) \ldots g_{a-1}(u_{a-1}(t)) = g_a(-u_a(t)).
\]
We change the parameter to \( w = -u_a(t) \) and obtain a geodesic asymptotic line \( g_a(w) \) on the \( (p-1) \)-parametric robot-manipulator 
\[
g_{a+1}(u_{a+1}) \ldots g_p(u_p) g_1(u_1) \ldots g_{a-1}(u_{a-1}).
\]

Lemma 6. A \( (p-1) \)-parametric robot-manipulator with additional asymptotic geodesic curve determines a \( p \)-parametric robot-manipulator with additional degree of freedom.  

Proof. Let \( u_i(t) \) be the parametric expression of the additional asymptotic geodesic curve, \( i = 1, \ldots, p-1 \). This yields \( g_1(u_1(t)) \ldots g_{p-1}(u_{p-1}(t)) = g(t) \), where \( g(t) \) is a rotation or translation. This means that the \( p \)-parametric robot-manipulator \( g_1(u_1) \ldots g_{p-1}(u_{p-1}) \cdot g^{-1}(u_p) \) has an additional degree of freedom \( g(u_p) \) must be different from \( g_{p-1}(u_{p-1}) \), because \( g = g_{p-1} \) leads to \( g_1(u_1(t)) \ldots g_{p-2}(u_{p-2}(t)) \cdot g_{p-1}(u_{p-1}(t) - t) = e \), which shows that the given robot-manipulator has an additional degree of freedom and the problem is trivial.  

Remark. The \( p \)-parametric robot-manipulator from Lemma 6 can be singular.

As the dimension of the vector space generated by vectors \( Y_1, \ldots, Y_p \) is constant on an open and dense set in \( \mathbb{R}^p \), we can define this dimension as the rank of the robot-manipulator. It is the maximal dimension of the tangent space of the robot-manipulator. Obviously, singular robot-manipulators have rank less than the number of parameters.  

Theorem 2. Let \( g \) be a \( p \)-parametric robot-manipulator of rank \( p - s \). Then \( g \) has \( s \) independent additional degrees of freedom.  

Proof. Let at first \( s = 1 \). Vectors \( Y_1, \ldots, Y_p \) are linearly dependent for all \( u \in \mathbb{R}^p \). This means that there exist functions \( m_1(u_1, \ldots, u_p), \ldots, m_p(u_1, \ldots, u_p) \) such that
\[
\sum_{i=1}^{p} m_i(u_a)Y_i(u_a) = 0,
\]
where we can suppose that \( m_i \) are differentiable on an open subset of \( \mathbb{R}^p \).

Let us consider the following system of ordinary differential equations

\[
\frac{du_i}{dt} = m_i(u_a).
\]

Let \( u_i(t) \) be an arbitrary solution of (4). Let us consider the following expression:

\[
[g_1(u_1(t)) \ldots g_p(u_p(t))]', [g_1(u_1(t)) \ldots g_p(u_p(t))]^{-1} =
\]

\[
= \frac{dg_1}{du_1}g_1^{-1}u_1' + \frac{dg_2}{du_2}g_2^{-1}g_1^{-1}u_2' + \ldots + g_1 \ldots g_{p-1} \frac{dg_p}{du_p}g_p^{-1}g_1^{-1}u_p' =
\]

\[
= X_1 m_1 + \ldots + Ad(g_1 \ldots g_{p-1})X_p m_p = 0
\]

because \( \sum_{i=1}^{p} m_iY_i = 0 \). This yields \( [g_1(u_1(t)) \ldots g_p(u_p(t))]' = 0 \) and \( g_1(u_1(t)) \ldots g_p(u_p(t)) = \gamma \), where \( \gamma \in C_6 \) is constant. We can change the representation of the robot-manipulator to have \( \gamma = e \). (If \( g_1(u_1(t)) \ldots g_p(u_p(t)) = \gamma \), we have \( \gamma = g_1(u_1(t_0)) \ldots g_p(u_p(t_0)) \) for some \( t_0 \), let us denote \( g_i(u_i(t_0)) = \gamma_i \).) Let for simplicity \( p = 3 \), other cases are similar. We obtain

\[
e = g_1g_2g_3\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1} = g_1\gamma_1^{-1} \cdot \gamma_1(g_2\gamma_2^{-1}) \cdot \gamma_1\gamma_2(g_3\gamma_3^{-1})(\gamma_1\gamma_2)^{-1}.
\]

Because \( g_i\gamma_i^{-1} = g_i(u_i(t) - u_i(t_0)) \), we obtain an another position of the same robot-manipulator and we have a solution for the additional degree of freedom. Let us remark that the choice of initial conditions \( u_i(t_0) = u_i^0 \) for (4) shows that any position of the robot-manipulator leads to an additional degree of freedom.

The proof for \( s > 1 \) is similar, but we have to show at first that the corresponding system of partial differential equations satisfies the integrability conditions. To simplify denotations, let \( s = 2, p = 6 \). This means that the dimension of the vector space generated by \( Y_1, \ldots, Y_6 \) is equal to 4 on some open subset of \( \mathbb{R}^4 \). This yields

\[
\sum_{i=1}^{6} Y_i m_i(u_a) = 0, \quad \sum_{i=1}^{6} Y_i n_i(u_a) = 0,
\]

where we can suppose \( m_1 = 1, m_2 = 0, n_1 = 0, n_2 = 1 \). Let us consider the following system of partial differential equations
\begin{align}
\frac{\partial u_i}{\partial t} &= m_i(u_a), \frac{\partial u_i}{\partial r} = n_i(u_a).
\end{align}

Differentiation of (6) yields
\begin{align}
\frac{\partial^2 u_i}{\partial t \partial r} &= \sum_{a=1}^{6} \frac{\partial m_i}{\partial u_a} \cdot \frac{\partial u_a}{\partial r} = \sum_{a=1}^{6} \frac{\partial m_i}{\partial u_a} n_a = \sum_{a=1}^{6} \frac{\partial n_i}{\partial u_a} m_a.
\end{align}

Integrability conditions are
\begin{align}
\sum_{a=1}^{6} \left( \frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \right) &= 0 \text{ for } i = 1, \ldots, 6.
\end{align}

Differentiation of (5) yields
\begin{align}
\sum_{i=1}^{6} \left( \frac{\partial Y_i}{\partial u_a} m_i + Y_i \frac{\partial m_i}{\partial u_a} \right) &= 0; \quad \sum_{i=1}^{6} \left( \frac{\partial Y_i}{\partial u_a} n_i + Y_i \frac{\partial n_i}{\partial u_a} \right) &= 0 \text{ for } a = 1, \ldots, 6.
\end{align}

We multiply the first equation of (7) by \( n_a \), the second one by \( m_a \), add them over \( a \) and subtract the results. The result is
\begin{align}
\sum_{i=1}^{6} Y_i \left( \sum_{a=1}^{6} \left( \frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \right) \right) + \sum_{i,a=1}^{6} \frac{\partial Y_i}{\partial u_a} (m_i n_a - n_i m_a) &= 0.
\end{align}

From [2] we know that
\begin{align}
\frac{\partial Y_i}{\partial u_a} = 0 \text{ for } a \leq i, \quad \frac{\partial Y_i}{\partial u_a} = [Y_i, Y_a] \text{ for } i > a.
\end{align}

This yields
\begin{align}
\sum_{i,a=1}^{6} \frac{\partial Y_i}{\partial u_a} (m_i n_a - n_i m_a) &=
= \sum_{i>a} [Y_i, Y_a] m_i n_a + \sum_{i<a} [Y_i, Y_a] m_i n_a = \sum_{i,a=1}^{6} [Y_i, Y_a] m_i n_a = 0,
\end{align}

because
\begin{align}
0 = \left[ \sum_{i=1}^{6} Y_i m_i, \sum_{a=1}^{6} Y_a n_a \right] = \sum_{i,a=1}^{6} [Y_i, Y_a] m_i n_a.
\end{align}

We have obtained
\begin{align}
\sum_{i=1}^{6} Y_i \left( \sum_{a=1}^{6} \left( \frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \right) \right) &= 0.
\end{align}

The dimension of the vector space generated by \( Y_1, \ldots, Y_6 \) is four and therefore there exist functions \( \lambda(u_a), \mu(u_a) \) such that
\[
\sum_{a=1}^{6} \left( \frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \right) = \lambda m_i + \mu n_i.
\]

Substitution for \( i = 1 \) and \( i = 2 \) yields \( \lambda = \mu = 0 \). This finishes the proof. \( \diamond \)

So far we have proved the following implications for properties of robot-manipulators:

- singular \( \rightarrow \) additional degree of freedom at each position;
- totally geodesic \( \rightarrow \) additional asymptotic geodesic curves;
- totally geodesic \( \rightarrow \) rank equal to 3;
- additional degree of freedom \( \equiv \) additional asymptotic geodesic curve in a partial robot-manipulator.

It is not difficult to show that the only singular robot-manipulators of rank 3 (and therefore \( p > 3 \)) are spherical and planar robot-manipulators. The problem of the classification of all robot-manipulators with additional degree of freedom remains open. (It includes the classification of all singular robot-manipulators and the classification of robot-manipulators with additional geodesic asymptotic curve.)

There is a 1-1 correspondence between motions of robot-manipulators with additional degrees of freedom and closed kinematical chains, which have possibility to move. To show it we prove the following

**Lemma 7.** Let \( g_1(u_1(t)) \ldots g_p(u_p(t)) = e \) for functions \( u_i(t), i = 1, \ldots, p \). Let us denote \( d_p \) the offset between axes \( Y_{p-1}, Y_p \) and \( Y_p, Y_1 \) and similarly for \( d_1 \). Then \( \langle Y_1(t), Y_p(t) \rangle, \langle K(Y_1(t), Y_p(t)) \rangle, d_p \) and \( d_1 \) are constant.

**Proof.** We have \( Y_k = Ad(g_1 \ldots g_{k-1})X_k \), where \( X_k \) is the initial position of \( Y_k, k = 1, \ldots, p \). This yields \( Y_1 = X_1, Y_p = Ad(g_1 \ldots g_{p-1})X_p = Ad(g_{p-1})X_p = X_p \) and therefore \( \langle Y_1, Y_p \rangle = \langle X_1, X_p \rangle, K(Y_1, Y_p) = K(X_1, X_p) \). Let now \( S_p = \sin \alpha_p, C_p = \cos \alpha_p \), where \( \alpha_p \) is the angle between \( Y_p \) and \( Y_1 \). Calculation yields

\[
-d_p S^2_{p-1} S_p^2 = S_p^2 |z_{p-1}, v_{p-1}, z_p| + |z_p, v_p, z_{p-1}|C_{p-1} |C_{p-1} - S^2_{p-1} |z_1, v_1, z_p| + |z_p, v_p, z_1|C_p|
\]

where \( Y_i = (z_i; v_i) \) for \( i = 1, \ldots, p \). As \( Y_{p-1} = Ad(g_1 \ldots g_{p-2})X_{p-1} = Ad(g_1 \ldots g_{p-1})X_{p-1} = Ad(g_{p-1})X_{p-1} \), it is easy to show that \( |z_{p-1}, v_{p-1}, z_p| + |z_p, v_p, z_{p-1}|C_{p-1} \) is invariant with respect to \( u_p \). \( \diamond \)
References


