POLYNOMIAL IDENTITIES FOR TENSOR PRODUCTS OF GRASSMANN ALGEBRAS

Onofrio M. DI VINCENZO

Dipartimento di Matematica, Università di Messina, Salita Sperone 31, 98166 Messina, Italia

Vesselin DRENSKY*)

Institute of Mathematics, Bulgarian Academy of Sciences, Akad. Georgy Bonchev Str. block 8, 1113 Sofia, Bulgaria

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Abstract: Let $E$ be the Grassmann (or exterior) algebra of an infinite-dimensional vector space over a field of characteristic 0 and let $E_k$ be the Grassmann algebra of a $k$-dimensional vector space. We describe the $S_n$-cocharacters and the asymptotic behaviour of the codimensions for the T-ideals of the polynomial identities for the tensor products $E_k \otimes E_l$ and $E \otimes E_l$, $k, l \geq 2$. As a consequence, we obtain a necessary and sufficient condition for the inclusion of the T-ideals $T(E_k \otimes E_l) \subseteq T(E_{k'} \otimes E_{l'})$.

Introduction

Let $K\langle X \rangle$ be the free unitary associative algebra freely generated by a countable set of variables $X = \{x_1, x_2, \ldots\}$ over a field $K$ of characteristic 0. For any unitary PI-algebra $R$ we denote by $T(R)$ the

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ideal of $K(X)$ consisting of all polynomial identities for $R$; $T(R)$ is called a T-ideal. Kemer [7] has discovered the structure theory of T-ideals. It turns out that all T-prime ideals correspond to algebras obtained by constructions with the $n \times n$ matrix algebra $M_n(K)$ and the Grassmann (or exterior) algebra $E$. The set of T-prime ideals is closed under tensor products over $K$. If $T(R_1)$ and $T(R_2)$ are T-prime ideals, then $T(R_1 \otimes R_2)$ is also T-prime. The largest T-prime ideals are $T(K) = T(M_1(K))$, $T(M_2(K))$, $T(E)$ and $T(E \otimes E)$ with inclusions $T(K) \supset T(M_2(K))$ and $T(K) \supset T(E) \supset T(E \otimes E)$. The structure of $T(K)$ is very simple, that of $T(E)$ is also well known [8]. Since $T(E \otimes E)$ is the minimal T-prime ideal which is not contained in $T(M_2(K))$, it is an important object in the investigation of the non-matrix polynomial identities. Popov [10] has obtained a generating set for $T(E \otimes E)$ and has computed its $S_n$-cocharacters. The T-ideals $T(E \otimes E)$ and $T(M_2(K))$ have some similar properties and can be treated with the same combinatorial techniques. The second author [5] has computed the codimensions of $T(E \otimes E)$ and jointly with Luisa Carini [1] the Hilbert (or Poincaré) series of $T(E \otimes E)$. Recently the first author [3] has described the $\mathbb{Z}_2$-graded polynomial identities for $E \otimes E$.

In this paper we describe the polynomial identities for the tensor product $E_k \otimes E_l$ of two finite-dimensional Grassmann algebras and, as a consequence, the polynomial identities of $E \otimes E_l$. The algebras $E_{2k}$ and $E_{2k+1}$ have the same polynomial identities and it is sufficient to consider the algebras $E_{2k} \otimes E_{2l}$ and $E \otimes E_{2l}$, $k \geq l \geq 1$. Since we work with unitary algebras only, we study the proper (or commutator) polynomial identities introduced by Specht [11]. Our main result is the computing of the proper $S_n$-cocharacter sequence of $E_{2k} \otimes E_{2l}$ and $E \otimes \otimes E_{2l}$. There exists a simple relationship between the proper and the ordinary $S_n$-cocharacters [4] and our result allows to obtain also the usual cocharacters. As a consequence we give a sufficient and necessary condition for the inclusion $T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'})$. This holds if and only if $k + l \geq k' + l'$ and $l \geq l'$. We also determine the exact asymptotic behaviour of the codimension sequences of $E_{2k} \otimes E_{2l}$ and $E \otimes E_{2l}$.

1. Proper identities

We fix a field $K$ of characteristic 0. All algebras which we con-
sider are unitary $K$-algebras, all vector spaces and tensor products are also over $K$. We use the following notation: $K\langle X \rangle$ is the free associative algebra generated by $X = \{x_1, x_2, \ldots \}$, $K\langle x_1, \ldots , x_m \rangle$ is the free subalgebra of rank $m$, $P_n$ is the space of the multilinear polynomials of degree $n$ in $K\langle x_1, \ldots , x_n \rangle$. For an algebra $R$ we denote by $T(R)$ the set of all polynomial identities for $R$.

A self-contained background and references on the proper (or commutator) polynomial identities can be found in [6]. We follow the notation in [6]. We define commutators of length $\geq 2$ by

$$[x_1, x_2] = x_1 \text{ad} x_2 = x_1 x_2 - x_2 x_1, \quad [x_1 \ldots , x_{n-1}, x_n] = [[x_1 \ldots , x_{n-1}], x_n].$$

An element $f(x_1, \ldots , x_m) \in K\langle X \rangle$ is called proper if $f$ is a linear combination of products of commutators $[x_{i_1}, \ldots ][\ldots , x_{i_n}]$. We denote by $\Gamma_n$ the space of the multilinear proper polynomials of degree $n$. For a PI-algebra $R$ we denote

$$P_n(R) = P_n/(P_n \cap T(R)), \quad \Gamma_n(R) = \Gamma_n/(\Gamma_n \cap T(R)).$$

The vector spaces $P_n(R)$ and $\Gamma_n(R)$ are $S_n$-modules, where $S_n$ is the symmetric group of degree $n$. Their $S_n$-characters are called respectively the $n$-th cocharacter and the $n$-th proper cocharacter of $T(R)$ (or of $R$). The degrees of these characters, i.e. the dimensions

$$c_n(R) = \dim P_n(R), \quad \gamma_n(R) = \dim \Gamma_n(R),$$

are called the $n$-th codimension and the $n$-th proper codimension of $T(R)$.

We fix a partition $\lambda = (\lambda_1, \ldots , \lambda_r)$ of $n$ (notation $\lambda \vdash n$). We denote by $M(\lambda)$ the irreducible $S_n$-module corresponding to $\lambda$ and by $T_\lambda(\tau)$ the $\lambda$-tableau corresponding to $\tau \in S_n$.

| $\tau(1)$ | $\tau(r_1 + 1)$ | $\ldots$ | $\ldots$ | $\tau(n - r_k + 1)$ \\
|----------|----------------|--------|--------|----------------|
| $\tau(2)$ | $\tau(r_1 + 2)$ | $\vdots$ | $\vdots$ | $\vdots$       \\
| $\vdots$ | $\vdots$        | $\ldots$ | $\ldots$ | $\tau(n)$       \\
| $\tau(r_2)$ | $\tau(r_1 + r_2)$ |        |        |                \\

$\tau(r_2 + 1)$  \\
$\vdots$  \\
$\tau(r_1)$  \\

$T_\lambda(\tau)$

Let $P$ and $\Xi$ be the row and the column stabilizers of $T_\lambda(\tau)$, respectively.
Up to a multiplicative constant the element
\[ t_{\lambda r} = \sum_{\rho \in \mathcal{P}} (-1)^{\rho} \rho \xi \in KS_n, (-1)^{\xi} = \text{sign}\xi \]
is a minimal idempotent of $KS_n$ and generates an $S_n$-module $M(\lambda) \subset KS_n$.

Let $d = [x_1, \ldots] \ldots [\ldots, x_n]$ be a product of commutators of length $\geq 2$ and let
\[ V_d = KS_n(d) = \text{sp}\{\pi d = [x_{\pi(1)}, \ldots] \ldots [\ldots, x_{\pi(n)}] | \pi \in S_n\}. \]
Then $\Gamma_n = \sum V_d$, where the sum is on all possible products $d$ of length $n$. If the polynomial
\[ \phi_\lambda = \phi_\lambda(x_1, \ldots, x_n) = t_{\lambda r} d \]
is non-zero is $V_d$, then $\phi_\lambda$ generates an $S_n$-submodule $M(\lambda)$ of $\Gamma_n$.
Replacing by the same variable $x_p$ all the variables of $\phi_\lambda(x_1, \ldots, x_n)$ whose indices are in the $p$-th row of $T_\lambda(r)$, $p = 1, \ldots, r$, we obtain a proper polynomial
\[ f_\lambda = f_\lambda(x_1, \ldots, x_r) \]
which is the highest weight vector of the polynomial representation of the general linear group corresponding to the partition $\lambda$ and the linearization of $f_\lambda$ equals $\phi_\lambda$ up to a multiplicative constant.

**Lemma 1.1.** If $n \geq m$ and $\mu = (\mu_1, \ldots, \mu_r)$ and $\lambda = (\mu_1 + 1, \ldots, \nu_r + 1, 1^{n-m-r})$ are partitions of $m$ and $n$, respectively, then
\[ \dim M(\lambda) = \frac{1}{m!} \dim M(\mu) \psi_\mu(n), \]
where $\psi_\mu(n) \in \mathbb{Q}[n]$ is a polynomial of degree $m$ in $n$ and the leading term of $\psi_\mu(n)$ is equal to 1.

**Proof.** The dimension of $M(\lambda)$, $\lambda \vdash n$, is given by the hook formula
\[ \dim M(\lambda) = n! \prod h_{ij}^{-1}(\lambda), \]
where $h_{ij}(\lambda)$ is the length of the $(i, j)$-th hook of the Young diagram of $\lambda$, i.e. $h_{ij}(\lambda) = \lambda_i + \lambda_j' - (i + j) + 1$, where $\lambda_j'$ is the length of the $j$-th column of the diagram. The hooks of $\lambda$ are equal to
\[ h_{i1}(\lambda) = n - m + 1 - i + \mu_i, i = 1, \ldots, r, \]
\[ h_{i1}(\lambda) = n - m + 1 - i, i = r + 1, \ldots, n - m, \]
\[ h_{ij}(\lambda) = h_{i,j-1}(\mu), j > 1, i = 1, \ldots, r. \]
Hence
\[ \dim M(\lambda) = \frac{1}{m!} \prod_{i=1}^{n!} h_{ij}^{-1}(\mu) \frac{n!}{(n-m-r)!} \prod_{i=1}^{r} (n-m+1-i+\mu_i)^{-1} = \]

\[ = \frac{1}{m!} \dim M(\mu) \psi_\mu(n) \]

and

\[ \psi_\mu(n) = n(n-1) \ldots (n-m-r+1) \prod_{i=1}^{r} (n-m+1-i+\mu_i)^{-1} \]

is a polynomial of degree \( m \) in \( n \) with leading term equal to 1. \( \diamond \)

**Proposition 1.2.** [4, 5] Let \( R \) be a PI-algebra.

(i) If \( P_n(R) = \sum_{\lambda} m(\lambda) M(\lambda), \Gamma_n(R) = \sum m'(\mu) M(\mu), \) then \( m(\lambda) = \sum m'(\mu), \) where for \( \lambda = (\lambda_1, \ldots, \lambda_r) \) the summation runs over all partitions \( \mu = (\mu_1, \ldots, \mu_r) \) such that \( \lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_r \geq \mu_r. \)

(ii) The codimension sequence \( c_n(R) \) and the proper codimension sequence \( \gamma_n(R), n = 0, 1, 2, \ldots, \) are related by the equality

\[ c_n(R) = \sum_{m=0}^{n} \binom{n}{m} \gamma_m(R); \]

(iii) The codimension series \( c(R, t) = \sum c_t(R) t^n \) and the proper codimension series \( \gamma(R, t) = \sum \gamma_t(R) t^n \) satisfy the equation

\[ c(R, t) = \frac{1}{1 - t} \gamma(R, \frac{t}{1 - t}). \]

**Proposition 1.3.** [10] \( \Gamma_n(E \otimes E) = \sum M(a + 2, 2^b, 1^c) + \varepsilon_n M(1^n), \) where \( (a + 2, 2^b, 1^c) \vdash n, a \geq 0, b + c > 0; \varepsilon_n = 0 \) for \( n \) odd and \( \varepsilon_n = 1 \) for \( n \) even. Here \( (a + 2, 2^b, 1^c) \) is a short notation for the partition

\[ (a + 2, 2^{\frac{b}{c}}, 1^{\frac{c}{b}}). \]

Since \( T(E_k \otimes E_l) \supset T(E \otimes E) \supset T(E \otimes E), \) we obtain that \( \Gamma_n(E_k \otimes E_l) \) and \( \Gamma_n(E \otimes E) \) are factor modules of \( \Gamma_n(E \otimes E). \) In order to obtain the proper cocharacters of \( E_k \otimes E_l \) it is sufficient to establish for which irreducible \( S_n \)-modules \( M(\lambda) \subset \Gamma_n(E \otimes E), \lambda = (\lambda_1, \ldots, \lambda_r), \) the corresponding polynomial \( f_\lambda(x_1, \ldots, x_r) \) vanishes on \( E_k \otimes E_l. \) We fix the following polynomials \( f_\lambda = f_\lambda(x_1, \ldots, x_{c+1}) \) for \( \lambda = (a + 2, 1^c), a \geq 0, c \geq 1: \)

\[ f_\lambda = \sum (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(2p)} (x_{\sigma(2p+1)} ad^r+1)x_1 \]

for \( \lambda = (r + 2, 1^{2p}). \)
\[ f_\lambda = \sum (-1)^{\sigma} x_{\sigma(1)} \ldots x_{\sigma(2p-2)} ([x_{\sigma(2p-1)}, x_{\sigma(2p)}] \text{ad}^{r+1} x_1) \]

for \( \lambda = (r + 2, 1^{2p-1}) \); and \( f_\lambda = f_\lambda(x_1, \ldots, x_{b+c+1}) \) for \( \lambda = (a+2, 2^b, 1^c) \), \( a, c \geq 0, b > 0 \):

\[ f_\lambda = \sum (-1)^{\sigma} (-1)^r x_{\sigma(1)} \ldots x_{\sigma(2p)} x_{\tau(1)} \ldots x_{\tau(2q-2)} ([x_{\tau(2q-1)}, x_{\tau(2q)}] \text{ad}^r x_1) \]

for \( \lambda = (r + 2, 2^{2q-1}, 1^{2(p-q)}) \);

\[ f_\lambda = \sum (-1)^{\sigma} (-1)^r x_{\sigma(1)} \ldots x_{\sigma(2p-2)} ([x_{\sigma(2p-1)}, x_{\sigma(2p)}] \text{ad}^{2r} x_1), x_{\tau(1)}] x_{\tau(2)} \ldots x_{\tau(2q+1)} \]

for \( \lambda = (2r + 2, 2^{2q}, 1^{2(p-q)-1}) \) and \( \lambda = (2r + 2, 2^{2q-1}, 1^{2(q-p)+1}) \);

\[ f_\lambda = \sum (-1)^{\sigma} (-1)^r x_{\sigma(1)} \ldots x_{\sigma(2p-2)} ([x_{\sigma(2p-1)}, x_{\sigma(2p)}, x_{\tau(1)}] \text{ad}^{2r+1} x_1), x_{\tau(2)} \ldots x_{\tau(2q+1)} \]

for \( \lambda = (2r + 3, 2^{2q}, 1^{2(p-q)-1}) \) and \( \lambda = (2r + 3, 2^{2q-1}, 1^{2(q-p)+1}) \);

\[ f_\lambda = \sum (-1)^{\sigma} (-1)^r [([x_{\sigma(1)}, x_{\sigma(2)}] \text{ad}^{2r} x_1), x_{\tau(1)}] [x_{\tau(2)}, x_{\tau(3)}, x_{\sigma(3)}] \times x_{\sigma(4)} \ldots x_{\sigma(2p+1)} x_{\tau(4)} \ldots x_{\tau(2q+1)} \]

for \( \lambda = (2r + 2, 2^{2q}, 1^{2(p-q)}) \);

\[ f_\lambda = \sum (-1)^{\sigma} (-1)^r x_{\sigma(1)} \ldots x_{\sigma(2p)} x_{\tau(1)} \ldots x_{\tau(2q)} ([x_{\tau(2q+1)} \text{ad}^{2r+1} x_1], x_{\sigma(2p+1)}) \]

for \( \lambda = (2r + 3, 2^{2q}, 1^{2(p-q)}) \);

\[ f_\lambda = s_{2p}(x_1, \ldots, x_{2p}) \]

for \( \lambda = (1^{2p}), p \geq 1 \), where

\[ s_m(x_1, \ldots, x_m) = \sum (-1)^{\sigma} x_{\sigma(1)} \ldots x_{\sigma(m)} \]

is the standard polynomial of degree \( m \). Since the standard polynomial of even degree is proper, it is easy to see that all the polynomials \( f_\lambda \) are also proper. Up to a multiplicative constant the linearization of each \( f_\lambda \) is equal to \( \phi_\lambda = t_{\lambda d} \) for some \( \tau \in S_n \) and a product of commutators \( d \) and generates a submodule \( M(\lambda) \) of \( \Gamma_n \), \( \lambda \vdash n \). Some of the \( f_\lambda \)s are as in the paper by Popov [10] but some of them are replaced by more convenient polynomials.
2. Preliminary reductions

Let \( V \) be a countably dimensional vector space with basis \( \{e_1, e_2, \ldots\} \). The Grassmann (or exterior) algebra \( E = E(V) \) of \( V \) is the algebra generated by \( e_1, e_2, \ldots \) with defining relations \( e_i e_j = -e_j e_i, \; i, j = 1, 2, \ldots \). The Grassmann algebra \( E_k \) of a \( k \)-dimensional vector space is generated by \( e_1, \ldots, e_k \). Let \( E \otimes E \) be the tensor square of \( E \). We fix generators \( e_1 \otimes 1, e_2 \otimes 1, \ldots \) of \( E \otimes 1 \) and \( 1 \otimes \tilde{e}_1, 1 \otimes \tilde{e}_2, \ldots \) of \( 1 \otimes E \) and write the elements \( u \otimes v \in E \otimes E \) as \( uv \) without the symbol \( \otimes \) between \( u \) and \( v \).

Lemma 2.1. For \( \delta, \varepsilon = 0, 1 \),

\[
T(E_{2k+\delta} \otimes E_{2l+\varepsilon}) = T(E_{2k} \otimes E_{2l}), \; T(E \otimes E_{2l+\varepsilon}) = T(E \otimes E_{2l}).
\]

Proof. By [9, Th. 1], if \( A_1, A_2, B_1 \) and \( B_2 \) are PI-algebras such that \( T(A_1) = T(A_2), T(B_1) = T(B_2) \), then \( T(A_1 \otimes B_1) = T(A_2 \otimes B_2) \). Since the T-ideals \( T(E_{2k+1}) \) and \( T(E_{2k}) \) are equal (see e.g. [2]), this gives immediately the proof of the lemma. \( \diamond \)

In the sequel we fix \( k \geq l \geq 1 \) and study the polynomial identities for \( E_{2k} \otimes E_{2l} \) and \( E \otimes E_{2l} \). We use an idea from [3]. The algebra \( E \) has a natural \( \mathbb{Z}_2 \)-grading \( E = E[0] \oplus E[1] \), where \( E[0] \) and \( E[1] \) are spanned on the products of even and odd length, respectively. We denote by \( y, y_1, y_2, \ldots \) arbitrary elements of \( E[1] \otimes E[1] \) and by \( z_1, z_2, \ldots \) arbitrary elements of \( E[1] \otimes E[0] \oplus E[0] \otimes E[1] \).

Lemma 2.2. The elements \( y, y_1, y_2, \ldots, z_1, z_2, \ldots \) satisfy:

(i) \( y y_1, [z_1, z_2] \) and \( y(z_1 \circ z_2) \) are central in \( E \otimes E \), where \( z_1 \circ z_2 = z_1 z_2 + z_2 z_1 \);

(ii) \( y y_1 = y_1 y, \; z_1 y = -y z_1, \; z_1(\text{ad}^r y) = (-2)^r y^r z_1 \);

(iii) \( [y^{2r} z_1, z_2] = y^{2r} [z_1, z_2], \; [y^{2r+1} z_1, z_2] = y^{2r+1} (z_1 \circ z_2) \);

(iv) \( \bar{x}_1 = y, \; \bar{x}_i = z_i, \; i = 2, \ldots, q \), then

\[
\sum (-1)^{\sigma(1)} \cdots (-1)^{\sigma(q)} = q y \sum (-1)^{\sigma(2)} \cdots (-1)^{\sigma(q)};
\]

(v) \( z_2 \text{ad}^r (y + z_1) = (-2)^{-1} y^{r-1} (-2 y z_2 + (-1)^r z_1 z_2 + z_2 z_1) \equiv (-2)^r y^r z_2 \text{ modulo the centre of } E \otimes E ;
\]

(vi) \( z_3 z_2 z_1 = -z_1 z_2 z_3 \);

(vii) \( z_1 z_2 z_1 = 0, z_2 z_1^2 = -z_2 z_2, z_3 z_4 z_1 z_2 = z_1 z_2 z_3 z_4, z_2 z_1^2 = z_1^2 z_2^2 \);

(viii) \( y u y_1 = y_1 u y, \; z_1 u z_1 v z_1 = 0 \) for all \( u, v \in E \otimes E \).

Proof. The case (i) is obvious because \( y y_1, [z_1, z_2], y(z_1 \circ z_2) \in E[0] \otimes E[0] \), the centre of \( E \otimes E \). Since the identity \( z_1 y = -y z_1 \) from (ii) is multilinear in \( y \) and \( z_1 \), it suffices to consider only the cases \( y = e_1 \tilde{e}_1 \),
\(z_1 = e_2\) and \(y = e_1 \bar{e}_1, z_1 = \bar{e}_2,\) similarly for \(yy_1 = y_1 y.\) The verification is trivial. This gives also (iii) and (iv) which are consequences of (i) and (ii). For example,
\[
[y^{2r+1}z_1, z_2] = (y^2)^r[yz_1, z_2] = y^{2r}(yz_1z_2 - z_2y_1z_1) = y^{2r}(yz_1z_2 + yz_2z_1) = y^{2r+1}(z_1 \circ z_2).
\]

(v) We use induction on \(r.\) For \(r = 1, z_2 \text{ad}(y + z_1) = -2yz_2 + [z_2, z_1] \equiv -2yz_2 \) modulo the centre \(E^{[0]} \otimes E^{[0]}\) of \(E \otimes E.\) Let \(z_2 \text{ad}^r(y + z_1) \equiv (-2)^r y^r z_2 \mod E^{[0]} \otimes E^{[0]}\). Then
\[
z_2 \text{ad}^{r+1}(y + z_1) = (-2)^r[y^r z_2, y + z_1] =\]
\[
= (-2)^r(y^r[z_2, y] + y^r z_2 z_1 - z_1 y^r z_2) =\]
\[
= (-2)^r(-2y^{r+1}z_2 + y^r((-1)^{r+1}z_1 z_2 + z_2 z_1)).
\]

In both the cases \(r\) even and \(r\) odd, \(y^r((-1)^{r+1}z_1 z_2 + z_2 z_1) \in E^{[0]} \otimes E^{[0]}\). For (vi) it is sufficient to consider the cases \(z_i \in \{e_i, \bar{e}_i\}, i = 1, 2, 3,\) and (vi) can be easily checked. The identities from (vii) are consequences of (vi); (viii) follows from (ii) and (vii). \(\diamondsuit\)

By the convention of Section 1, for \(\lambda = (\lambda_1, \ldots, \lambda_r) = (a + 2, b, 1) \vdash n, r = b + c + 1, a \geq 0, b + c > 0, \tau \in S_n\) and a product of commutators \(d\) we consider the polynomial \(\phi_\lambda(x_1, \ldots, x_n) = t_\lambda d\) and its symmetrization \(f_\lambda(x_1, \ldots, x_r).\)

**Lemma 2.3.** If \(f_\lambda(x_1, \ldots, x_r)\) is not a polynomial identity for \(E_{2k} \otimes E_{2l},\) then
\[
2a + 2b + c + 2 \leq 2(k + l), a + b + 1 \leq 2l.
\]

**Proof.** Every variable of \(f_\lambda(x_1, \ldots, x_r)\) is in a commutator. Since \(E^{[0]} \otimes E^{[0]}\) is the centre of \(E \otimes E,\) there exist elements \(y_1 + z_1, \ldots, y_r + z_r\) such that
\[
\tilde{f}_\lambda = f_\lambda(y_1 + z_1, \ldots, y_r + z_r) \neq 0,
\]
\[
y_i \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}, \quad z_i \in E_{2k}^{[0]} \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]}.
\]

Let \(\alpha = (\alpha_1, \ldots, \alpha_r)\) and let \(f_\lambda^{(\alpha)}\) be the homogeneous component of \(\tilde{f}_\lambda\) of degree \(\alpha_i\) in \(y_i, i = 1, \ldots, r.\) By Lemma 2.2 (viii), \(\tilde{f}_\lambda = \sum f_\lambda^{(\alpha)}\), where the summation runs over all \(\alpha\) with \(\alpha_i \leq 2.\) Using Lemma 2.2 (ii) and (vii) we can write every non-zero \(f_\lambda^{(\alpha)}\) in the form
\[
f_\lambda^{(\alpha)} = y_1^{\beta_1} \cdots y_r^{\beta_r} z_i z_i^2 \cdots z_i^{a_i} g_\alpha(z_{j_1}, \ldots, z_{j_s}),
\]
where \(\beta_i = \lambda_i - \alpha_i\) and \(\alpha_{i_1} = \cdots = \alpha_{i_s} = 2, \alpha_{j_1} = \cdots = \alpha_{j_t} = 1.\) The non-zero element \(y_1^{\beta_1} \cdots y_r^{\beta_r}\) is a linear combination of pro-
ducts $e_{m_1} \ldots e_{m_k} \tilde{e}_{n_1} \ldots \tilde{e}_{n_{s'}}$, where $\beta, \beta' \geq \beta_1 + \ldots + \beta_r$ and $e_{m_i}, \tilde{e}_{n_j}$ are pairwise different generators of $E_{2k} \otimes 1$ and $1 \otimes E_{2l}$, respectively. Similarly, we need at least $2s + t = \alpha_1 + \ldots \alpha_r$ generators for $z_{i_1}^2 \ldots z_{i_s}^2 g_\alpha(z_{j_1}, \ldots, z_{j_t})$. Therefore

\[
2(k + l) \geq 2(\beta_1 + \ldots + \beta_r) + \alpha_1 + \ldots \alpha_r \geq \\
\beta_1 + (\alpha_1 + \beta_1) + \ldots (\alpha_r + \beta_r) = \beta_1 + \lambda_1 + \ldots + \lambda_r.
\]

Since $\beta_1 = \lambda_1 - \alpha_1 \geq a$ and $\lambda_1 + \ldots + \lambda_r = a + 2b + c + 2$, we obtain $2(k + l) \geq 2a + 2b + c + 2$. If $z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]}$, then $z_i^2 = 0$. Hence we need at least one generator $\tilde{e}_{n_j}$ for each product $z_i^2$. Since $f^{(a)}_\lambda$ is equal to $y_1^{\lambda_1} h_0, y_2^{\lambda_2} h_1, \ldots$ or $y_1^{\lambda_1-2} z_i^2 h_2$ for some $h_0, h_1, h_2 \in E_{2k} \otimes E_{2l}$, we need at least $\lambda_i - 1$ generators of $1 \otimes E_{2l}$ for each $i = 1, \ldots, r$, i.e. $2l \geq (\lambda_1 - 1) + \ldots + (\lambda_r - 1) = a + b + 1$.

For a polynomial $f(x_1, x_2, \ldots, x_r) \in \mathcal{K}(X)$ we denote by $f^{(j)}$ the homogeneous component of degree $j$ in $z_1$ of the element $f(y + z_1, z_2, \ldots, z_r)$, $j = 0, 1, 2$. In virtue of Lemma 2.2 (viii), $f(y + z_1, z_2, \ldots, z_r) = f^{(0)} + f^{(1)} + f^{(2)}$.

**Lemma 2.4.** If $f_\lambda(x_1, \ldots, x_r)$ is not a polynomial identity for $E_{2k} \otimes \otimes E_{2l}$,

\[
\lambda = (a + 2, 2^b, 1^c) \quad \text{and} \quad 2a + 2b + c + 2 = 2(k + l),
\]

then $f^{(2)}(y + z_1, z_2, \ldots, z_r) = 0$ for some $y \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}, z_i \in E_{2k}^{[1]} \otimes \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]}$.

**Proof.** As in the proof of Lemma 2.3, let

\[
f^{(a)}_\lambda = y_1^{\beta_1} \ldots y_r^{\beta_r} z_{i_1}^2 \ldots z_{i_s}^2 g_\alpha(z_{j_1}, \ldots, z_{j_t})
\]

be a non-zero homogeneous component of $f_\lambda(y_1 + z_1, \ldots, y_r + z_r) \in E_{2k} \otimes E_{2l}$. Since $2(k + l) = 2a + 2b + c + 2 = a + \lambda_1 + \ldots + \lambda_r$, we obtain from the inequalities

\[
2(k + l) \geq (\beta_1 + \ldots + \beta_r) + (\lambda_1 + \ldots + \lambda_r) \geq \\
\beta_1 + \lambda_1 + \ldots + \lambda_r \geq a + \lambda_1 + \ldots + \lambda_r
\]

that $\beta_1 = a, \beta_2 = \ldots = \beta_r = 0$, i.e. $f^{(a)}_\lambda(y_1 + z_1, \ldots, y_r + z_r) = 0$. All the sums in the sequel are on $\sigma \in S_m$, where the symmetric group $S_m$ acts on the set of symbols $\{d + 1, \ldots, d + m\}$ and the values of $d$ and $t$ are clear from the context.

**Lemma 2.5.** The elements $z_1, z_2, \ldots$ satisfy the following identities:
(i) \[ \sum (-1)^\sigma z_{\sigma(1)} \ldots z_{\sigma(2p)} [z_{\sigma(2p+1)}, z_1] = 0; \]
(ii) \[ \sum_{\sigma(2p+1) \neq \sigma} (-1)^\sigma z_{\sigma(1)} \ldots z_{\sigma(2p+1)} z_1 = p z_1^2 s_{2p}(z_2, \ldots, z_{2p+1}); \]
(iii) \[ \sum (-1)^\sigma z_{\sigma(2)} \ldots z_{\sigma(2p-1)} [z_{\sigma(2p)}, z_1] = -p^{-1} s_{2p}(z_1, \ldots, z_{2p}); \]
(iv) \[ s_{2p}(z_1, \ldots, z_{2p}) s_{2q}(z_1, \ldots, z_{2q}) =
= (2q)! (p!)^2 ((p-q)!)^{-1} z_1^2 \ldots z_2^2 s_{2(p-q)}(z_{2q+1}, \ldots, z_{2p}), \quad p \geq q; \]
(v) \[ s_{2p}(z_1, \ldots, z_{2p}) \sum (-1)^\tau z_{\tau(2)} \ldots z_{\tau(2q-1)}(z_{\tau(q)} o z_1) = 0, \quad p \geq q; \]
(vi) \[ s_{2p-1}(z_1, \ldots, z_{2p-1}) s_{2q-1}(z_1, \ldots, z_{2q-1}) =
= s_{2q-1}(z_1, \ldots, z_{2q-1}) s_{2p-1}(z_1, \ldots, z_{2p-1}) =
= (2q-1)! p! (p-1)! ((p-q)!)^{-1} z_1^2 \ldots z_{2q-2} z_{2q-1} s_{2(p-q)}(z_{2q}, \ldots, z_{2p-1}), \quad p \geq q; \]
(vii) \[ s_{2p-1}(z_1, \ldots, z_{2p-1}) s_{2q}(z_1, \ldots, z_{2q}) =
= (2p-1)! (q!)^2 ((q-p)!)^{-1} z_1^2 \ldots z_{2p-2} s_{2(q-p)+1}(z_2, \ldots, z_{2q}), \quad p \leq q; \]
(viii) \[ s_{2p-1}(z_2, \ldots, z_{2p}) o s_{2q+1}(z_1, \ldots, z_{2q+1}) =
= (2q+1)! (p-1)! ((p-q)!)^{-1} z_2^2 \ldots z_{2q+1} s_{2(p-q)+1}(z_{2q+2}, \ldots, z_{2p}) o z_1, \quad p > q. \]

**Proof.** (i) Let \( h = \sum (-1)^\sigma z_{\sigma(1)} \ldots z_{\sigma(2p)} [z_{\sigma(2p+1)}, z_1] \). Since \([z_i, z_j]\) are central elements,

\[ h = 2^{-p} \sum (-1)^\sigma [z_{\sigma(1)}, z_{\sigma(2)}] \ldots [z_{\sigma(2p-1)}, z_{\sigma(2p)}] [z_{\sigma(2p+1)}, z_1] = \]

\[ = 2^{-(p+1)} (p+1)^{-1} \sum (-1)^\sigma [z_{\sigma(1)}, z_{\sigma(2)}] \ldots [z_{\sigma(2p-1)}, z_{\sigma(2p)}] [z_{\sigma(2p+1)}, z_{\sigma(2p+2)}] \]

for \( z_{2p+2} = z_1 \) and \( h = (p+1)^{-1} s_{2p+2}(z_1, \ldots, z_{2p+1}, z_1) = 0. \)

(ii) By Lemma 2.2 (viii) \( z_1 z_{i_1} \ldots z_{i_{2q-1}} z_1 = 0, \quad q \geq 1, \) and the only non-zero summands of \( h \) are for \( 1 \in \{\sigma(1), \sigma(3), \ldots, \sigma(2p-1)\} \). Using the identity (vi) from Lemma 2.2 we obtain

\[ h = p z_1 \sum (-1)^\sigma z_{\sigma(2)} \ldots z_{\sigma(2p+1)} z_1 = p z_1^2 s_{2p}(z_2, \ldots, z_{2p+1}). \]

(iii) \[ \sum (-1)^\sigma z_{\sigma(2)} \ldots z_{\sigma(2p-1)} [z_{\sigma(2p)}, z_1] = \]

\[ = 2^{-(p-1)} \sum (-1)^\sigma [z_{\sigma(2)}, z_{\sigma(3)}] \ldots [z_{\sigma(2p-2)}, z_{\sigma(2p-1)}] [z_{\sigma(2p)}, z_1] = \]

\[ = 2^{-p} p^{-1} s_{2p}(z_1, \ldots, z_{2p}). \]
(iv) Let
\[ h = h(z_1, \ldots, z_{2p}) = s_{2p}(z_1, \ldots, z_{2p})z_1 \cdots z_{2q} = \]
\[ = \sum (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}z_1 \cdots z_{2q}. \]
In virtue of the first identity from Lemma 2.2 (vii), the only non-zero summands of \( h \) are for
1, 3, \ldots, 2q - 1 \in \{\sigma(2), \sigma(4), \ldots, \sigma(2p)\},
2, 4, \ldots, 2q \in \{\sigma(1), \sigma(4), \ldots, \sigma(2p - 1)\} and by Lemma 2.2 (vi) and (vii)
\[ h = (-1)^q(p(p - 1) \cdots (p - q + 1))^{2} \sum (-1)^{\sigma} z_{2q}z_1(z_4z_3) \cdots \]
\[ \cdots(z_{2q}z_{2q-1})z_{\sigma(2p)}(z_1z_{2q}) \cdots(z_{2q-1}z_{2q}) = \]
\[ = (-1)^q(p!)^{2}((p - q)!)^{-2}(z_{2q}^2z_1z_2)(z_4z_3^3z_4) \cdots \]
\[ \cdots(z_{2q}z_{2q-1}z_{2q})s_{2p}(z_{p-q})(z_{2q+1}, \ldots, z_{2p}) = \]
\[ = (p!)^{2}((p - q)!)^{-2}z_{1}^2 \cdots z_{2q}^2s_{2p}(z_{p-q})(z_{2q+1}, \ldots, z_{2p}). \]
Now we extend the action of \( S_{2q} \) trivially on \( \{2q + 1, \ldots, 2p\} \). For a fixed \( \tau \in S_{2q}, S_{2p} \tau = S_{2p} \) Hence
\[ s_{2p}(z_1, \ldots, z_{2p})s_{2q}(z_1, \ldots, z_{2q}) = \]
\[ = \sum (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}(z_1z_{2q}) \cdots z_{\sigma(2q)} = \]
\[ = \sum (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}z_{\tau(1)} \cdots z_{\tau(2q)} = \]
\[ = \sum s_{2p}(z_{\tau(1)}, \ldots, z_{\tau(2p)})z_{\tau(1)} \cdots z_{\tau(2q)} = \]
\[ = \sum h(z_{\tau(1)}, \ldots, z_{\tau(2p)}) = (2q)!h(z_1, \ldots, z_{2p}). \]
(v) Using the polynomial \( h \) defined in the proof of (iv), we obtain
\[ s_{2p}(z_1, \ldots, z_{2p}) \sum (-1)^{\tau} z_{\tau(1)} \cdots z_{\tau(2q-1)}(z_{\tau(2q)} \circ z_1) = \]
\[ = \sum (-s_{2p}(z_{\tau(2)}, \ldots, z_{\tau(2q-1)}z_{\tau(2q)}, z_{1}z_{2q+1}, \ldots, z_{2p})z_{\tau(2)} \cdots \]
\[ \cdots z_{\tau(2q-1)}z_{1} = \]
\[ + s_{2p}(z_{\tau(2)}, \ldots, z_{\tau(2q-1)}, z_{1}, z_{\tau(2q)}, z_{2q+1}, \ldots, z_{2p})z_{\tau(2)} \cdots \]
\[ \cdots z_{\tau(2q-1)}z_{1}z_{\tau(1)} \]
\[ = \sum (2q - 1)!(p!)^{2}((p - q)!)^{-2}z_{\tau(2)}^2 \cdots z_{\tau(2q-1)}^2 \times \]
\[ x(z_{2q}^2 z_1^2 + z_1^2 z_{2q}^2) s_{2(p-q)}(z_{2q+1}, \ldots, z_{2p}) = 0. \]

(vi) The non-zero summands of \( \sum (-1)^{\sigma} z_{\sigma(1)} \cdot z_{\sigma(2p-1)} z_1 \ldots z_{2q-1} \) are for

\[ 1, 3, \ldots, 2q - 1 \in \{ \sigma(1), \sigma(3), \ldots, \sigma(2p - 1) \}, \]
\[ 2, 4, \ldots, 2q - 2 \in \{ \sigma(2), \sigma(4), \ldots, \sigma(2p - 2) \} \]
and

\[ s_{2p-1}(z_1, \ldots, z_{2p-1}) z_1 \ldots z_{2q-1} = \]
\[ = p!(p - 1)!(p - q)!^{-2} z_1 \ldots z_{2q-1} s_{2(p-q)}(z_{2q}, \ldots, z_{2p-1}) z_1 \ldots z_{2q-1} = \]
\[ = p!(p - 1)!(p - q)!^{-2} (z_1 \ldots z_{2q-1})^2 s_{2(p-q)}(z_{2q}, \ldots, z_{2p-1}), \]
\[ (z_1 \ldots z_{2q-1})^2 = z_1(z_2 z_3) \ldots (z_{2q-2} z_{2q-1})(z_1 z_2) \ldots (z_{2q-3} z_{2q-2}) z_{2q-1} = \]
\[ = z_1(z_2 z_3) \ldots (z_{2q-3} z_{2q-2})(z_{2q-2} z_{2q-1}) z_{2q-1} = z_1^2 \ldots z_{2q-1}^2. \]

As in (iv)

\[ s_{2p-1}(z_1, \ldots, z_{2p-1}) s_{2q-1}(z_1, \ldots, z_{2q-1}) = \]
\[ = (2q - 1)! s_{2p-1}(z_1, \ldots, z_{2p-1}) z_1 \ldots z_{2q-1}. \]

The calculations for \( s_{2q-1}(z_1, \ldots, z_{2q-1}) s_{2p-1}(z_1, \ldots, z_{2p-1}) \) are similar.

(vii) is similar to (vi).

(viii) \( s_{2q+1}(z_1, \ldots, z_{2q+1}) = \)
\[ (q + 1) z_1 \sum (-1)^r z_{\tau(2)} \ldots z_{\tau(2q+1)} - q \sum (-1)^r z_{\tau(2)} z_1 z_{\tau(3)} \ldots z_{\tau(2q+1)}, \]
\[ h_1 = s_{2p-1}(z_2, \ldots, z_{2p}) o (((q + 1) z_1 z_2 - q z_2 z_1) z_3 \ldots z_{2q+1}) = \]
\[ = (q + 1)(s_{2p-1}(z_2, \ldots, z_{2p}) z_1)(z_2 z_3 \ldots z_{2q+1}) - \]
\[ - q(z_2 z_1)(s_{2p-1}(z_2, \ldots, z_{2p}) z_3 \ldots z_{2q+1}) + \]
\[ +(q + 1)(z_1 z_2)(z_3 \ldots z_{2q+1} s_{2p-1}(z_2, \ldots, z_{2p})) - \]
\[ - q(s_{2p-1}(z_2, \ldots, z_{2p}) z_2)(z_1 z_3 \ldots z_{2q+1}). \]

Since all elements in the parenthieses are of even length, Lemma 2.2 (vii) gives
\[ h_1 = (q + 1) z_2 h_2 z_1 - q h_2 z_2 z_1 + (q + 1) h_2 z_1 z_2 - q z_1 h_2 z_2, \]
where \( h_2 = z_3 \ldots z_{2q+1} s_{2p-1}(z_2, \ldots, z_{2p}). \) As in the proof of (vi)
\[ h_2 = - z_3 \ldots z_{2q+1} s_{2p-1}(z_3, \ldots, z_{2q+1}, z_2, z_2 z_2 + 2, \ldots, z_{2p}) = \]
\[ = -p!(p - 1)!(p - q)!^{-2} s_{2(p-q)}(z_2, z_{2q+2}, \ldots, z_{2p}) z_3^2 \ldots z_{2q+1}^2. \]

Since \( z_i^2 z_j = -z_j z_i^2, i = 3, \ldots, 2q + 1, j = 1, 2, \) and the standard polynomial of even length is central,
\[ h_1 = p!(p - 1)!((p - q)!)^{-2}(2q + 1) \times (z_2 S_{2(p-q)}(z_2, z_{2q+2}, \ldots, z_{2p}) z_1 - \\
- z_1 z_2 S_{2(p-q)}(z_2, z_{2q+2}, \ldots, z_{2p})) z_3^2 \ldots z_{2q+1}^2 = \\
p!(p - 1)!((p - q)!)^{-2}(2q + 1)(p - q) \times \\
x(z_2 S_{2(p-q)} - 1(z_{2q+2}, \ldots, z_{2p}) z_1 - \\
- z_1 z_2 S_{2(p-q)} - 1(z_{2q+2}, \ldots, z_{2p})) z_3^2 \ldots z_{2q+1}^2 = \\
p!(p - 1)!((p - q)!)^{-2}(2q + 1)(p - q) z_2^2 \ldots \\
\ldots z_{2q+1}^2 (s_{2(p-q)} - 1(z_{2q+2}, \ldots, z_{2p}) o z_1). \]

Hence

\[ s_{2p-1}(z_2, \ldots, z_{2p}) o s_{2q+1}(z_1, \ldots, z_{2q+1}) = \\
= (2q + 1)!p!(p - 1)!((p - q)!(p - q - 1))^{-1} z_2^2 \ldots \\
\ldots z_{2q+1}^2 (s_{2(p-q)} - 1(z_{2q+2}, \ldots, z_{2p}) o z_1). \]

Lemma 2.6. Let \( \lambda = (a + 2, 2b, c) \), \( a \geq 0, \ b + c > 0 \) and let \( f_\lambda(x_1, \ldots, x_{b+c+1}) \) be the polynomials from Section 1. If \( f_\lambda^{(i)} = f_\lambda^{(i)}(y + z_1, z_2, \ldots, \\
x_{b+c+1}), y \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}, z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]} \otimes E_{2k}^{[0]} \otimes E_{2l}^{[1]}, \) then there exist non-zero constants \( \alpha_\lambda \) from \( Q \) such that:

(i) \( f_\lambda^{(2)} = 0, \ f_\lambda^{(0)} = \alpha_\lambda y^2 + 2 s_{2p}(z_2, \ldots, z_{2p+1}), \lambda = (2r + 2, 1^{2p}); \)

(ii) \( f_\lambda^{(2)} = \alpha_\lambda y^2 + 1 z_1^2 s_{2p}(z_2, \ldots, z_{2p+1}), \lambda = (2r + 3, 1^{2p}); \)

(iii) \( f_\lambda^{(1)} = \alpha_\lambda y^2 - 1 (s_{2p-1}(z_2, \ldots, z_{2p}) o z_1), \lambda = (2r + 2, 1^{2p-1}); \)

(iv) \( f_\lambda^{(1)} = \alpha_\lambda y^2 + 2 s_{2p}(z_1, \ldots, z_{2p}), \lambda = (2r + 3, 1^{2p-1}); \)

(v) \( f_\lambda^{(2)} = \alpha_\lambda y^2 z_1^2 \ldots z_{2q}^2 s_{2(p-q)}(z_{2q+1}, \ldots, z_{2p}), \\
\lambda = (2r + 2, 2^{2q-1}, 1^{2(p-q)}); \)

(vi) \( f_\lambda^{(2)} = 0, \ f_\lambda^{(0)} = \alpha_\lambda y^2 z_1^2 \ldots z_{2q}^2 s_{2(p-q)}(z_{2q+1}, \ldots, z_{2p}), \\
\lambda = (2r + 3, 2^{2q-1}, 1^{2(p-q)}); \)

(vii) \( f_\lambda^{(1)} = \alpha_\lambda y^2 - 1 z_2^2 \ldots z_{2q+1}^2 (s_{2(p-q)} - 1(z_{2q+2}, \ldots, z_{2p}) o z_1) \) for \\
\lambda = (2r + 2, 2^{2q-1}, 1^{2(p-q)-1}) and \\
f_\lambda^{(1)} = \alpha_\lambda y^2 z_1^2 \ldots z_{2p}^2 s_{2(p-q)}(z_1, z_{2p+1}, \ldots, z_{2q+1}) for \\
\lambda = (2r + 2, 2^{2p-1}, 1^{2(p-q)-1}); \)

(viii) \( f_\lambda^{(1)} = \alpha_\lambda y^2 z_1^2 \ldots z_{2q+1}^2 s_{2(p-q)}(z_1, z_{2q+2}, \ldots, z_{2p}) \) for \\
\lambda = (2r + 3, 2^{2q-1}, 1^{2(p-q)-1}) and \\
f_\lambda^{(1)} = \alpha_\lambda y^2 z_1^2 \ldots z_{2p}^2 s_{2(p-q)}(z_1, z_{2q+1}, \ldots, z_{2q+1}) o z_1 \) for \\
\lambda = (2r + 3, 2^{2p-1}, 1^{2(p-q)-1}); \
\[(x)\quad f^{(2)}_{\lambda} = \alpha \gamma^{2r+2} \bar{x}_1^2 \cdots \bar{x}_{2q+1}^2 \gamma(\bar{z}_{2q+1}, \ldots, \bar{z}_{2p+1}), \quad \lambda = (2r + 3, 2^{2q+1}, 1^2)\]

\[(x)\quad f^{(2)}_{\lambda} = \alpha \gamma^{2r+1} \bar{x}_1^2 \cdots \bar{x}_{2q+1}^2 \gamma(\bar{z}_{2q+2}, \ldots, \bar{z}_{2p+1}), \quad \lambda = (2r + 3, 2^{2q+1}, 1^2)\]

**Proof.** Let \(\bar{x}_1 = y + z_1, \bar{x}_i = z_i, i > 1\), and let \(\bar{f}_{\lambda} = f_{\lambda}(\bar{x}_1, \ldots, \bar{x}_{b+c+1})\) for \(\lambda = (a + 2, 2^b, 1^c), a, b, c \geq 0\).

1. \(\bar{f}_{\lambda} = \sum_{\sigma} (-1)^{\sigma} \bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(2p)}(\bar{x}_{\sigma(2p+1)}, y + z_1) = 2^{2r}y^{2r} \sum_{\sigma} (-1)^{\sigma} \bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(2p)}(\bar{z}_{\sigma(2p+1)}, y + z_1)\)

and \(f^{(2)}_{\lambda} = 2^{2r}y^{2r} \sum_{\sigma} (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}(\bar{z}_{\sigma(2p+1)}, z_1) = 0\) by Lemma 2.5 (i);

\(f^{(0)}_{\lambda} = -2^{2r+1}y^{2r} \sum_{\sigma(2p+1) \neq 1} (-1)^{\sigma} (\bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(2p)})^{(0)}(\bar{z}_{\sigma(2p+1)}, z_1)\)

and by Lemma 2.2 (iv)

\(f^{(0)}_{\lambda} = -2^{2r+2}p^2 y^{2r+1} \sum_{\sigma} (-1)^{\sigma} z_{\sigma(2)} \cdots z_{\sigma(2p)}(\bar{z}_{\sigma(2p+1)}(y + z_1) = 2^{2r+2}p^2 y^{2r+1} z_{\sigma(2)} \cdots z_{\sigma(2p+1)}\).

2. Since \(z_2 \gamma^{2r+2}(y + z_1) = -2^{2r+1}y^{2r+1}(-2yz_2 + z_2 \circ z_1)\), we obtain that

\(f^{(2)}_{\lambda} = -2^{2r+1}y^{2r+1} \sum_{\sigma(2p+1) \neq 1} (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}(z_{\sigma(2p+1) \circ z_1}) = -2^{2r+1}y^{2r+1} \sum_{\sigma(2p+1) \neq 1} (-1)^{\sigma} z_{\sigma(1)} \cdots z_{\sigma(2p)}(2z_{\sigma(2p+1)} z_1 - [z_{\sigma(2p+1)}, z_1])\)

and the identity follows from Lemma 2.5 (i) and (ii).

3. \(\bar{f}_{\lambda} = \sum_{\sigma} (-1)^{\sigma} \bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(2p-2)}([\bar{x}_{\sigma(2p-1)}, \bar{x}_{\sigma(2p)}] \gamma^{2r+1} \bar{x}_1) = -2 \sum_{\sigma} (-1)^{\sigma} z_{\sigma(2)} \cdots z_{\sigma(2p-1)}([z_{\sigma(2)}, y] \gamma^{2r}, y + z_1],\)

\(f^{(1)}_{\lambda} = 2^{2r+2}y^{2r+1} \sum_{\sigma} (-1)^{\sigma} z_{\sigma(2)} \cdots z_{\sigma(2p-1)} z_{\sigma(2p})(z_{\sigma(2p) \circ z_1}).\)

Since \(\sum_{\sigma} (-1)^{\sigma} z_{\sigma(2)} \cdots z_{\sigma(2p-1)}\) is central, we obtain for some \(\alpha_\lambda \neq 0, \alpha_\lambda \in Q,\)

\(f^{(1)}_{\lambda} = \alpha_\lambda y^{2r+1} \sum_{\sigma} (-1)^{\sigma} (z_{\sigma(2)} \cdots z_{\sigma(2p-1)} z_{\sigma(2p)} z_1 + z_1 z_{\sigma(2)} \cdots z_{\sigma(2p-1)}) = \alpha_\lambda (s_{2p-1}(z_2, \ldots, z_{2p}) \circ z_1)\)

4. As in (iii)
\[ f^{(1)}_{\lambda} = -2 \sum (-1)^{\sigma} z_{\sigma(2)} \ldots z_{\sigma(2p-1)} \left\{ [z_{\sigma(2p)}, y] \text{ad}^{2r+1} y, z_1 \right\} = \]
\[ = -2^{2r+3} y^{2r+2} \sum (-1)^{\sigma} z_{\sigma(2)} \ldots z_{\sigma(2p-1)} [z_{\sigma(2p)}, z_1] = \]
\[ = \alpha_{\lambda} y^{2r+2} s_{2p}(z_1, \ldots, z_{2p}) \]
by Lemma 2.5 (iii).

(v) For \( r = 0 \), \( f^{(2)}_{\lambda} = 2 s_{2p}(z_1, \ldots, z_{2p}) s_{2q}(z_1, \ldots, z_{2q}) \). Let \( r > 0 \).
The non-zero summands of \( f^{(2)}_{\lambda} \) are for \( \tau(2q-1) = 1 \) or \( \tau(2q) = 1 \) and
\[ f^{(2)}_{\lambda} = -2 \sum (-1)^{\sigma} (-1)^{r} z_{\sigma(1)} \ldots z_{\sigma(2p)} z_{\tau(2)} \ldots \]
\[ \ldots z_{\tau(2q-1)} [z_{\tau(2q)}, y] \text{ad}^{2r-1} y, z_1 ] = \]
\[ = -2^{2r+1} y^{2r} \sum (-1)^{\sigma} (-1)^{r} z_{\sigma(1)} \ldots z_{\sigma(2p)} z_{\tau(2)} \ldots z_{\tau(2q-1)} [z_{\tau(2q)}, z_1] . \]

By Lemma 2.5 (iii) and (iv),
\[ f^{(2)}_{\lambda} = 2^{2r+1} q^{-1} y^{2r} s_{2p}(z_1, \ldots, z_{2p}) s_{2q}(z_1, \ldots, z_{2q}) = \]
\[ \alpha_{\lambda} y^{2r} z_1^{2} \ldots z_{2q}^{2} s_{2(p-q)}(z_{2q+1}, \ldots, z_{2p}) . \]

(vi) \( f^{(2)}_{\lambda} = -2 s_{2p}(z_1, \ldots, z_{2p}) \sum (-1)^{r} z_{\tau(2)} \ldots \]
\[ \ldots z_{\tau(2q-1)} [z_{\tau(2q)}, \text{ad}^{2r+1} y, z_1] = \]
\[ = 2^{2r+2} y^{2r+1} s_{2p}(z_1, \ldots, z_{2p}) \sum (-1)^{r} z_{\tau(2)} \ldots z_{\tau(2q-1)} (z_{\tau(2q)} \circ z_1) \]
and \( f^{(2)}_{\lambda} = 0 \) by Lemma 2.5 (v).
\[ f^{(0)}_{\lambda} = -2 p y s_{2p-1}(z_2, \ldots, z_{2p}) z_{\tau(2)} \ldots z_{\tau(2q-1)} (z_{\tau(2q)} \text{ad}^{2r+1} y) = \]
\[ = 2^{2r+4} p y^{2r+3} s_{2p-1}(z_2, \ldots, z_{2p}) s_{2q-1}(z_2, \ldots, z_{2q}) \]
and we apply Lemma 2.5 (vi).

(vii) \( f^{(1)}_{\lambda} = -2 \sum (-1)^{\sigma} (-1)^{r} z_{\sigma(2)} \ldots \]
\[ \ldots z_{\sigma(2p-1)} [(z_{\sigma(2p)} \text{ad}^{2r+1} y), z_{\tau(1)}] z_{\tau(2)} \ldots z_{\tau(2q+1)} = \]
\[ = 2^{2r+2} y^{2r+1} \sum (-1)^{\sigma} (-1)^{r} z_{\sigma(2)} \ldots \]
\[ \ldots z_{\sigma(2p-1)} (z_{\sigma(2p)} \circ z_{\tau(1)}) z_{\tau(2)} \ldots z_{\tau(2q+1)} = \]
\[ = 2^{2r+2} y^{2r+1} (s_{2p-1}(z_2, \ldots, z_{2p}) \circ s_{2q+1}(z_1, \ldots, z_{2q+1})) \]
and we apply Lemma 2.5 (viii). For \( p \leq q \),
\[ s_{2q+1}(z_1, \ldots, z_{2q+1}) = -s_{2q+1}(z_2, \ldots, z_{2q}, z_1, z_{2p+1}, \ldots, z_{2q+1}) \]
and we apply Lemma 2.5 (vi).
(viii) \( f^{(1)}_\lambda = -2 \sum (-1)^\sigma (-1)^r z_{\sigma(2)} \ldots \]
\[ \ldots z_{\sigma(2p-1)}[z_{\sigma(2p)ad^{2r+2}}y, z_1]z_{r(2)} \ldots z_{r(2q+1)} = \]
\[-2^{2r+3}y^{2r+2} \sum (-1)^\sigma z_{\sigma(2)} \ldots z_{\sigma(2p-1)}[z_{\sigma(2p)}, z_1]s_{2q}(z_2, \ldots, z_{2q+1}).\]

For \( p > q \) we apply Lemma 2.5 (iii) and (iv). Let \( p \leq q \).
\[ f^{(1)}_\lambda = -2^{2r+3}y^{2r+2}[s_{2p-1}(z_2, \ldots, z_{2p}), z_1]s_{2q}(z_2, \ldots, z_{2q+1}) = \]
\[-2^{2r+3}y^{2r+2}[s_{2p-1}(z_2, \ldots, z_{2p}), s_{2q}(z_2, \ldots, z_{2q+1}), z_1]. \]

By Lemma 2.5 (vii)
\[ s_{2p-1}(z_2, \ldots, z_{2p})s_{2q}(z_2, \ldots, z_{2q+1}) = \]
\[ = (2p-1)!(q!)^2((q-p)!(q-p+1))^{-1}z_2^2 \ldots \]
\[ \ldots z_{2p}^2 s_{2(q-p)+1}(z_{2p+1}, \ldots, z_{2q+1}). \]

Bearing in mind that \( z_1 z_2^2 \ldots z_{2p}^2 = -z_2^2 \ldots z_{2p}^2 z_1 \), we obtain
\[ [s_{2p-1}(z_2, \ldots, z_{2p}), s_{2q}(z_2, \ldots, z_{2q+1}), z_1] = \]
\[ = \alpha_\lambda z_2^2 \ldots z_{2p}^2(s_{2(q-p)+1}(z_{2p+1}, \ldots, z_{2q+1}) \circ z_1). \]

(ix) \( f_\lambda = -4 \sum (-1)^\sigma (-1)^r[z_{\sigma(2)}ad^{2r+1}y, z_{r(2)}][z_{r(3)}, y, z_{\sigma(3)}] \times \]
\[ \times z_{\sigma(4)} \ldots z_{\sigma(2p+1)}z_{r(4)} \ldots z_{r(2q+1)} = f^{(0)}_\lambda. \]

Hence \( f^{(2)}_\lambda = 0 \).
\[ f^{(0)}_\lambda = -2^{2r+4}y^{2r+2} \sum (-1)^\sigma (-1)^r(z_{\sigma(2)} \circ z_{r(2)})(z_{\sigma(3)} \circ z_{r(3)}) \times \]
\[ \times z_{\sigma(4)} \ldots z_{\sigma(2p+1)}z_{r(4)} \ldots z_{r(2q+1)} = \]
\[ = -2^{2r+4}y^{2r+2} \sum (-1)^\sigma (-1)^r(z_{\sigma(2)}z_{r(2)}z_{r(3)}z_{\sigma(3)} + \]
\[ + z_{r(2)}z_{\sigma(2)}z_{\sigma(3)}z_{r(3)} + z_{r(2)}z_{\sigma(2)}z_{r(3)}z_{\sigma(3)} + \]
\[ + z_{\sigma(2)}z_{r(2)}z_{\sigma(3)}z_{r(3)}z_{\sigma(4)} \ldots z_{\sigma(2p+1)}z_{r(4)} \ldots z_{r(2q+1)}. \]

Since \( \sum (-1)^r z_{r(2)}z_{r(3)} \) and \( \sum (-1)^\sigma z_{\sigma(2)}z_{\sigma(3)} \) are central elements, we obtain
\[ \sum (-1)^\sigma (-1)^r(z_{\sigma(2)}z_{r(2)}z_{r(3)}z_{\sigma(3)} + z_{r(2)}z_{\sigma(2)}z_{\sigma(3)}z_{r(3)}) \times \]
\[ \times z_{\sigma(4)} \ldots z_{\sigma(2p+1)}z_{r(4)} \ldots z_{r(2q+1)} = \]
\[ = 2s_{2p}(z_2, \ldots, z_{2p+1})s_{2q}(z_2, \ldots, z_{2q+1}) = \]
\[ = 2(2q)!(p!)^2((p-q)!)^{-2}z_2^2 \ldots z_{2q+1}^2s_{2(p-q)}(z_{2q+2}, \ldots, z_{2p+1}). \]
\[
\sum (-1)^\sigma (-1)^r z_{\sigma(2)} z_{\sigma(3)} z_{\sigma(4)} \cdots z_{\sigma(2p+1)} z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= \sum (-1)^{\sigma_r} (z_{\sigma_r(2)} z_{\sigma_r(3)} z_{\sigma_r(4)} \cdots z_{\sigma_r(2p+1)}) z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= (p + 1) \sum z_{\tau(2)}^2 s_{2p-1}(z_{\tau(3)}, \cdots, z_{\tau(2p+1)}) z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= (p + 1) p! (p - 1)! ((p - q)!)^{-2} \sum z_{\tau(2)}^2 \cdots \\
\cdots z_{\tau(2q+1)}^2 s_{2(p-q)}(z_{2q+1}, \cdots, z_{2p+1}) = \\
= (2q)! (p + 1)! (p - 1)! ((p - q)!)^{-2} z_{2q+1}^2 \cdots z_{2q+1}^2 s_{2(p-q)}(z_{2q+1}, \cdots, z_{2p+1}); \\
\sum (-1)^\sigma (-1)^r z_{\tau(2)} z_{\sigma(2)} z_{\tau(3)} z_{\sigma(3)} \cdots z_{\sigma(2p+1)} z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= - \sum (-1)^{\sigma_r} z_{\tau(3)} (z_{\sigma_r(2)} z_{\tau(2)} z_{\sigma_r(3)} \cdots z_{\sigma_r(2p+1)}) z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= -(p + 1) \sum z_{\tau(3)} z_{\tau(2)}^2 s_{2p-1}(z_{\tau(3)}, \cdots, z_{\tau(2p+1)}) z_{\tau(4)} \cdots z_{\tau(2q+1)} = \\
= (p + 1) \sum z_{\tau(2)}^2 z_{\tau(3)} \cdots z_{\tau(2q+1)}^2 s_{2p-1}(z_{\tau(3)}, \cdots, z_{\tau(2p+1)}) = \\
= (2q)! (p + 1)! (p - 1)! ((p - q)!)^{-2} z_{2q+1}^2 \cdots z_{2q+1}^2 s_{2(p-q)}(z_{2q+1}, \cdots, z_{2p+1}); \\
f^{(0)}_\lambda = -2^{2r+5} y^{2r+2} (2q)! ((p!)^2 + (p + 1)! (p - 1)!)((p - q)!)^{-2} \times \\
\times z_{2q+1}^2 \cdots z_{2q+1}^2 s_{2(p-q)}(z_{2q+1}, \cdots, z_{2p+1}). \\
(x) f^{(2)}_\lambda = \sum_{r(2q+1) \neq 1} (-1)^{\sigma_r} (-1)^r z_{\sigma(1)} \cdots z_{\sigma(2p)} \times \\
\times z_{\tau(1)} \cdots z_{\tau(2q)} [(z_{\tau(2q+1)} \text{ad}^r y, z_{\sigma(2p+1)}) = \\
= -2^{2r+1} y^{2r+1} \sum_{r(2q+1) \neq 1} (-1)^{\sigma_r} (-1)^r z_{\sigma(1)} \cdots \\
\cdots z_{\sigma(2p)} z_{\tau(1)} \cdots z_{\tau(2q)} (z_{\tau(2q+1)} \circ z_{\sigma(2p+1)}) = \\
= -2^{2r+1} y^{2r+1} \sum_{r(2q+1) \neq 1} (-1)^r (z_{\tau(1)} \cdots z_{\tau(2q+1)} \circ s_{2p+1}(z_1, \cdots, z_{2p+1}));
\[ = 2(2q + 1)! (p + 1)! p! ((p - q)!)^{-2} z_1^2 \ldots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \ldots, z_{2p+1}); \]

\[ s_{2p+1}(z_1, \ldots, z_{2p+1}) \circ (s_{2q}(z_2, \ldots, z_{2q+1}) z_1) = \]

\[ = (s_{2p+1}(z_1, \ldots, z_{2p+1}) \circ z_1) s_{2q}(z_2, \ldots, z_{2q+1}) = \]

\[ = 2(p + 1) z_1^2 s_{2p}(z_2, \ldots, z_{2p+1}) s_{2q}(z_2, \ldots, z_{2q+1}) = \]

\[ = 2(2q)! (p + 1)! p! ((p - q)!)^{-2} z_1^2 \ldots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \ldots, z_{2p+1}). \]

Hence

\[ f^{(2)}_{\lambda} = -2^{2r+2} y^{2r+1} ((2q + 1)! (p + 1)! p! ((p - q)!)^{-2} -\]

\[ -(2q)! (p + 1)! p! ((p - q)!)^{-2} z_1^2 \ldots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \ldots, z_{2p+1}) = \]

\[ = -2^{2r+3} y^{2r+1} q(2q)! (p + 1)! p! ((p - q)!)^{-2} z_1^2 \ldots \]

\[ \ldots z_{2q+1}^2 s_{2(p-q)}(z_{2q+2}, \ldots, z_{2p+1}). \]

The results of Lemma 2.6 can be summarized in the following way.

**Lemma 2.7.** (i) If \( a + b + 1 \equiv c \equiv 0 \pmod{2} \), then

\[ f^{(2)}_{\lambda} = \alpha_{\lambda} y^{a} z_1^2 \ldots z_{b+1}^2 s_c(z_{b+2}, \ldots, z_{b+c+1}); \]

(ii) If \( a + b + 1 \equiv 1, c \equiv 0 \pmod{2}, \) then

\[ f^{(2)}_{\lambda} = 0, \quad f^{(0)}_{\lambda} = \alpha_{\lambda} y^{a+2} z_2^2 \ldots z_{b+1}^2 s_c(z_{b+2}, \ldots, z_{b+c+1}); \]

(iii) If \( a + b + 1 \equiv 0, c \equiv 1 \pmod{2}, \) then

\[ f^{(1)}_{\lambda} = \alpha_{\lambda} y^{a+1} z_2^2 \ldots z_{b+1}^2 s_{c+1}(z_1, z_{b+2}, \ldots, z_{b+c+1}); \]

(iv) If \( a + b + 1 \equiv c \equiv 1 \pmod{2}, \)

\[ f^{(1)}_{\lambda} = \alpha_{\lambda} y^{a+1} z_2^2 \ldots z_{b+1}^2 (s_c(z_{b+2}, \ldots, z_{b+c+1}) \circ z_1). \]

**Proof.** The assertion (i) follows from Lemma 2.6 (ii), (v) and (x); (ii) is a consequence of Lemma 2.6 (i), (vi) and (ix); (iii) is derived from Lemma 2.6 (iv), (vii) and (viii); (iv) from Lemma 2.6 (iii), (vii) and (viii). \( \diamond \)

3. Cocharacters and codimensions

In this section we prove the main results of the paper.

**Theorem 3.1.** Let \( h_{ij}(\lambda) \) denote the \((i, j)\)-th hook of the Young diagram of the partition \( \lambda \). If \( k \geq l \geq 1 \), then
\[ \Gamma_n(E_{2k} \otimes E_{2l}) = \sum M(\lambda) + \varepsilon_n M(1^n), \]

where \( \varepsilon_n = 1 \) for \( n \) even and \( n \leq 2(k+l) \) and \( \varepsilon_n = 0 \) otherwise and the summation is over all partitions \( \lambda = (a+2, 2^b, 1^c) \) of \( n \) such that \( a \geq 0, b + c > 0, h_{12}(\lambda) = a + b + 1 \leq 2l \) and one of the following conditions holds:

(i) \( h_{11}(\lambda) + h_{12}(\lambda) - 1 = 2a + 2b + c + 2 < 2(k+l); \)
(ii) \( h_{11}(\lambda) + h_{12}(\lambda) - 1 = 2(k+l) \) and \( h_{12}(\lambda) \equiv 0 \) (mod 2).

**Proof.** Let \( M(1^n) \subset \Gamma_n(E_{2k} \otimes E_{2l}) \). Then \( n \) is even, for example \( n = 2p, \)

\[ s_{2p}(x_1, \ldots, x_{2p}) = 2^{-p} \sum [x_{\sigma(1)}, x_{\sigma(2)}] \ldots [x_{\sigma(2p-1)}, x_{\sigma(2p)}] \]

generates \( M(1^n) \) and \( s_n(u_1, \ldots, u_n) \neq 0 \) for some \( u_i \in E_{2k} \otimes E_{2l} \). As in the proof of Lemma 2.3 we need at least \( n \) different generators \( e_i \) and \( \bar{e}_j \) for the elements \( u_1, \ldots, u_n \), i.e. \( n \leq 2(k+l) \).

If \( n \leq 2(k+l) \), then it is easy to see that

\[ s_{2p}(e_1, \ldots, e_{2k}, \bar{e}_1, \ldots, \bar{e}_{2(p-k)}) = \]

\[ = \binom{p}{k} (2k)! (2(p-k))! e_1 \ldots e_{2k} \bar{e}_1 \ldots \bar{e}_{2(p-k)} \neq 0. \]

Let \( M(a + 2, 2^b, 1^c) \subset \Gamma_n(E_{2k} \otimes E_{2l}), a \geq 0, b + c > 0 \). In virtue of Lemma 2.3, \( 2a + 2b + c + 2 \leq 2(k + l) \) and \( a + b + 1 \leq 2l \).

First, let \( a + b + 1 \leq 2l \) and \( 2a + 2b + c + 2 = 2(k + l) \). Hence \( c \equiv 0 \) (mod 2). By Lemma 2.4 \( f_\lambda = 0 \) is a polynomial identity for \( E_{2k} \otimes \otimes E_{2l} \) if and only if \( f^{(2)}_\lambda (y + z_1, z_2, \ldots, z_{b+c+1}) = 0 \) for all \( y \in E_{2k}^{[1]} \otimes E_{2l}^{[1]}, z_i \in E_{2k}^{[1]} \otimes E_{2l}^{[0]} \oplus E_{2k}^{[0]} \otimes E_{2l}^{[1]} \). If \( h_{12}(\lambda) = a + b + 1 \equiv 0 \) (mod 2), then Lemma 2.7 (i) gives

\[ f^{(2)}_\lambda = \alpha_\lambda y^2 z_1^2 \ldots z_{b+1}^2 s_c(\bar{z}_{b+2}, \ldots, z_{b+c+1}). \]

Let \( y = e_1 \bar{e}_1 + \ldots + e_a \bar{e}_a, z_1 = e_{a+1} + \bar{e}_{a+1}, \ldots, z_{b+1} = e_{a+b+1} + \bar{e}_{a+b+1} \). We use even number of generators from each set \( \{e_1, \ldots, e_{2k}\} \) and \( \{\bar{e}_1, \ldots, \bar{e}_{2l}\} \). Hence we still have available even numbers of elements in each set and \( s_c(e_{a+b+2}, \ldots, e_{2k}, \bar{e}_{a+b+2}, \ldots, e_{2l}) \neq 0 \). If \( h_{12}(\lambda) \equiv 1 \) (mod 2), then by Lemma 2.7 (ii) \( f^{(2)}_\lambda = 0 \), i.e. \( f_\lambda = 0 \) is a polynomial identity for \( E_{2k} \otimes \otimes E_{2l} \).

Now, let \( a + b + 1 \leq 2l \) and \( 2a + 2b + c + 2 < 2(k + l) \). Depending on the parity of \( a + b + 1 \) and \( c \) we consider four different cases.

(1) \( a + b + 1 \equiv c \equiv 0 \) (mod 2). The proof in this case is similar to the case \( 2a + 2b + c + 2 = 2(k + l) \) and \( f^{(2)}_\lambda \neq 0 \) for suitable \( y, z_i \in E_{2k} \otimes E_{2l} \).
(2) \( a + b + 1 \equiv 1, c \equiv 0 \pmod{2} \). By Lemma 2.7 (ii),
\[
f^{(0)} = \alpha \lambda y^{a+1} z_2^2 \ldots z_{b+1}^2 s_c(z_{b+2}, \ldots, z_{b+c+1}).
\]
Clearly \( a + b + 1 \leq 2l - 1 \) and \( 2a + 2b + c + 2 \leq 2(k + l - 1) \). Hence we use \( a + b + 2 \) generators of both \( E_{2k} \otimes 1 \) and \( 1 \otimes E_{2l} \) for \( y = e_1 \tilde{e}_1 + \ldots + e_{a+2} \tilde{e}_{a+2}, \ z_2 = e_{a+3} + \tilde{e}_{a+3}, \ldots, z_{b+1} = e_{a+b+2} + \tilde{e}_{a+b+2} \) and we still have even sets of generators \( \{ e_{a+b+3}, \ldots, e_{2k} \}, \ \{ \tilde{e}_{a+b+3}, \ldots, \tilde{e}_{2l} \} \) in order to obtain \( s_c(z_{b+2}, \ldots, z_{b+c+1}) \neq 0 \).

(3) \( a + b + 1 \equiv 0, c \equiv 1 \pmod{2} \). By Lemma 2.7 (iii)
\[
f^{(1)} = \alpha \lambda y^{a+1} z_2^2 \ldots z_{b+1}^2 s_{c+1}(z_1, z_{b+2}, \ldots, z_{b+c+1}).
\]
Let \( y = e_1 \tilde{e}_1 + \ldots + e_{a+1} \tilde{e}_{a+1}, \ z_2 = e_{a+2} + \tilde{e}_{a+2}, \ldots, z_{b+1} = e_{a+b+1} + + \tilde{e}_{a+b+1} \). Then we have left \( 2k - (a + b + 1) \equiv 0 \pmod{2} \) elements \( e_{a+b+2}, \ldots, e_{2k} \) and \( 2l - (a + b + 1) \equiv 0 \pmod{2} \) elements \( \tilde{e}_{a+b+2}, \ldots, \tilde{e}_{2l} \).

Since \( 2a + 2b + c + 2 \leq 2(k + l) - 1 \), we can choose \( z_1, z_{b+2}, \ldots, z_{b+c+1} \) in such a way that \( s_{c+1}(z_1, z_{b+2}, \ldots, z_{b+c+1}) \neq 0 \) and \( f^{(1)} \neq 0 \).

(4) \( a + b + 1 \equiv c \equiv 1 \pmod{2} \). By Lemma 2.7 (iv)
\[
f^{(1)} = \alpha \lambda y^{a+1} z_2^2 \ldots z_{b+1}^2 s_c(z_{b+2}, \ldots, z_{b+c+1}) z_1.
\]
Again \( y = e_1 \tilde{e}_1 + \ldots + e_{a+1} \tilde{e}_{a+1}, \ z_2 = e_{a+2} + \tilde{e}_{a+2}, \ldots, z_{b+1} = e_{a+b+1} + + \tilde{e}_{a+b+1} \) and we have on disposal odd number of elements \( e_{a+b+2}, \ldots, e_{2k} \in E_{2k} \otimes 1 \) and \( \tilde{e}_{a+b+2}, \ldots, \tilde{e}_{2l} \in 1 \otimes E_{2l} \).

Since \( e_j \circ \tilde{e}_j = 0, e_i \circ \tilde{e}_j = = 2e_i \tilde{e}_j \) and
\[
s_{2m+1}(x_1, \ldots, x_{2m+1}) = \sum (-1)^{i-1} s_{2m}(x_1, \ldots, \hat{x}_i, \ldots, x_{2m+1}) x_i,
\]
it is easy to see that
\[
s_{2(p+q)+1}(e_{i_1}, \ldots, e_{i_{2p}}, \tilde{e}_{j_1}, \ldots, \tilde{e}_{i_{2q+1}}) \circ e_{i_{2p+1}} = \]
\[
= \begin{pmatrix} p + q \\ p \end{pmatrix} s_{2p}(e_{i_1}, \ldots, e_{i_{2p}}) (s_{2q+1}(\tilde{e}_{j_1}, \ldots, \tilde{e}_{i_{2q+1}}) \circ e_{i_{2p+1}}) = \]
\[
= 2 \begin{pmatrix} p + q \\ p \end{pmatrix} (2p)! (2q + 1)! e_{i_1} \ldots e_{i_{2p+1}} \tilde{e}_{j_1} \ldots \tilde{e}_{i_{2q+1}}
\]
and \( f^{(1)} \neq 0 \).

**Remark 3.2.** Using the proper cocharacters of \( E_{2k} \otimes E_{2l} \) we can obtain the ordinary cocharacter sequence. For example
\[
\Gamma_0(E_2 \otimes E_2) = M(0), \ \Gamma_2(E_2 \otimes E_2) = M(1^2),
\]
\[
\Gamma_3(E_2 \otimes E_2) = M(2, 1), \ \Gamma_4(E_2 \otimes E_2) = M(2^2) + M(1^4)
\]
and \( \Gamma_n(E_2 \otimes E_2) = 0 \) for all other \( n \). Applying Prop. 1.2 (i) we obtain for \( n \geq 6 \).
\[ P_n(E_2 \otimes E_2) = M(n) + 2M(n-1,1) + 2M(n-2,2) + 2M(n-2,1^2) + \\
+ 2M(n-3,2,1) + M(n-3,1^3) + M(n-4,2^2) + M(n-4,1^4). \]

**Corollary 3.3.** Let \( k \geq l \geq 1 \), \( k' \geq l' \geq 1 \). Then \( T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'}) \) if and only if \( k + l \geq k' + l' \) and \( l \geq l' \).

**Proof.** Since the \( S_n \)-module \( \Gamma_n(E \otimes E) \) is a sum of pairwise non-isomorphic irreducible submodules, \( T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'}) \) if and only if \( \Gamma_n(E_{2k'} \otimes E_{2l'}) \) is isomorphic to a submodule of \( \Gamma_n(E_{2k} \otimes E_{2l}) \) for all \( n \).

Let \( k + l \geq k' + l' \) and \( l \geq l' \). Applying Th. 3.1 we obtain that every irreducible submodule of \( \Gamma_n(E_{2k'} \otimes E_{2l'}) \) participates in the decomposition of \( \Gamma_n(E_{2k} \otimes E_{2l}) \), i.e. \( \Gamma_n(E_{2k'} \otimes E_{2l'}) \subset \Gamma_n(E_{2k} \otimes E_{2l}) \) and \( T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'}) \).

Let \( T(E_{2k} \otimes E_{2l}) \subset T(E_{2k'} \otimes E_{2l'}) \). Hence \( \Gamma_n(E_{2k'} \otimes E_{2l'}) \subset \Gamma_n(E_{2k} \otimes E_{2l}) \) for all \( n \). Since

\[
M(2^{b+1},1^c) = M(2^{2l'},1^{2(k'-l')}) \subset \Gamma_{2(k'+l')}E_{2k'} \otimes E_{2l'},
\]

Th. 3.1 gives \( 2b + c + 2 = 2(k' + l') \leq 2(k + l) \), \( b + 1 = 2l' \leq 2l \), i.e. \( k + l \geq k' + l' \), \( l \geq l' \).  

**Theorem 3.4.** Let \( l \geq 1 \). Then

\[
\Gamma_n(E \otimes E_{2l}) = \sum M(a + 2, b, 1^c) + \varepsilon_n M(1^n),
\]

where the sum is over all partitions \( (a + 2, b, 1^c) \) of \( n \), such that \( a \geq 0 \), \( b + c > 0 \) and \( a + b + 1 \leq 2l; \varepsilon_n = 1 \) for \( n \) even and \( \varepsilon_n = 0 \) for \( n \) odd.

**Proof.** Considering \( \Gamma_n(E \otimes E_{2l}) \) and \( \Gamma_n(E_{2k} \otimes E_{2l}) \) as \( S_n \)-submodules of \( \Gamma_n(E \otimes E) \) we obtain

\[ \Gamma_n(E \otimes E_{2l}) = \bigcup_{k \geq 1} \Gamma_n(E_{2k} \otimes E_{2l}). \]

Hence by Th. 3.1 \( M(1^n) \subset \Gamma_n(E \otimes E_{2l}) \) for \( n \) even. Let \( \lambda = (a + 2, b, 1^c) \vdash n \) and let \( k \) be large enough. Then the condition \( h_{11}(\lambda) + h_{12}(\lambda) - 1 < 2(k + l) \) from Th. 3.1 is satisfied automatically and \( M(\lambda) \subset \Gamma_n(E_{2k} \otimes E_{2l}) \) if and only if \( h_{12}(\lambda) = a + b + 1 \leq 2l \).

**Theorem 3.5.** Let \( k \geq l \geq 1 \).

(i) The codimension sequence \( c_n(E_{2k} \otimes E_{2l}) \) is a polynomial with rational coefficients of degree \( 2(k + l) \) in \( n \).

(ii) For \( n > 0 \)

\[
c_n(E \otimes E_{2l}) = 2^{n-1} \xi(n) + \eta(n),
\]

where \( \xi(n) \) and \( \eta(n) \) are polynomials with rational coefficients in \( n \), \( \deg \xi(n) = 2l, \deg \eta(n) \leq 4l - 1 \) and the leading term of \( \xi(n) \) is equal to \( (2l)! \)^{-1}.  

\[ \]
Proof. (i) Let $\lambda = (a+2, 2^b, 1^c) \vdash n$ and let $M(\lambda) \subset \Gamma_n(E_{2k} \otimes E_{2l})$. By Th. 3.1, $h_{11}(\lambda) + h_{12}(\lambda) - 1 \leq 2(k + l)$. Since $n \leq h_{11}(\lambda) + h_{12}(\lambda) - 1$, we obtain that $n \leq 2(k + l)$. Similarly, $M(1^n) \subset \Gamma_n(E_{2k} \otimes E_{2l})$ if and only if $n$ is even and $n \leq 2(k + l)$. Hence $\Gamma_{2(k+l)}(E_{2k} \otimes E_{2l}) \neq 0$ and $\Gamma_n(E_{2k} \otimes E_{2l}) = 0$ for $n > 2(k + l)$. Equivalently, $\gamma_{2(k+l)}(E_{2k} \otimes E_{2l}) > 0$ and $\gamma_n(E_{2k} \otimes E_{2l}) = 0$ for $n > 2(k + l)$ and the assertion follows from Prop. 1.2 (ii).

(ii) By Th. 3.4, $\Gamma_n(E \otimes E_{2l}) = \sum M(\lambda) + \epsilon_n M(1^n)$, where $\lambda = (a+2, 2^b, 1^c) \vdash n$, $a \geq 0$, $b + c > 0$, $a + b + 1 \leq 2l$ and $\epsilon_n = 0, 1$. The dimension of $M(1^n)$ is equal to 1. By Lemma 1.1, for fixed $a + b + 1$

$$\dim M(a+2, 2^b, 1^c) = \psi_{ab}(n) = \frac{1}{(a + b + 1)!} \dim M(a+1, 1^b)n^{a+b+1} + \ldots,$$

where $\psi_{ab}(n) \in \mathbb{Q}[n]$ and $\deg \psi_{ab}(n) = a + b + 1$. Hence for $n \geq 4l$

$$\gamma_n = \gamma_n(E \otimes E_{2l}) = \epsilon_n + \sum_{a+b+1 \leq 4l} \psi_{ab}(n),$$

$\psi_l(n) = \sum \psi_{ab}(n)$ is a polynomial of degree $2l$ and with leading term

$$\tilde{\gamma}_n = \frac{1}{(2l)!} \sum_{p=0}^{2l} \dim M(2l - p, 1^p).$$

The polynomials $(\binom{n+m}{m})_m = 0, 1, 2, \ldots$, form a basis of $\mathbb{Q}[n]$ and we rewrite $\psi_l(n)$ in the form

$$\psi_l(n) = \sum_{m=0}^{2l} \gamma_m'(n + m) \binom{n + m}{m}$$

for some $\gamma_m' \in \mathbb{Q}$ and

$$\gamma_{2l}' = (2l)! \tilde{\gamma}_n = \sum_{p=0}^{2l} \dim M(2l - p, 1^p).$$

Therefore

$$\gamma_n = \sum_{m=0}^{2l} \gamma'_m \binom{n + m}{m} + \epsilon_n, \ n \geq 4l,$$

$$\gamma_n = \nu_n + \sum_{m=0}^{2l} \gamma'_m \binom{n + m}{m} + \epsilon_n, \ \nu_n \in \mathbb{Q}, \ n < 4l,$$
\[ \gamma(t) = \gamma(E \otimes E_{2l}, t) = \sum_{n \geq 0} \sum_{m=0}^{2l} \gamma'_m \binom{n+m}{m} t^n + \sum_{m=0}^{4l-1} \nu_n t^n + \sum_{n \geq 0} t^{2n} = \]
\[ = \sum_{m=0}^{2l} \frac{\gamma'_m}{(1-t)^{m+1}} + \theta_l(t) + \frac{1}{1-t^2}, \]

where \( \theta_l(t) \in \mathbb{Q}[t] \) and \( \deg \theta_l(t) \leq 4l - 1 \). Applying Prop. 1.2 (iii) we obtain
\[ c(t) = c(E \otimes E_{2l}, t) = \sum c_n(E \otimes E_{2l}) t^n = \]
\[ = \sum_{m=0}^{2l} \frac{\gamma'_m (1-t)^m}{(1-2t)^{m+1}} + \frac{1}{1-t} \theta_l \left( \frac{t}{1-t} \right) + \frac{1}{2(1-2t)} + \frac{1}{2}, \]
\[ \frac{(1-t)^m}{(1-2t)^{m+1}} = \frac{(1 + (1-2t))^m}{2^m(1-2t)^{m+1}} = \frac{1}{2^m(1-2t)^{m+1}} = \rho_m \left( \frac{1}{1-2t} \right), \]

where \( \rho_m(t) \in \mathbb{Q}[t] \), \( \deg \rho_m(t) < m \). Similarly
\[ \frac{1}{1-t} \theta_l \left( \frac{t}{1-t} \right) = \tau_l \left( \frac{1}{1-t} \right), \quad \tau_l(t) = \sum \tau_m t^m \in \mathbb{Q}[t], \]
\[ \deg \tau_l(t) \leq 4l - 1. \]

Hence
\[ c(t) = \frac{\gamma''_{2l}}{2^{2l}(1-2t)^{2l+1}} + \sum_{m=0}^{2l-1} \frac{\gamma''_m}{(1-2t)^{m+1}} + \sum_{m=0}^{4l-1} \frac{\tau_m}{(1-t)^{m+1}} + \frac{1}{2} = \]
\[ = \sum_{n \geq 0} \left[ \left( \frac{\gamma''_{2l}}{2^{2l}} \binom{n+2l}{2l} + \sum_{m=0}^{2l-1} \gamma''_m \binom{n+m}{m} \right) 2^n + \right. \]
\[ + \sum_{m=0}^{4l-1} \tau_m \binom{n+m}{m} \right] t^n + \frac{1}{2} \]

and \( c_n = \xi_l(n) 2^n + \eta_l(n), \ n > 0 \), where \( \xi_l(n), \eta_l(n) \in \mathbb{Q}[n], \ \deg \xi_l(n) = 2l, \ \deg \eta_l(n) \leq 4l - 1 \) and the leading term of \( \xi_l(n) \) is equal to
\[ \frac{\gamma''_{2l}}{2^{2l}(2l)!} = \frac{1}{2^{2l}(2l)!} \sum_{p=0}^{2l} \dim M(2l - p, 1^p). \]

Using the hook formula it is easy to see that
\[
\dim M(2^l - p, 1^p) = \binom{2p - 1}{p}, \quad \sum_{p=0}^{2^l} \dim M(2^l - p, 1^p) = 2^{2^l - 1}
\]

and this completes the proof of the theorem. ◊

References


