RELATIONSHIPS BETWEEN DISTANCE DOMINATION PARAMETERS

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Abstract: For any integer $n \geq 2$ a set $D$ of vertices of a graph $G$ of order $p$ is defined to be a $P_{\leq n}$-dominating set (total $P_{\leq n}$-dominating set) of $G$ if every vertex in $V(G) - D$ (respectively $V(G)$) is at distance at most $n - 1$ from some vertex in $D$ other than itself. The $P_{\leq n}$-domination number, $\gamma_n(G)$ (total $P_{\leq n}$-domination number $\gamma^t_n(G)$) is the minimum cardinality among all $P_{\leq n}$-dominating sets (total $P_{\leq n}$-dominating sets) of $G$. It is shown that if $G$ is a connected graph on $p \geq 2n$ vertices, then $\gamma_n(G) + \gamma^t_n(G) \leq 2p/n$. A set $I$ of vertices in a graph $G$ is $P_{\leq n}$-independent if the distance between every two vertices of $I$ is at least $n$. A $P_{\leq n}$-dominating set that is also $P_{\leq n}$-independent is called a $P_{\leq n}$-independent dominating set. The minimum cardinality among all $P_{\leq n}$-independent dominating sets in a graph $G$ is the $P_{\leq n}$-independent domination number of $G$ and is denoted by $i_n(G)$. It is shown that if $G$ is a connected graph of order $p \geq n$, then $i_n(G) + (n - 1)\gamma_n(G) \leq p$. 
The terminology and notation of [2] will be used throughout. Recall that a dominating set (total dominating set) \( D \) of a graph \( G \) is a set of vertices of \( G \) such that every vertex of \( V(G) - D \) (respectively, \( V(G) \)) is adjacent to some vertex of \( D \). The domination number (total domination number) of \( G \) is the minimum cardinality of a dominating set (total dominating set) of \( G \). Further, the distance \( d(u,v) \) between two vertices \( u \) and \( v \) of \( G \) is the length of a shortest \( u-v \) path if one exists, otherwise \( d(u,v) = \infty \). In [5] generalizations of the above-mentioned domination parameters are defined and studied. For an integer \( n \geq 2 \), a set \( D \) of vertices of a graph \( G \) is defined to be a \( P_{\leq n} \)-dominating set (total \( P_{\leq n} \)-dominating set) of \( G \) if every vertex in \( V(G) - D \) (respectively \( V(G) \)) is at distance at most \( n-1 \) from some vertex in \( D \) other than itself. The \( P_{\leq n} \)-domination number \( \gamma_n(G) \) (total \( P_{\leq n} \)-domination number \( \gamma_t^n(G) \)) is the minimum cardinality of a \( P_{\leq n} \)-dominating set (total \( P_{\leq n} \)-dominating set) of \( G \). Hence \( \gamma_2(G) = \gamma(G) \) and \( \gamma_t^2(G) = \gamma_t(G) \).

In [5] sharp bounds for the \( P_{\leq n} \)-domination number and total \( P_{\leq n} \)-domination number of a graph are established. In particular the following two results were obtained.

**Theorem A.** If \( G \) is a connected graph of order \( p \geq n \), then \( \gamma_n(G) \leq p/n \).

**Theorem B.** If \( G \) is a connected graph of order \( p \geq 2 \), then

\[
\gamma_t^n(G) = 2 \quad \text{for} \quad 2 \leq p \leq 2n - 1
\]

and

\[
\gamma_t^n(G) \leq \frac{2p}{2n - 1} \quad \text{for} \quad p \geq 2n - 1.
\]

We now investigate relationships between these two generalized domination parameters. Observe that if \( G \) is a connected graph on \( p \) vertices with \( 2 \leq p \leq 2n - 1 \), then \( \text{rad}(G) \leq n - 1 \) and so \( \gamma_n(G) + \gamma_t^n(G) = 3 \). We thus consider graphs of order \( p \geq 2n \). Allan, Laskar and Hedetniemi [1] showed that, if \( G \) is a connected graph of order \( p \geq 3 \), then \( \gamma(G) + \gamma_t(G) \leq p \). The following theorem generalizes this result.

**Theorem 1.** For an integer \( n \geq 2 \), if \( G \) is a connected graph of order \( p \geq 2n \), then

\[
\gamma_n(G) + \gamma_t^n(G) \leq 2p/n.
\]

**Proof.** Let \( n \geq 2 \) be an integer. If \( T \) is a spanning tree of a connected graph \( G \) of order at least \( 2n \) and \( \gamma_n(T) + \gamma_t^n(T) \leq 2p(G)/n \), then
\( \gamma_n(G) + \gamma_n^t(G) \leq \gamma_n(T) + \gamma_n^t(T) \leq 2p(G)/n \). Hence we shall prove the theorem by establishing its validity for a tree \( G \). We proceed by induction on the order of a tree of order at least \( 2n \).

Let \( T \) be a tree of order \( 2n \). Then \( \text{diam} \, T \leq 2n - 1 \), and so \( \text{rad} \, T \leq n - 1 \) or \( T \) is bicentral with \( \text{rad} \, T \leq n \). If \( \text{rad} \, T \leq n - 1 \), then a central vertex of \( T \) is within distance \( n - 1 \) from every vertex of \( T \), while a central vertex, together with any other vertex of \( T \), forms a total \( P_{\leq n} \)-dominating set of \( T \). Hence in this case, \( \gamma_n(T) + \gamma_n^t(T) = 3 < 2p(T)/n \). If, however, \( \text{rad} \, T = n \), then the central vertices of \( T \) form a total \( P_{\leq n} \)-dominating set (and hence certainly a \( P_{\leq n} \)-dominating set) of \( T \) and so \( \gamma_n(T) + \gamma_n^t(T) = 4 = 2p(T)/n \). Hence the theorem is true for a tree of order \( 2n \).

Assume that \( \gamma_n(T') + \gamma_n^t(T') \leq 2p(T')/n \) for all trees \( T' \) with \( 2n \leq p(T') < k \), and let \( T \) be a tree of order \( k \). If \( \text{diam} \, T \leq 2n - 1 \), then \( \gamma_n(T) + \gamma_n^t(T) \leq 4 < 2p(T)/n \). So we may assume that \( \text{diam} \, T \geq 2n \).

Suppose that there exists an edge \( e \) of \( T \) such that both components of \( T - e \) are of order at least \( 2n \). Let \( T_1 \) and \( T_2 \) be the components of \( T - e \). Then \( 2n \leq p(T_i) < k \) and so, by the induction hypothesis, for \( i \in \{1, 2\} \), \( T_i \) has a \( P_{\leq n} \)-dominating set \( D_i \) and a total \( P_{\leq n} \)-dominating set \( D_i' \) with \( |D_i| + |D_i'| = \gamma_n(T_i) + \gamma_n^t(T_i) \leq 2p(T_i)/n \). Then \( D_1 \cup D_2 \) is a \( P_{\leq n} \)-dominating set of \( T \) and \( D_1' \cup D_2' \) is a total \( P_{\leq n} \)-dominating set of \( T \) with \( \gamma_n(T) + \gamma_n^t(T) \leq |D_1 \cup D_2| + |D_1' \cup D_2'| \leq 2p(T)/n \). For the remainder of the proof we shall therefore assume that, for each edge \( e \) of \( T \), at least one of the (two) components of \( T - e \) is of order less than \( 2n \). In particular, we note that \( 2n \leq \text{diam} \, T \leq 4n - 2 \). Let \( \text{diam} \, T = d \) and let \( u, v \) be two vertices of \( T \) such that \( d(u, v) = d \geq 2n \). Let the \( u - v \) path in \( T \) be denoted by \( P : u = u_0, u_1, \ldots, u_d = v \). To complete the proof we consider four lemmas.

**Lemma 1.** If \( 2n < p(T) \leq 3n - 2 \), then \( \gamma_n(T) + \gamma_n^t(T) < 2p(T)/n \).

**Proof.** Let \( T_1, T_2 \) and \( T_3 \) denote the components of \( T - u_{n-1}u_n, T - u_{d-n}u_{d-n+1} \) and \( T - \{u_{n-1}u_n, u_{d-n}u_{d-n+1}\} \), respectively, containing \( u, v \) and \( u_n \) respectively. Since \( p(T) \leq 3n - 2 \), it follows that \( d \leq 3n - 3 \); so \( d(u_{n-1}, u_{d-n+1}) = d + 2 - 2n \leq n - 1 \). Moreover, since \( P \) is a longest path in \( T \), the vertex \( u_{n-1} \) (\( u_{d-n+1} \)) is at distance at most \( n - 1 \) from every vertex in \( T_1 \) (\( T_2 \), respectively). As \( p(T_3) = p(T) - (p(T_1) + p(T_2)) \leq 3n - 2 - 2n = n - 2 \), every vertex of \( T_3 \) is within distance \( n - 2 \) from both \( u_{n-1} \) and \( u_{d-n+1} \) in \( T \). It follows that \( \gamma_n(T) = \gamma_n^t(T) = |\{u_{n-1}, u_{d-n+1}\}| = 2 \); so \( \gamma_n(T) + \gamma_n^t(T) = 4 < 2p(T)/n \). This completes the proof of Lemma 1. \( \square \)
Lemma 2. If \( p(T) \geq 3n - 1 \) and \( 2n \leq d \leq 3n - 3 \), then \( \gamma_n(T) + \gamma^t_n(T) \leq 2p(T)/n \).

**Proof.** Let \( T_1, T_2 \) and \( T_3 \) be defined as in the proof of Lemma 1. Since \( d \leq 3n - 3 \), \( d(u_{n-1}, u_{d-n+1}) \leq n - 1 \). Moreover, as \( P \) is a longest path in \( T \), \( u_{n-1}(u_{d-n+1}) \) is at distance at most \( n - 1 \) from every vertex in \( T_1 \) (\( T_2 \), respectively).

If \( p(T_3) \leq n - 1 \), then every vertex of \( T_3 \) is within distance \( n - 1 \) from both \( u_{n-1} \) and \( u_{d-n+1} \); consequently, \( \gamma_n(T) + \gamma^t_n(T) = 4 < 2p(T)/n \).

Suppose that \( n \leq p(T_3) \leq 2n - 1 \). Then \( p(T) \geq 3n \) and \( \text{diam} \ T_3 \leq 2n - 2 \); so \( \text{rad} \ T_3 \leq n - 1 \). We show that there exists a central vertex of \( T_3 \) that is distance at most \( n - 1 \) from \( u_{n-1} \) or \( u_{d-n+1} \). If this is not the case, then, for \( w \) a central vertex of \( T_3 \), \( w \) is at distance \( n - 1 \) from both \( u_n \) and \( u_{d-n} \). Since \( d(u_n, u_{d-n}) = d - 2n \leq n - 3 \), \( w \) is not a vertex of the \( u_n - u_{d-n} \) path. Let \( Q : v = w_0, w_1, \ldots, w_s \) be the shortest path from \( w \) to a vertex of the \( u_n - u_{d-n} \) path. Then, necessarily, \( w_s = u_j \) for some \( j \in \{ n + 1, \ldots, d - n - 1 \} \) and \( V(Q) \cap V(P) = \{ u_j \} \). Let \( T' \) and \( T'' \) denote the components of \( T_3 - ww_1 \) containing \( w_1 \) and \( w \) respectively. Since the \( w_1 - u_n \) path (of order \( n - 1 \)) does not contain the vertex \( u_{d-n} \), we observe that \( p(T') \geq n \). Further, if \( p(T'') \leq n - 1 \), then it follows that \( w_1 \) is a central vertex of \( T_3 \) at distance \( n - 1 \) from both \( u_{n-1} \) and \( u_{d-n+1} \), which contradicts our assumption. Hence \( p(T'') \geq n \), and so \( p(T_3) \geq 2n \), which again produces a contradiction. Hence there exists a central vertex \( w \) (say) of \( T_3 \) that is at distance at most \( n - 1 \) from \( u_{n-1} \) or \( u_{d-n+1} \), and from each vertex of \( T_3 \). Thus \( D = \{ u_{n-1}, u_{d-n+1}, w \} \) is a total \( P_{\leq n} \)-dominating set (and so certainly a \( P_{\leq n} \)-dominating set) of \( T \); so \( \gamma_n(T) + \gamma^t_n(T) \leq 6 \leq 2p(T)/n \).

If \( p(T_3) \geq 2n \), then it follows from the induction hypothesis that \( T_3 \) has a \( P_{\leq n} \)-dominating set \( D' \) and a total \( P_{\leq n} \)-dominating set \( D'' \) with \( |D'| + |D''| = \gamma_n(T_3) + \gamma^t_n(T_3) \leq 2p(T_3)/n \). So \( D_1 = D' \cup \{ u_{n-1}, u_{d-n+1} \} \) is a \( P_{\leq n} \)-dominating set of \( T \) and \( D_2 = D'' \cup \{ u_{n-1}, u_{d-n+1} \} \) is a total \( P_{\leq n} \)-dominating set of \( T \) with \( \gamma_n(T) + \gamma^t_n(T) \leq |D_1| + |D_2| + 4 \leq 2p(T_3)/n + 2(p(T_1) + p(T_2))/n = 2p(T)/n \). This completes the proof of Lemma 2. \( \square \)

**Lemma 3.** If \( 3n - 2 \leq d \leq 4n - 3 \), then \( \gamma_n(T) + \gamma^t_n(T) \leq 2p(T)/n \).

**Proof.** Necessarily there exists an integer \( i, 1 \leq i \leq d - 1 \), such that the components of \( T - u_{i-1}u_i \) and \( T - u_iu_{i+1} \) containing \( u \) are, respectively, of order less than \( 2n \) and of order at least \( 2n \). From the assumption
that, for every edge \( e \) of \( T \), \( T - e \) contains a component of order at most \( 2n - 1 \), it follows that \( d - 2n + 1 \leq i \leq 2n - 1 \).

Let \( T'_1 \) and \( T'_2 \) be the components of \( T - u_i \) containing \( u \) and \( v \), respectively. We note that \( T'_1 \) and \( T'_2 \) are both of order less than \( 2n \).

Further, let \( \deg u_i = r \) and denote by \( T'_1, T'_2, \ldots, T'_r \) the components of \( T - u_i \) and by \( w_i \) the vertex in \( T'_i \) adjacent to \( u_i \) in \( T(i = 1, 2, \ldots, r) \). We note that \( w_1 = u_{i-1} \) and \( w_2 = u_{i+1} \). If \( r \geq 3 \), then for \( j \in \{3, \ldots, r\} \) we observe that, since one component of \( T - u_i w_j \) contains \( P \) and is therefore of order at least \( 2n \), the component \( T'_j \) is of order at most \( 2n - 1 \).

We consider two possibilities.

**Case 1:** Suppose that \( i = 2n - 1 \) or \( i = d - 2n + 1 \). Without loss of generality, we may assume (relabelling the path \( P \) by \( v = u_0, u_1, \ldots, u_d = u \) if necessary) that \( i = 2n - 1 \). Since \( p(T'_1) \leq 2n - 1 \), \( T'_1 \cong P_{2n-1} \) and \( \{u_{n-1}\} \) is a \( P_{\leq n} \)-dominating set of \( T'_1 \). We consider two possibilities.

**Case 1.1:** Suppose that \( d = 3n - 2 \). Then \( u_{2n-1} = u_{d-n+1} \) and every vertex of \( T'_2 \) is within distance \( n - 1 \) from \( u_{2n-1} \). Consequently, if \( r = 2 \), then \( \gamma_n(T) + \gamma_i(T) \leq |\{u_{n-1}, u_{2n-1}\}| + \{|u_{n-1}, u_{2n-2}, u_{2n-1}\}| = 5 \leq 2(3n - 1)/n \leq 2p(T)/n \). We now consider the case where \( r \geq 3 \).

Let \( \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \) where

\[
\begin{align*}
I_1 &= \{j \in I \mid p(T'_j) \leq n - 1\}, \\
I_2 &= \{j \in I \mid n \leq p(T'_j) \leq 2n - 2\}, \\
I_3 &= \{j \in I \mid p(T'_j) = 2n - 1\}.
\end{align*}
\]

If \( j \in I_1 \), then \( u_{2n-1} \) is within distance \( n - 1 \) from every vertex of \( T'_j \). If \( j \in I_2 \), then since \( p(\{V(T'_j) \cup \{u_{2n-1}\}\}) \leq 2n - 1 \), \( T'_j \) contains a vertex \( z_j \) such that \( \{z_j\} \) is a \( P_\leq n \)-dominating set of \( T'_j \) and \( d(u_{2n-1}, z_j) \leq n - 1 \). If \( j \in I_3 \), then \( d(T'_j) \leq n - 1 \). Let \( x_j \) be a central vertex of \( T'_j \). It follows, therefore, that

\[
\gamma_n(T) \leq |\{u_{n-1}, u_{2n-1}\}| + |\bigcup_{j \in I_2} \{z_j\}| + |\bigcup_{j \in I_3} \{x_j\}| = 2 + |I_2| + |I_3| \quad \text{and} \quad \gamma'_n(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_3} \{x_j, w_j\}| = 3 + |I_2| + 2|I_3|;
\]

so \( \gamma_n(T) + \gamma'_n(T) \leq 5 + 2|I_2| + 3|I_3| \).

However, \( p(T) \geq d + 1 + n|I_2| + (2n - 1)|I_3| = 3n - 1 + n|I_2| + (2n - 1)|I_3| \).

Hence \( 2p(T)/n \geq 6 - 2/n + 2|I_2| + (4 - 2/n)|I_3| \geq 5 + 2|I_2| + 3|I_3| \geq \gamma_n(T) + \gamma'_n(T) \).

**Case 1.2:** Suppose that \( 3n - 1 \leq d \leq 4n - 3 \). Then \( d - n + 1 > 2n - 1 \) and so \( u_{d-n+1} \in V(T'_2) \). Further, since \( p(T'_2) \leq 2n - 1 \),
\{u_{d-n+1}\} is a \(P_{\leq n}\)-dominating set of \(T'_2\). Since \(d \leq 4n - 3\), we observe that \(d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2 \leq n - 1\).

If \(r = 2\), then

\[
\gamma_n(T) + \gamma_n^t(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| = 6 \leq 2(3n)/n \leq 2p(T)/n.
\]

If \(r \geq 3\), then let \(I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4\) where

\[
I_1 = \{j \in I \mid p(T'_j) \leq 4n - d - 3\},
I_2 = \{j \in I \mid 4n - d - 2 \leq p(T'_j) \leq n - 1\},
I_3 = \{j \in I \mid n \leq p(T'_j) \leq 2n - 2\},
I_4 = \{j \in I \mid p(T'_j) = 2n - 1\}.
\]

If \(j \in I_1\), then, since \(d(u_{d-n+1}, u_{2n-1}) = d - 3n + 2\), it follows that \(u_{d-n+1}\) is within distance \(n - 1\) from every vertex of \(T'_j\). If \(j \in I_2\), then \(u_{2n-1}\) is within distance \(n - 1\) from every vertex of \(T'_j\). If \(j \in I_3\), then \(T'_j\) contains a vertex \(z_j\) such that \(\{z_j\}\) is a \(P_{\leq n}\)-dominating set of \(T'_j\) and \(d(u_{2n-1}, z_j) \leq n - 1\). If \(j \in I_4\), then rad \(T'_j \leq n - 1\). Let \(x_j\) be a central vertex of \(T'_j\). We now consider two possibilities.

**Case 1.2.1:** Suppose that \(|I_2| \geq 1\). Then it follows that \(\gamma_n(T) \leq |\{u_{n-1}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j, w_j\}| = 3 + |I_3| + |I_4|\) and

\[
\gamma_n^t(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{d-n+1}\}| + |\bigcup_{j \in I_3} \{z_j\}| + |\bigcup_{j \in I_4} \{x_j, w_j\}| = 4 + |I_3| + 2|I_4|;
\]

so \(\gamma_n(T) + \gamma_n^t(T) \leq 7 + 2|I_3| + 3|I_4|\). However, \(p(T) \geq (d+1) + (4n-d-2)|I_2| + n|I_3| + (2n-1)|I_4| \geq 4n - 1 + n|I_3| + (2n - 1)|I_4|\). Hence \(2p(T)/n \geq 8 - 2/n + 2|I_3| + (4 - 2/n)|I_4| \geq 7 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)\).

**Case 1.2.2:** Suppose that \(|I_2| = 0\). Then it follows that \(\gamma_n(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |I_3| + |I_4| = 2 + |I_3| + |I_4|\) and \(\gamma_n^t(T) \leq 4 + |I_3| + 2|I_4|\); so \(\gamma_n(T) + \gamma_n^t(T) \leq 6 + 2|I_3| + 3|I_4|\). However, \(p(T) \geq d + 1 + n|I_3| + (2n - 1)|I_4| \geq 3n + n|I_3| + (2n - 1)|I_4|\). Hence \(2p(T)/n \geq 6 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T)\).

**Case 2:** Suppose that \(d - 2n + 2 \leq i \leq 2n - 2\). Then, since \(d \geq 3n - 2\), \(n \leq d - 2n + 2 \leq i \leq 2n - 2 \leq d - n\). Hence \(u_{n-1}(u_{d-n+1})\) is a vertex of \(T'_1\) (\(T'_2\), respectively). In fact, as \(P\) is a longest path in \(T\) and as \(p(T'_i) \leq 2n - 1\) (\(1 \leq i \leq 2\)), \(\{u_{n-1}\}\) (\(\{u_{d-n+1}\}\)) is a \(P_{\leq n}\)-dominating set of \(T'_1\) (\(T'_2\), respectively). Furthermore, since \(i \leq 2n - 2\), \(d(u_{n-1}, u_i) = i - n + 1 \leq n - 1\) and since \(i \geq d - 2n + 2\), \(d(u_{d-n+1}, u_i) = d - n + 1 - i \leq n - 1\). Consequently, if \(r = 2\),
then \( \gamma_n(T) + \gamma_n^t(T) \leq |\{u_{n-1}, u_{d-n+1}\}| + |\{u_{n-1}, u_i, u_{d-n+1}\}| = 5 \leq 2(3n-1)/n \leq 2p(T)/n. \)

If \( r \geq 3 \), then let \( I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4 \), where

\[
I_1 = \{j \in I \mid p(T'_j) < \max(2n - i - 1, 2n + i - d - 1)\},
\]

\[
I_2 = \{j \in I \mid \max(2n - i - 1, 2n + i - d - 1) \leq p(T'_j) \leq n - 1\},
\]

\[
I_3 = \{j \in I \mid n \leq p(T'_j) \leq 2n - 2\},
\]

\[
I_4 = \{j \in I \mid p(T'_j) = 2n - 1\}.
\]

If \( j \in I_1 \), then \( p(T'_j) \leq 2n - i - 2 \) or \( p(T'_j) \leq 2n + i - d - 2 \). If \( p(T'_j) \leq 2n - i - 2 \), then since \( d(u_{n-1}, u_i) = i - n + 1 \), it follows that \( u_{n-1} \) is within distance \( n - 1 \) from every vertex of \( T'_j \). If \( p(T'_j) \leq 2n + i - d - 2 \), then, since \( d(u_{d-n+1}, u_i) = d - n + 1 - i \), it follows that \( u_{d-n+1} \) is within distance \( n - 1 \) from every vertex of \( T'_j \). If \( j \in I_2 \), then \( u_i \) is within distance \( n - 1 \) from every vertex of \( T'_j \). If \( j \in I_3 \), then \( T'_j \) contains a vertex \( z_j \) such that \( \{z_j\} \) is a \( P_{\leq n} \)-dominating set of \( T'_j \) and \( d(u_i, z_j) \leq n - 1 \). If \( j \in I_4 \), then \( \text{rad}(T'_j) \leq n - 1 \). Let \( x_j \) be a central vertex of \( T'_j \).

**Case 2.1:** Suppose that \( |I_2| \geq 1 \). Then it follows that \( \gamma_n(T) \leq \sum_{i \in I_3} |\{u_{n-1}, u_i, u_{d-n+1}\}| + \sum_{j \in I_3} |\{z_j\}| + \sum_{j \in I_4} |\{x_j, w_j\}| = 3 + |I_3| + |I_4| \) and \( \gamma_n^t(T) \leq |\{u_{n-1}, u_i, u_{d-n+1}\}| + \sum_{j \in I_3} |\{z_j\}| + \sum_{j \in I_4} |\{x_j, w_j\}| = 3 + |I_3| + 2|I_4| \); so \( \gamma_n(T) + \gamma_n^t(T) \leq 6 + 2|I_3| + 3|I_4| \).

If \( \max(2n - i - 1, 2n + i - d - 1) = 2n - i - 1 \), then \( p(T) \geq (d + 1) + (2n - i - 1)|I_2| + n|I_3| + (2n - 1)|I_4| \geq 2n + d - i + n|I_3| + (2n - 1)|I_4| \geq 3n + n|I_3| + (2n - 1)|I_4| \), since \( d - i \geq n \). Hence \( \frac{2p(T)}{n} \leq 6 + 2|I_3| + (4 - 2/n)|I_4| \geq 2n + 3|I_3| + 3|I_4| \). Hence \( \gamma_n(T) + \gamma_n^t(T) \geq \gamma_n(T) + \gamma_n^t(T) \).

If \( \max(2n - i - 1, 2n + i - d - 1) = 2n + i - d - 1 \), then \( p(T) \geq \frac{2p(T)}{n} \leq 6 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma_n^t(T) \).

**Case 2.2:** Suppose that \( |I_2| = 0 \). Then it follows that \( \gamma_n(T) \leq \sum_{i \in I_3} |\{u_{n-1}, u_i, u_{d-n+1}\}| + |I_3| + |I_4| \geq 2 + |I_3| + |I_4| \) and \( \gamma_n^t(T) \leq 3 + |I_3| + 2|I_4| \); so \( \gamma_n(T) + \gamma_n^t(T) \leq 5 + 2|I_3| + 3|I_4| \). However, \( p(T) \geq d + 1 + n|I_3| + (2n - 1)|I_4| \geq 3n - 1 + n|I_3| + (2n - 1)|I_4| \). Hence \( \frac{2p(T)}{n} \geq 6 - 2/n + 2|I_3| + (4 - 2/n)|I_4| \geq 5 + 2|I_3| + 3|I_4| \). This completes the proof of Lemma 3. \( \diamond \)

**Lemma 4.** If \( d = 4n - 2 \), then \( \gamma_n(T) + \gamma_n^t(T) \leq 2p(T)/n. \)
Proof. Suppose that $d = 4n - 2$. Then, using the notation introduced in the first two paragraphs of the proof of Lemma 3, it follows that $i = 2n - 1$. Furthermore, since $p(T'_i) \leq 2n - 1$, we therefore have $T'_i \cong P_{2n-1}$ $(1 \leq i \leq 2)$ and so $\{u_{n-1}\} \{u_{3n-1}\}$ is a $P_{\leq n}$-dominating set of $T'_1$ $(T'_2$, respectively). We observe, however, that $u_{2n-1}$ is at distance $n$ from both $u_{n-1}$ and $u_{3n-1}$. Consequently, if $r = 2$, then $\gamma_n(T) + \gamma^t_n(T) = |\{u_{n-1}, u_{2n-1}, u_{3n-1}\}| + |\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}| = 7 \leq 2(4n - 1)/n = 2p(T)/n$.

If $r \geq 3$, then let $I = \{3, \ldots, r\} = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{j \in I \mid p(T'_i) \leq n - 2\},$$
$$I_2 = \{j \in I \mid p(T'_i) = n - 1\},$$
$$I_3 = \{j \in I \mid n \leq p(T'_i) \leq 2n - 2\},$$
$$I_4 = \{j \in I \mid p(T'_i) = 2n - 1\}.$$

If $j \in I_1$, then every vertex of $T'_j$ is within distance $n - 1$ from the vertices $u_{2n-2}, u_{2n-1}$ and $u_{2n}$. If $j \in I_2$, then $u_{2n-1}$ is within distance $n - 1$ from every vertex of $T'_j$. If $j \in I_3$, then $T'_j$ contains a vertex $z_j$ such that $\{z_j\}$ is a $P_{\leq n}$-dominating set of $T'_j$ and $d(z_j, u_{2n-1}) \leq n - 1$. If $j \in I_4$, then $\text{rad} T'_j \leq n - 1$. Let $x_j$ be a central vertex of $T'_j$. We now consider two possibilities.

Case 1: Suppose that $|I_2| \geq 1$. Then it follows that $\gamma_n(T) \leq |\{u_{n-1}, u_{2n-1}, u_{3n-1}\}| + \bigcup_{j \in I_3} \{z_j\} + \bigcup_{j \in I_4} \{x_j\} = 3 + |I_3| + |I_4|$ and $\gamma^t_n(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}| + \bigcup_{j \in I_3} \{z_j\} + \bigcup_{j \in I_4} \{w_j, x_j\}$.

so $\gamma_n(T) + \gamma^t_n(T) \leq 8 + 2|I_3| + 3|I_4|$. However, $p(T) \geq 4n - 1 + (n - 1)|I_2| + n|I_3| + (2n - 1)|I_4| \geq 5n - 2 + n|I_3| + (2n - 1)|I_4|$. Hence $2p(T)/n \geq 10 - 2n + 2|I_3| + (4 - 2n) > 8 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma^t_n(T)$.

Case 2: Suppose that $|I_2| = 0$. Then, if $|I_3| \geq 1$, it follows that $\gamma_n(T) \leq |\{u_{n-1}, u_{3n-1}\}| + \bigcup_{j \in I_4} \{z_j\} + \bigcup_{j \in I_4} \{x_j\} = 2 + |I_3| + |I_4|$ and $\gamma^t_n(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n-1}, u_{2n}, u_{3n-1}\}| + |I_3| + 2|I_4| \leq 5 + |I_3| + 2|I_4|$, so $\gamma_n(T) + \gamma^t_n(T) \leq 7 + 2|I_3| + 3|I_4|$. However, $p(T) \geq 4n - 1 + n|I_3| + (2n - 1)|I_4|$. Hence $2p(T)/n \geq 8 - 2n + 2|I_3| + (4 - 2n)|I_4| \geq 7 + 2|I_3| + 3|I_4| \geq \gamma_n(T) + \gamma^t_n(T)$.

If $|I_3| = 0$, then it follows that $\gamma_n(T) \leq |\{u_{n-1}, u_{2n-1}, u_{3n-1}\}| + |\bigcup_{j \in I_4} \{x_j\}| = 3 + |I_4|$ and $\gamma^t_n(T) \leq |\{u_{n-1}, u_{2n-2}, u_{2n}, u_{3n-1}\}| + 2|I_4| = 4 + 2|I_4|$, so $\gamma_n(T) + \gamma^t_n(T) \leq 7 + 3|I_4|$. However, $p(T) \geq 4n - 1 +
+ (2n - 1)|I_4|. Hence 2p(T)/n \geq 8 - 2/n + (4 - 2/n)|I_4| \geq 7 + 3|I_4| \geq \gamma_n(T) + \gamma^I_n(T).

This completes the proof of Lemma 4 and thus of Th. 1. ◇

That the bound in Th. 1 is best possible may be seen as follows: Let G be obtained from a connected graph H by attaching a path of length n - 1 to each vertex of H. (The graph G is shown in Fig. 1.) Then \gamma_n(G) + \gamma^I_n(G) = 2p(H) = 2p(G)/n.

![Diagram](image.png)

Fig. 1.

The fact that every maximal independent set of vertices in a graph is also a dominating set motivated Cockayne and Hedetniemi [3] in 1974 to initiate the study of another domination parameter. A dominating set of vertices in a graph that is also an independent set is called an independent dominating set. The minimum cardinality among all independent dominating sets of a graph G is called the independent domination number of G and is denoted by i(G).

The independent domination number of a graph and the distance domination parameters introduced earlier suggest yet another distance domination parameter. A set I of vertices in a graph G is P_n-independent in G if every two vertices of I are at distance at least n apart in G. A P_n-independent set of vertices in a graph that is also a P_n-dominating set is called a P_n-independent dominating set. The minimum cardinality among all P_n-independent dominating sets of a graph G is called the P_n-independent domination number of G and is denoted by i_n(G). Hence i_2(G) = i(G).

Before investigating relationships between the distance domination parameter i_n and the distance domination parameters \gamma_n and \gamma^I_n we need some additional concepts. A set of vertices X \subset V(G) has property \pi_n(n \geq 2) if and only if every nontrivial path of length \ell \leq n - 1 in G contains at least \ell vertices of X. A set of vertices with
property \( \pi_n \) is called a \( P_{\leq n} \)-cover of \( G \). So a \( P_{\leq 2} \)-cover of \( G \) is simply a cover of \( G \). The minimum cardinality among all \( P_{\leq n} \)-covers of \( G \) is called the \( P_{\leq n} \)-covering number of \( G \) and is denoted by \( \alpha_n(G) \). The maximum cardinality among all \( P_{\leq n} \)-independent sets is called the \( P_{\leq n} \)-independence number of \( G \) and is denoted by \( \beta_n(G) \). Hence \( \alpha_2(G) \) is simply the covering number \( \alpha(G) \) and \( \beta_2(G) \) is the independence number \( \beta(G) \). The next Gallai-type result generalizes a well-known relationship between the covering number and independence number of a graph [4].

**Theorem 2.** If \( G \) is a connected graph of order \( p \geq n \), then

\[
\alpha_n(G) + \beta_n(G) = p.
\]

**Proof.** We note that \( X \) is a \( P_{\leq n} \)-cover if and only if \( V(G) \setminus X \) is a \( P_{\leq n} \)-independent set of vertices. So if \( X \) is a \( P_{\leq n} \)-cover of cardinality \( \alpha_n(G) \), then \( \alpha_n(G) = |X| \) and \( |V(G) \setminus X| = p - \alpha_n(G) \leq \beta_n(G) \). Similarly if \( Y \) is a \( P_{\leq n} \)-independent set of vertices of cardinality \( \beta_n(G) \), \( p - \beta_n(G) = |V(G) \setminus Y| \geq \alpha_n(G) \). Thus \( \alpha_n(G) + \beta_n(G) = p \).

Allan, Laskar and Hedetniemi [1] showed that if \( G \) is a graph of order \( p \) that has no isolated vertices, then \( \gamma(G) + i(G) \leq p \). We now present a generalization of this result.

**Theorem 3.** If \( G \) is a connected graph of order \( p \geq n \geq 2 \), then

\[
i_n(G) + (n - 1)\gamma_n(G) \leq p.
\]

**Proof.** Let \( X \) be a \( P_{\leq n} \)-cover such that \( \langle X \rangle \) contains as few components as possible of order less than \( n - 1 \). We show that \( \langle X \rangle \) has no components of order less than \( n - 1 \). Suppose \( \langle X \rangle \) has a component \( G_1 \) of order \( p_1 \leq n - 2 \). Since \( G \) is connected, and \( p \geq n \), there is a vertex \( s \in S = V(G) \setminus X \) that is adjacent to a vertex \( y \) in \( G_1 \) and a vertex \( z \) in \( V(G) \setminus V(G_1) \). Since \( S \) is \( P_{\leq n} \)-independent, \( z \) must belong to some component \( G_2 \neq G_1 \) of \( \langle X \rangle \). Note that \( s \) is the only vertex of \( S \) which is adjacent to a vertex (or vertices) in \( G_1 \), for if \( t \) is any other vertex of \( S \) that is adjacent to a vertex of \( G \), \( d(t, s) \leq n - 1 \), which is not possible since \( S \) is \( P_{\leq n} \)-independent.

Now if \( p(G_1) = 1 \), let \( S' = (S \setminus \{s\}) \cup \{y\} \). Otherwise if \( p(G_1) \geq 2 \), let \( x \neq y \) be a vertex of \( G_1 \) which is not a cut-vertex of \( G_1 \) and set \( S' = (S \setminus \{s\}) \cup \{x\} \). Then \( S' \) is a \( P_{\leq n} \)-independent set of cardinality \( |V(G) \setminus X| \). Since \( X \) is a \( P_{\leq n} \)-cover of \( \alpha_n(G) \), it follows from Th. 2, that \( |V(G) \setminus X| = p - \alpha_n(G) = \beta_n(G) \), i.e., \( |S'| = \beta_n(G) \). However, then \( X' = V(G) \setminus S' \) is a \( P_{\leq n} \)-cover of \( G \) of cardinality \( \alpha_n(G) \).
such that \( (X') \) contains fewer components of order less than \( n - 1 \) than \( (X) \). This contradicts our choice of \( X \). Hence \( (X) \) has no components of order less than \( n - 1 \).

Since \( G \) is connected, every vertex in \( V(G) - X \) is adjacent with a vertex in \( X \) and, consequently

\[
\gamma_n(G) \leq \gamma_{n-1}(\langle X \rangle).
\]

Since \( (X) \) has no component of order smaller than \( n - 1 \), it follows from Th. A that

\[
\gamma_n(G) \leq \frac{p((X))}{n-1} = \frac{|X|}{n-1} = \frac{\alpha_n(G)}{n-1}.
\]

The fact that \( \beta_n(G) = |V(G) - X| \geq i_n(G) \) and Th. 2 now imply that

\[
i_n(G) + (n - 1)\gamma_n(G) \leq \alpha_n(G) + \beta_n(G) = p. \quad \square
\]

The bound given in Th. 3 is best possible as we now see. Let \( G \) be the graph shown in Fig. 1. Then \( i_n(G) = \gamma_n(G) = p(H) \) and \( i_n(G) + (n - 1)\gamma_n(G) = np(H) = p(G) \). It is shown in [6] that if \( T \) is a tree of order \( p \geq 2n - 1 \), then \( i_n(T) + (n - 1)\gamma_n(T) \leq p \).

References


