A SANDWICH WITH CONVEXITY

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Abstract: We prove that real functions $f$ and $g$, defined on a real interval $I$, satisfy

$$f(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)$$

for all $x, y \in I$ and $t \in [0, 1]$ iff there exists a convex function $h : I \rightarrow \mathbb{R}$ such that $f \leq h \leq g$. Using this sandwich theorem we characterize solutions of two functional inequalities connected with convex functions and we obtain also the classical one-dimensional Hyers-Ulam Theorem on approximately convex functions.

Introduction

It is the aim of this note to characterize real functions which can be separated by a convex function. This leads us to functional inequality
(1) \[ f(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y). \]

Using this characterization we describe also solutions of the inequalities

(2) \[ f(tx + (T - t)y) \leq tf(x) + (T - t)f(y) \]

and

(3) \[ f(tx + (T - t)y + (1 - T)z_0) \leq tf(x) + (T - t)f(y) + (1 - T)f(z_0). \]

Functions fulfilling (2) appear in a connection with the converse of Minkowski’s inequality in the case where the measure of the space considered is less than 1 (see [4; pp. 671–672] and [5; Remark 16]).

1. A sandwich theorem

Our main result reads as follows.

**Theorem 1.** Real functions \( f \) and \( g \), defined on a real interval \( I \), satisfy

(1) for all \( x, y \in I \) and \( t \in [0, 1] \) iff there exists a convex function \( h : I \to \mathbb{R} \) such that

(4) \[ f \leq h \leq g. \]

**Proof.** We argue as in [1; proof of Th. 2]. Assume that functions \( f, g : I \to \mathbb{R} \) satisfy (1) and denote by \( E \) the convex hull of the epigraph of \( g \):

\[ E = \text{conv} \{ (x, y) \in I \times \mathbb{R} : g(x) \leq y \}. \]

Let \((x, y) \in E\). It follows from the Carathéodory Theorem (see [3; Cor. 17.4.2] or [6; Th. 31E] or [7; the lemma on p. 88]) that \((x, y)\) belongs to a two-dimensional simplex \( S \) with vertices in the epigraph of \( g \). Denote \( y_0 = \inf \{ z \in \mathbb{R} : (x, z) \in S \} \).

Then \( y \geq y_0 \) and \((x, y_0)\) belongs to the boundary of \( S \). Consequently \((x, y_0) = t(x_1, y_1) + (1 - t)(x_2, y_2)\) with some \( t \in [0, 1] \) and \((x_1, y_1), (x_2, y_2) \in I \times \mathbb{R} \) such that \( g(x_1) \leq y_1 \) and \( g(x_2) \leq y_2 \). Hence, using also (1), we get

\[ y \geq y_0 = ty_1 + (1 - t)y_2 \geq tg(x_1) + (1 - t)g(x_2) \geq f(tx_1 + (1 - t)x_2) = f(x). \]

This allows us to define a function \( h : I \to \mathbb{R} \) by the formula

\[ h(x) = \inf \{ y \in \mathbb{R} : (x, y) \in E \} \]

and gives \( f \leq h \). Moreover, since \((x, g(x)) \in E\) for every \( x \in I \), we have also \( h \leq g \). It remains to show that \( h \) is convex. To this end fix arbitrarily \( x_1, x_2 \in I \) and \( t \in [0, 1] \). Then, for any reals \( y_1, y_2 \) such that
(x_1, y_1), (x_2, y_2) \in E we have (tx_1 + (1 - t)x_2, ty_1 + (1 - t)y_2) \in E, whence h(tx_1 + (1 - t)x_2) \leq ty_1 + (1 - t)y_2. Passing to infimum we obtain the desired inequality: h(tx_1 + (1 - t)x_2) \leq th(x_1) + (1 - t)h(x_2). This ends the proof (of the "only if" part but the "if" part is obvious). ⊢

The following example shows that Th. 1 cannot be generalized for functions defined on a convex subset of the (complex) plane.

**Example 1.** Let \( D \subseteq \mathbb{C} \) be the open ball centered at zero and with the radius 2, and let \( z_1, z_2, z_3 \) be the (different) third roots of the unity. Define the functions \( f \) and \( g \) on \( D \) by the formulas

\[
 f(z) = \begin{cases} 
 0 & \text{if } z \neq 0 \\
 1 & \text{if } z = 0 
\end{cases} \quad g(z) = \begin{cases} 
 0 & \text{if } z \in \{z_1, z_2, z_3\} \\
 3 & \text{if } z \in D \setminus \{z_1, z_2, z_3\}. 
\end{cases}
\]

It is easy to check that (1) holds for all \( x, y \in D \) and \( t \in [0, 1] \). Suppose that there exists a convex function \( h : D \to \mathbb{R} \) satisfying (4). Then

\[
 1 = f(0) = f \left( \frac{1}{3}(z_1 + z_2 + z_3) \right) \leq h \left( \frac{1}{3}(z_1 + z_2 + z_3) \right) \leq \frac{1}{3}(h(z_1) + h(z_2) + h(z_3)) \leq \frac{1}{3}(g(z_1) + g(z_2) + g(z_3)) = 0,
\]

a contradiction.

Arguing as in the proof of Th. 1 we can get however the following results.

**Theorem 1a.** Real functions \( f \) and \( g \), defined on a convex subset \( D \) of an \((n - 1)\)-dimensional real vector space, satisfy

\[
 f \left( \sum_{j=1}^{n} t_j x_j \right) \leq \sum_{j=1}^{n} t_j g(x_j)
\]

for all vectors \( x_1, \ldots, x_n \in D \) and reals \( t_1, \ldots, t_n \in [0, 1] \) summing up to 1 iff there exists a convex function \( h : D \to \mathbb{R} \) satisfying (4).

**Theorem 1b.** Real functions \( f \) and \( g \), defined on a convex subset \( D \) of a vector space, satisfy (5) for each positive integer \( n \), vectors \( x_1, \ldots, x_n \in D \) and reals \( t_1, \ldots, t_n \in [0, 1] \) summing up to 1 iff there exists a convex function \( h : D \to \mathbb{R} \) satisfying (4).

### 2. Applications

We start with an application of Th. 1 connected with approximately convex functions.
If \( \varepsilon \) is a positive real number and a real function \( f \), defined on a real interval \( I \), satisfies
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon
\]
for all \( x, y \in I \) and \( t \in [0, 1] \), then (1) holds with \( g = f + \varepsilon \) and it follows from Th. 1 that there exists a convex function \( h : I \to \mathbb{R} \) such that
\[
f(x) \leq h(x) \leq f(x) + \varepsilon \quad \text{for} \quad x \in I.
\]
Putting \( \varphi(x) = h(x) - \varepsilon/2 \) we obtain a convex function \( \varphi : I \to \mathbb{R} \) such that
\[
|\varphi(x) - f(x)| \leq \varepsilon/2 \quad \text{for} \quad x \in I.
\]
This is the classical one-dimensional Hyers-Ulam Stability Theorem (see [2; Th. 2]; cf. also [1; Th. 2] and [3; Th. 17.4.2]).

Further applications of our Th. 1 concern solutions of the inequalities (2) and (3). Denote by \( J \) either \([0, +\infty)\) or \((0, +\infty)\). Given \( T > 0 \) and \( f : J \to \mathbb{R} \) we define the function \( f_T : J \to \mathbb{R} \) by the formula
\[
f_T(x) = T^{-1}f(Tx).
\]

**Theorem 2.** Let \( T \) be a positive real number. A function \( f : J \to \mathbb{R} \) satisfies (2) for all \( x, y \in J \) and \( t \in [0, T] \) iff there exists a convex function \( \varphi : J \to \mathbb{R} \) such that
\[
\varphi_T \leq f \leq \varphi.
\]

**Proof.** Assume that \( f : J \to \mathbb{R} \) satisfies (2). Putting \( T \cdot t \) in place of \( t \) in (2) we have
\[
f_T(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]
for all \( x, y \in J \) and \( t \in [0, 1] \). Applying Th. 1 we obtain a convex function \( h : J \to \mathbb{R} \) such that
\[
f_T \leq h \leq f.
\]
Define now \( \varphi : J \to \mathbb{R} \) by the formula
\[
\varphi(x) = Th(T^{-1}x).
\]
Then \( \varphi \) is convex and (6) holds.

Conversely, if (6) holds with a convex function \( \varphi : J \to \mathbb{R} \) then (9) defines a convex function \( h : J \to \mathbb{R} \) which satisfies (8) whence (7) follows for all \( x, y \in J \) and \( t \in [0, 1] \). But this means that (2) holds for all \( x, y \in J \) and \( t \in [0, T] \). \( \diamond \)

**Example 2.** If \( T \in (0, 1) \), then taking \( \varphi(x) = x^2 \) for \( x \in [0, +\infty) \) we get by Th. 2 that every function \( f : [0, +\infty) \to \mathbb{R} \) satisfying
\[
Tx^2 \leq f(x) \leq x^2 \quad \text{for} \quad x \in [0, +\infty)
\]
is a solution of (2). Similarly, if \( T \in (1, +\infty) \), then taking \( \varphi(x) = 1/x \) for \( x \in (0, +\infty) \) we see that every function \( f : (0, +\infty) \to \mathbb{R} \) such that
\[\frac{1}{(T^2x)} \leq f(x) \leq \frac{1}{x} \quad \text{for} \quad x \in (0, +\infty)\]
satisfies (2).

Now we pass to inequality (3). Fix a real interval \(I\) and a point \(z_0 \in I\). For \(T \in (0, 1)\) put
\[I_T^* = TI + (1 - T)z_0.\]

Given a real function \(\varphi\) with the domain containing \(I_T^*\), we define \(\varphi_T^* : I \to \mathbb{R}\) by the formula
\[\varphi_T^*(x) = T^{-1}(\varphi(Tx + (1 - T)z_0) - (1 - T)\varphi(z_0)).\]

**Theorem 3.** Let \(T \in (0, 1)\). A function \(f : I \to \mathbb{R}\) satisfies (3) for all \(x, y \in I\) and \(t \in [0, T]\) iff there exists a convex function \(\varphi : I_T^* \to \mathbb{R}\) such that
\[\varphi_T^*(x) \leq f(x) \quad \text{for} \quad x \in I \quad \text{and} \quad f(x) \leq \varphi(x) \quad \text{for} \quad x \in I_T^*.\]

**Proof.** Assume that \(f\) satisfies (3). Putting \(T \cdot t\) in place of \(t\) in (3) we have
\[f_T^*(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)\]
for all \(x, y \in I\) and \(t \in [0, 1]\). Applying Th. 1 we obtain a convex function \(h : I \to \mathbb{R}\) such that
\[f_T^* \leq h \leq f.\]

Since \(f_T^*(z_0) = f(z_0)\), we have \(h(z_0) = f(z_0)\). Define \(\varphi : I_T^* \to \mathbb{R}\) by the formula
\[\varphi(x) = Th(T^{-1}(x - (1 - T)z_0)) + (1 - T)f(z_0).\]

Then \(\varphi\) is a convex function, \(\varphi(z_0) = f(z_0)\),
\[\varphi_T^*(x) = h(x) \leq f(x) \quad \text{for} \quad x \in I\]
and
\[\varphi(x) \geq Tf_T^*(T^{-1}(x - (1 - T)z_0)) + (1 - T)f(z_0) = f(x) \quad \text{for} \quad x \in I_T^*.\]

Conversely, if (10) holds with a convex function \(\varphi : I_T^* \to \mathbb{R}\) then \(f(z_0) = \varphi(z_0)\) and (13) defines a convex function \(h : I \to \mathbb{R}\) which satisfies (12). This implies (11) for all \(x, y \in I\) and \(t \in [0, 1]\). Consequently \(f\) satisfies (3) for all \(x, y \in I\) and \(t \in [0, T]\). \(\diamondsuit\)

**References**


