MEROMORPHIC STARLIKE UNIVALENT FUNCTIONS WITH ALTERNATING COEFFICIENTS

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Abstract: Coefficient estimates and distortion theorems are obtained for meromorphic starlike univalent functions with alternating coefficients. Further class preserving integral operators are obtained.

1. Introduction

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

which are regular in the punctured disc $U^* = \{ z : 0 < |z| < 1 \}$. Define

$D^0 f(z) = f(z)$,

$$D^1 f(z) = \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \frac{(z^2 f(z))'}{z}.$$ 

$$D^2 f(z) = D(D^1 f(z)).$$
and for $n = 1, 2, 3, \ldots$

\[ D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m + 2)^n a_m z^m = \frac{(zD^{n-1} f(z))'}{z}. \]

In [4] Uralegaddi and Somanatha obtained a new criteria for meromorphic starlike univalent functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha), \quad 0 \leq \alpha < 1, \quad n \in \mathbb{N}_0 = \{0, 1, \ldots\}$, where $B_n(\alpha)$ is the class consisting of functions in $\sum$ satisfying

\[ \text{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\} < -\alpha, \quad |z| < 1, \quad 0 \leq \alpha < 1, \quad n \in \mathbb{N}_0. \]

The condition (1.2) is equivalent to

\[ D^{n+1} f(z) = \frac{1 + (3 - 2\alpha)w(z)}{1 + w(z)}, \]

\[ w(z) \in H = \{ w \text{ regular, } w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in U = \{ z : |z| < 1 \} \}, \]

or, equivalently,

\[ \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| < 2\alpha - 3. \]

We note that $B_0(\alpha) = \Sigma^*(\alpha)$, is the class of meromorphically starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) and $B_0(0) = \Sigma^*$, is the class of meromorphically starlike functions.

Let $\sigma_A$ be the subclass of $\Sigma$ which consists of functions of the form

\[ f(z) = \frac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 \ldots = \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^{m-1} a_m z^m, \quad a_m \geq 0 \]

and let $\sigma_{A,n}^*(\alpha) = B_n(\alpha) \cap \sigma_A$.

In this paper coefficient inequalities, distortion theorems for the class $\sigma_{A,n}^*(\alpha)$ are determined. Techniques used are similar to these of Silverman [2] and Uralegaddi and Ganigi [3]. Finally, the class preserving integral operators of the form
\[ F(z) = \frac{c}{z^{c+1}} \int_{0}^{z} t^{c} f(t) \, dt \quad (c > 0) \]

is considered.

2. Coefficient inequalities

**Theorem 1.** Let \( f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_{m} z^{m} \). If

\[ \sum_{m=1}^{\infty} (m + 2)^{n} (m + \alpha) |a_{m}| \leq (1 - \alpha), \]

then \( f(z) \in B_{n}(\alpha) \).

**Proof.** Suppose (2.1) holds for all admissible values of \( \alpha \) and \( n \). It suffices to show that

\[ \left| \frac{D^{n+1} f(z)}{D^{n} f(z)} - 1 \right| \left| \frac{D^{n} f(z)}{D^{n+1} f(z)} + 2\alpha - 3 \right| < 1 \quad \text{for} \quad |z| < 1. \]

We have

\[ \left| \frac{D^{n+1} f(z)}{D^{n} f(z)} - 1 \right| = \left| \frac{\sum_{m=1}^{\infty} (m + 2)^{n} (m + 1) a_{m} z^{m+1}}{2(1 - \alpha) - \sum_{m=1}^{\infty} (m + 2)^{n} (m - 1 + 2\alpha) a_{m} z^{m+1}} \right| \]

\[ \leq \frac{\sum_{m=1}^{\infty} (m + 2)^{n} (m + 1) |a_{m}|}{2(1 - \alpha) - \sum_{m=1}^{\infty} (m + 2)^{n} (m - 1 + 2\alpha) |a_{m}|} . \]

The last expression is bounded above by 1, provided

\[ \sum_{m=1}^{\infty} (m + 2)^{n} (m + 1) |a_{m}| \leq 2(1 - \alpha) - \sum_{m=1}^{\infty} (m + 2)^{n} (m - 1 + 2\alpha) |a_{m}| \]

which is equivalent to (2.1), and this is true by hypothesis. \( \diamond \)
For functions in $\sigma_{A,n}^*(\alpha)$ the converse of the above theorem is also true.

**Theorem 2.** A function $f(z)$ in $\sigma_A$ is in $\sigma_{A,n}^*(\alpha)$ if and only if

\[
\sum_{m=1}^{\infty} (m+2)^n (m+\alpha)a_m \leq (1-\alpha).
\]

(2.2)

**Proof.** In view of Th. 1 it suffices to show the only if part. Suppose

\[
\text{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} - 2 \right\} = \text{Re} \left\{ \frac{-\frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} (m+1)^n m a_m z^m}{\frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} (m+2)^n a_m z^m} \right\} < -\alpha.
\]

(2.3)

Choose values of $z$ on the real axis so that $\left( \frac{D^{n+1}f(z)}{D^n f(z)} - 2 \right)$ is real. Upon clearing the denominator in (2.3) and letting $z \to -1$ through real values, we obtain

\[
1 - \sum_{m=1}^{\infty} (m+2)^n m a_m \geq \alpha \left( 1 + \sum_{m=1}^{\infty} (m+2)^n a_m \right)
\]

which is equivalent to (2.2). \(\diamondsuit\)

**Corollary 1.** Let the function $f(z)$ defined by (1.5) be in the class $\sigma_{A,n}^*(\alpha)$. Then

\[a_m \leq \frac{(1-\alpha)}{(m+2)^n (m+\alpha)} \quad (m \geq 1).
\]

Equality holds for the functions of the form

\[f_m(z) = \frac{1}{z} + (-1)^{m-1} \frac{(1-\alpha)}{(m+2)^n (m+\alpha)} z^m.
\]

3. **Distortion theorems**

**Theorem 3.** Let the function $f(z)$ defined by (1.5) be in the class $\sigma_{A,n}^*(\alpha)$. Then for $0 < |z| = r < 1$,

\[
\frac{1}{r} - \frac{1-\alpha}{3^n (1+\alpha)^r} \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{3^n (1+\alpha)^r}
\]

(3.1)
with equality for the function

\[(3.2) \quad f(z) = \frac{1}{z} + \frac{1 - \alpha}{3^n(1 + \alpha)}z \quad \text{at } z = r, ir.\]

**Proof.** Suppose \(f(z)\) is in \(\sigma^*_\alpha,n(\alpha)\). In view of Th. 2, we have

\[3^n(1 + \alpha) \sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty} (m + 2)^n(m + \alpha)a_m \leq (1 - \alpha)\]

which evidently yields

\[\sum_{m=1}^{\infty} a_m \leq \frac{1 - \alpha}{3^n(1 + \alpha)}.\]

Consequently, we obtain

\[|f(z)| \leq \frac{1}{r} + \sum_{m=1}^{\infty} a_m r^m \leq \frac{1}{r} + r \sum_{m=1}^{\infty} a_m \leq \frac{1}{r} + \frac{1 - \alpha}{3^n(1 + \alpha)r}.\]

Also

\[|f(z)| \geq \frac{1}{r} - \sum_{m=1}^{\infty} a_m r^m \geq \frac{1}{r} - r \sum_{m=1}^{\infty} a_m \geq \frac{1}{r} - \frac{1 - \alpha}{3^n(1 + \alpha)r}.\]

Hence the results (3.1) follow. \(\Diamond\)

**Theorem 4.** Let the function \(f(z)\) defined by (1.5) be in the class \(\sigma^*_\alpha,n(\alpha)\). Then for \(0 < |z| = r < 1\),

\[(3.3) \quad \frac{1}{r^2} - \frac{1 - \alpha}{3^n(1 + \alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1 - \alpha}{3^n(1 + \alpha)}.\]

The result is sharp, the extremal function being of the form (3.2).

**Proof.** From Th. 2, we have

\[3^n(1 + \alpha) \sum_{m=1}^{\infty} ma_m \leq \sum_{m=1}^{\infty} (m + 2)^n(m + \alpha)a_m \leq (1 - \alpha)\]

which evidently yields

\[\sum_{m=1}^{\infty} ma_m \leq \frac{1 - \alpha}{3^n(1 + \alpha)}.\]
Consequently, we obtain
\[ |f'(z)| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m \leq \frac{1}{r^2} + \frac{1 - \alpha}{3^n(1 + \alpha)}. \]

Also
\[ |f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m \geq \frac{1}{r^2} - \frac{1 - \alpha}{3^n(1 + \alpha)}. \]

This completes the proof. \( \diamondsuit \)

Putting \( n = 0 \) in Th. 4, we get

**Corollary 2.** Let the function \( f(z) \) defined by (1.5) be in the class \( \sigma_{A,0}^*(\alpha) = \sigma_{A}^*(\alpha) \). Then for \( 0 < |z| = r < 1, \)
\[ \frac{1}{r^2} - \frac{1 - \alpha}{1 + \alpha} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1 - \alpha}{1 + \alpha}. \]

The result is sharp.

We observe that our result in Cor. 2 improves the result of Urale-gaddi and Ganigi [3, Th. 3 (Equation 4)].

**4. Class preserving integral operators**

In this section we consider the class preserving integral operators of the form (1.6).

**Theorem 5.** Let the function \( f(z) \) be defined by (1.5) be in the class \( \sigma_{A,n}^*(\alpha) \). Then
\[ F(z) = cz^{-c-1} \int_0^z t^{c} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{c}{c+m+1} a_m z^m, \quad c > 0 \]
belongs to the class \( \sigma_{A,n}^*(\beta(\alpha, n, c)), \) where
\[ \beta(\alpha, n, c) = \frac{(1 + \alpha)(c + 2) - c(1 - \alpha)}{(1 + \alpha)(c + 2) + c(1 - \alpha)}. \]

The result is sharp for
\[ f(z) = \frac{1}{z} + \frac{1 - \alpha}{3^n(1 + \alpha)} z. \]

**Proof.** Suppose \( f(z) \in \sigma_{A,n}^*(\alpha) \), then

\[
\sum_{m=1}^{\infty} (m + 2)^n(m + \alpha)a_m \leq (1 - \alpha).
\]

In view of Th. 2 we shall find the largest value of \( \beta \) for which

\[
\sum_{m=1}^{\infty} \frac{(m + 2)^n(m + \beta)}{(1 - \beta)(c + m + 1)} \cdot \frac{c}{c + m + 1} a_m \leq 1.
\]

It suffices to find the range of values of \( \beta \) for which

\[
\frac{c(m + 2)^n(m + \beta)}{(1 - \beta)(c + m + 1)} \leq \frac{(m + 2)^n(m + \alpha)}{(1 - \alpha)} \quad \text{for each } m.
\]

Solving the above inequality for \( \beta \) we obtain

\[
\beta \leq \frac{(m + \alpha)(c + m + 1) - mc(1 - \alpha)}{(m + \alpha)(c + m + 1) + c(1 - \alpha)}.
\]

For each \( \alpha \) and \( c \) fixed let

\[
F(m) = \frac{(m + \alpha)(c + m + 1) - mc(1 - \alpha)}{(m + \alpha)(c + m + 1) + c(1 - \alpha)}.
\]

Then

\[
F(m + 1) - F(m) = \frac{A}{B} > 0 \quad \text{for each } m,
\]

where

\[
A = c(m + 1)(m + 2)(1 - \alpha)
\]

and

\[
B = [(m + 1 + \alpha)(c + m + 2) + c(1 - \alpha)][(m + \alpha)(c + m + 1) + c(1 - \alpha)].
\]

Hence \( F(m) \) is an increasing function of \( m \). Since

\[
F(1) = \frac{(1 + \alpha)(c + 2) - c(1 - \alpha)}{(1 + \alpha)(c + 2) + c(1 - \alpha)}
\]

the result follows. \( \Box \)
Remark. Putting \( n = 0 \) in the above theorems, we have the results obtained by Uralegaddi and Ganigi [3].

References


