ON THE AREA SUM OF A CONVEX POLYGON AND ITS POLAR RECIPROCAL

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To o. Univ.-Prof. Dr. H. Vogler on his 60th birthday

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Abstract: Let $P$ be a plane convex polygon contained in the unit circle $K$, and let $P^*$ be the polar reciprocal of $P$ with respect to $K$. In this paper it is proved that the area sum of $P$ and $P^*$ is greater than or equal to 6 with equality if and only if $P$ is a square inscribed in $K$.

1. Introduction

Let $K$ be the unit circle centred at the origin $O$, and let $P$ be a convex polygon inscribed in $K$ and containing $O$ in its interior. We denote by $P^*$ the circumscribed polygon whose points of contact with $K$ are the vertices of $P$. J. Aczél and L. Fuchs [1] proved that

\[ a(P) + a(P^*) \geq 6, \]

where $a(X)$ denotes the area of the set $X$. Equality holds if and only if $P$ is a square. An alternative proof was given by E. Trost [5]. Complementary remarks to (1) were made by J. Rätz [4]. More generically, L. Kuipers and B. Meulenbeld [3] found the infimum of the weighted area sum $wa(P) + (1 - w)a(P^*)$ for any weight $w$ between 0 and 1, the infimum depending on $w$. They also obtained a similar result for the weighted perimeter sum of $P$ and $P^*$. 
In the present paper we shall extend inequality (1) to more general domains.

**Theorem.** Let $P$ be a convex polygon contained in the unit circle $K$. If $P^*$ is the polar reciprocal domain of $P$ with respect to $K$, then inequality (1) holds and equality occurs only if $P$ is a square inscribed in $K$.

2. **Proof of the Theorem**

We begin with a further proof of the theorem by Aczél and Fuchs. Let $P$ be a convex polygon inscribed in $K$, and let $P^*$ be the polar reciprocal domain of $P$. We may assume that $P$ contains the centre $O$ of $K$ in its interior, since otherwise $a(P^*) = \infty$. Let us denote the central angles spanned by the sides of $P$ by $2x_1, \ldots, 2x_n$, where

\[
0 < x_1 \leq x_2 \leq \ldots \leq x_n < \pi/2, \\
x_1 + \ldots + x_n = \pi.
\]  

(2)

If the function $f$ is defined by

\[
f(x) = \sin x \cos x + \tan x,
\]

we have to show that

\[
S = \sum_{i=1}^{n} f(x_i) \geq 6
\]  

(3)

with equality only for $n = 4$ and $x_1 = x_2 = x_3 = x_4 = \pi/4$.

From

\[
f'(x) = 2 \cos^2 x - 1 + \frac{1}{\cos^2 x}
\]

and

\[
f''(x) = 2 \frac{\sin x}{\cos^3 x} \left(1 - 2 \cos^4 x\right)
\]

we see that (i) $f$ is strictly increasing in $0 \leq x < \pi/2$; (ii) strictly concave in $0 \leq x \leq x_0$, and convex in $x_0 \leq x < \pi/2$, where

\[x_0 = \arccos 1/\sqrt{4|2} = 32.765 \ldots \circ.
\]

In the proof of (3) we may assume that

\[
x_0 \leq x_2.
\]  

(4)

If, on the contrary, $0 < x_1 \leq x_2 < x_0$, we can replace $x_1$ and $x_2$ by $x'_1$ and $x'_2$ such that

\[0 \leq x'_1 < x_1 \leq x_2 < x'_2 \leq x_0,
\]
\[ x'_1 + x'_2 = x_1 + x_2 \]

and \( x'_1 = 0 \) or \( x'_2 = x_0 \) or both. Since \( f \) is strictly concave in \([0, x_0]\), this process reduces the sum \( S \). Moreover, the number of the \( x_i \)'s contained in \((0, x_0)\) would decrease. After a finite number of steps we obtain a finite set of points, again denoted by \( \{x_1, \ldots, x_n\} \), which satisfies (2) and (4) and yields a smaller \( S \).

We now show that \( S \) can be diminished by displacing \( x_1 \) if

\[ 0 < x_1 < x_0 \leq x_2 \leq \ldots \leq x_n < \pi/2. \tag{5} \]

Since \( f \) is strictly convex in \([x_0, \pi/2]\), we have

\[ S \geq f(x_1) + (n - 1)f\left(\frac{\pi - x_1}{n - 1}\right) \equiv S(x_1) \tag{6} \]

with equality only if \( x_2 = \ldots = x_n = (\pi - x_1)/(n - 1) \). By (5), we note that \( (n - 1)x_0 < \pi \), whence

\[ n \leq 6. \]

From (6) it follows that

\[ S'(x_1) = \left(\cos^2 x_1 - \cos^2 \frac{\pi - x_1}{n - 1}\right) \left(2 - \cos^{-2} x_1 \cos^{-2} \frac{\pi - x_1}{n - 1}\right). \tag{7} \]

We now distinguish the following cases:

\( n = 3 \) or \( 4 \). For \( 0 < x_1 < x_0 \) we have

\[ \frac{\pi}{2} > \frac{\pi - x_1}{n - 1} > \frac{\pi - x_0}{3} > x_0, \]

which shows that

\[ \cos^2 x_1 - \cos^2 \frac{\pi - x_1}{n - 1} > 0, \]

and

\[ \cos \frac{\pi - x_1}{n - 1} < \cos \frac{\pi - x_0}{3} = 0.655 \ldots < \frac{1}{\sqrt{2}}, \]

whence

\[ 2 - \cos^{-2} x_1 \cos^{-2} \frac{\pi - x_1}{n - 1} < 0. \]

Thus

\[ S'(x_1) < 0 \]

and

\[ S(x_1) > S(x_0) \tag{8} \]

if \( x_1 < x_0 \).
\( n = 5 \) or \( 6 \). The function \( g \) defined by
\[
g(x_1) \equiv 2 \cos x_1 \cos \frac{\pi - x_1}{n - 1} = \cos \left( \frac{\pi - x_1}{n - 1} + x_1 \right) + \cos \left( \frac{\pi - x_1}{n - 1} - x_1 \right)
\]
has the derivatives
\[
g'(x_1) = \left(1 - \frac{1}{n - 1}\right) \sin \left( \frac{\pi - x_1}{n - 1} + x_1 \right) + \left(1 + \frac{1}{n - 1}\right) \sin \left( \frac{\pi - x_1}{n - 1} - x_1 \right),
\]
\[
g''(x_1) = \left(1 - \frac{1}{n - 1}\right)^2 \cos \left( \frac{\pi - x_1}{n - 1} + x_1 \right) - \left(1 + \frac{1}{n - 1}\right)^2 \cos \left( \frac{\pi - x_1}{n - 1} - x_1 \right).
\]
In view of \( \frac{\pi - x_1}{n - 1} < \frac{\pi}{4} \) and \( x_1 < x_0 < \frac{\pi}{4} \) we have \( g''(x_1) < 0 \) so that \( g \) is positive and strictly concave on \([0, x_0]\). This implies that \( \cos^2 x_1 \cos^2 \frac{\pi - x_1}{n - 1} \) is strictly convex and
\[
h(x_1) = 2 - \cos^2 x_1 \cos^2 \frac{\pi - x_1}{n - 1}
\]
is strictly concave in \([0, x_0]\).

\( n = 5 \). Since \( h(x_1) > 0 \) for \( x_1 \) close to \( 0 \), and \( \cos^2 \frac{\pi - x_0}{4} < \cos^2 x_0 = \frac{1}{\sqrt{2}} \), the function \( h \) passes from positive to negative values on \((0, x_0]\). By (7), \( S' \) and \( h \) have the same sign, since \( x_1 < \frac{\pi - x_1}{4} \) on \([0, x_0]\). Hence \( S \) attains its minimum only at one of the end points of the interval \([0, x_0]\). The fact that \( S(0) = 4f(\frac{\pi}{4}) = 6 \) and \( S(x_0) = 6.010\ldots \) shows that
\[
S(x_1) > S(0)
\]
for \( x_1 > 0 \).

\( n = 6 \). The supposition (5) restricts the variable \( x_1 \) to
\[
0 < x_1 \leq \pi - 5x_0,
\]
where \( \pi - 5x_0 < x_0 \). Since \( h \) is strictly concave on \([0, \pi - 5x_0]\), \( h(0) = 1 - \tan^2(\pi/5) > 0 \) and \( h(\pi - 5x_0) = 2 - \cos^2 x_0 \cos^2(\pi - 5x_0) > 2 - \cos^4 x_0 = 0 \) we conclude that \( h(x_1) > 0 \) for \( x_1 > 0 \). Because \( (\pi - x_1)/5 \geq x_0 > x_1 \), we have
\[
\cos^2 x_1 - \cos^2 \frac{\pi - x_1}{5} > 0.
\]
By (7), this shows that \( S'(x_1) > 0 \) and (9) is satisfied once more.
In conclusion, we state that

\[(10) \quad S \geq \inf m f \left( \frac{\pi}{m} \right) \]

for \( m = 3, 4, \ldots \), where \( \pi/m \geq \pi_0 \). But \( m \leq \pi/x_0 \) implies that \( m = 3, 4 \) or \( 5 \). The required inequality (3) follows from \( 3f(\pi/3) = 15\sqrt{3}/4 = 6.495 \ldots \), \( 4f(\pi/4) = 6 \) and \( 5f(\pi/5) = 6.010 \ldots \).

Let \( P \) be a convex polygon contained in the unit circle \( K \) with centre \( O \). To prove inequality (1) we may assume that \( O \) is an interior point of \( P \), since otherwise \( a(P^*) = \infty \). Let \( n \geq 3 \) be given. By a convex \( n \)-gon we mean a convex polygon with at most \( n \) sides. There exists a convex \( n \)-gon \( P \) contained in \( K \) and containing \( O \) in its interior and having the property that \( a(P) + a(P^*) \) attains its minimum. The proof of our theorem is completed by the following lemma.

**Lemma.** All the vertices of \( P \) are on the boundary of \( K \).

**Proof.** Let \( P = A_1A_2 \ldots A_n \) and \( P^* = B_1B_2 \ldots B_n \) be such that \( [4] B_i \vee B_{i+1} \) is the polar of \( A_i \), for \( i = 1, \ldots, n \). Suppose that \( A_1 \) is an inner point of \( K \). Then \( B_1 \vee B_2 \) does not intersect \( K \). We denote the interior angles of \( P^* \) at \( B_1 \) and \( B_2 \) by \( \beta_1 \) and \( \beta_2 \) respectively and distinguish the following two cases.

\( \beta_1 + \beta_2 > \pi \). The lines \( B_n \vee B_1 \) and \( B_3 \vee B_2 \) intersect outside \( P^* \) at a point \( U \) which is the pole of \( A_2 \vee A_n \). The segment joining \( O \) and \( U \) intersects \( B_1B_2 \) at an inner point \( T \). The polar \( t \) of \( T \) is parallel to \( A_2 \vee A_n \) and contains the vertex \( A_1 \). Since \( \overline{OT} < \overline{OU} \), the line \( A_2 \vee A_n \) separates \( O \) and \( A_1 \). Without loss of generality, we may assume that \( B_1T \leq \overline{TB_2} \). We displace \( A_1 \) on \( t \) through a small distance and obtain a new convex \( n \)-gon \( P' = A_1'A_2 \ldots A_n \) contained in \( K \). The polar \( n \)-gon \( P'^* = B_1'B_2'B_3 \ldots B_n \) arises from \( P^* \) by rotating \( B_1 \vee B_2 \) about \( T \). We choose the direction of the displacement of \( A_1 \) so that \( B'_2 \) lies on \( B_2B_3 \) and \( B'_1 \) on the elongated segment \( B_2B_1 \). Let \( p \) be the ray radiating from \( B_2 \), parallel to \( B_n \vee B_1 \) and intersecting the interior of \( P^* \) (this is possible because \( \beta_1 + \beta_2 > \pi \)). The segment \( B'_1B'_2 \) intersects \( p \) at a point \( B'_2' \). Then

\[ \overline{B'_1T} \leq \overline{TB'_2} < \overline{TB_2} \],

whence

\[ a(TB_1B'_1) < a(TB_2B'_2) \]

and

\[ a(P'^*) < a(P^*) \].
Since \( a(P') = a(P) \), we have a contradiction to the assumption that 
\( a(P) + a(P^*) \) is minimal.

\( \beta_1 + \beta_2 \leq \pi \). By displacing \( A_1 \) on the ray \( OA_1 \) towards the boundary of \( K \) through a small distance \( x \) we obtain a new convex \( n \)-gon \( P' = A'_1 A_2 \ldots A_n \). Let \( b \) be length of the orthogonal projection of \( A_2 A_n \) onto the perpendicular to \( O \vee A_1 \). Then

\[
a(P') - a(P) = a(A_1 A'_1 A_n) + a(A_1 A'_1 A_2),
\]

whence

\[
\frac{1}{x}(a(P') - a(P)) = \frac{1}{2} b.
\]

In view of \( b \leq \overline{A_2 A_n} \leq 2 \) this implies

\[(11) \quad \frac{1}{x}(a(P') - a(P)) \leq 1.\]

If \( \overline{OA_1} = d \), the polar of \( A'_1 \) has the distance \( 1/(d+x) \) from \( O \). Thus the polar \( n \)-gon of \( P' \), \( P'^* = B'_1 B'_2 B_3 \ldots B_n \), arises from \( P^* \) by displacing the side \( B_1 B_2 \) parallel to itself towards \( O \) through the distance

\[
\frac{1}{d} - \frac{1}{d+x} = \frac{x}{d(d+x)}.
\]

Hence

\[
a(P^*) - a(P'^*) = a(B_1 B_2 B'_2 B'_1)
= (\overline{B_1 B_2} + \overline{B'_1 B'_2}) \cdot x/2d(d+x).
\]

But clearly

\[
\overline{B_1 B_2} \geq \cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} \geq \cot \frac{\beta_1}{2} + \tan \frac{\beta_1}{2} \geq 2
\]

and also \( \overline{B'_1 B'_2} \geq 2 \). Since \( d < 1 \) and \( d + x \leq 1 \), we finally have

\[
\frac{1}{x}(a(P^*) - a(P'^*)) > 2.
\]

The combination with (11) yields

\[a(P') + a(P'^*) < a(P) + a(P^*)\]

which is impossible. Thus the lemma and the theorem are proved. ◊

**Corollary.** Let \( C \) be a closed convex set contained in the unit circle \( K \) and let \( C^* \) be its polar reciprocal. Then

\[(12) \quad a(C) + a(C^*) \geq 6.\]

If \( C \) is contained in the interior of \( K \), then strict inequality holds.
**Proof.** It suffices to consider a closed convex subset $C$ of $K$ having the centre $O$ of $K$ as an inner point. The sets $C$ and $C^*$ can be approximated by pairs of polar reciprocal convex polygons. Therefore, (12) is a consequence of the theorem. For any $r \in (0, 1)$, the set $rC$ is in the interior of $K$, and $(rC)^* = \frac{1}{r} C^*$. The function

$$f(r) = a(rC) + a(\frac{1}{r} C^*) = r^2 a(C) + \frac{1}{r^2} a(C^*)$$

has a negative derivative

$$f'(r) = \frac{2}{r^3} (r^4 a(C) - a(C^*)) < 0.$$

Hence

$$f(r) > f(1) = a(C) + a(C^*) \geq 6,$$

as required. ◊

3. Remarks

(i) It may be that in (12) equality holds only if $C$ is a square inscribed in $K$.

(ii) In the corollary, the assumption of convexity of $C$ is essential. If $C$ is the boundary of $K$, then $a(C) + a(C^*) = \pi$.

(iii) In Euclidean 3-space let $K$ be a solid unit sphere, $P$ a convex polyhedron inscribed in $K$ and $P^*$ the polar reciprocal of $P$ with respect to $K$. In the following list the values of $V(P) + V(P^*)$ are collected, where $P$ is a regular polyhedron (characterized by its number $n$ of vertices), and $V$ the volume

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V(P) + V(P^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>14.36960...</td>
</tr>
<tr>
<td>6</td>
<td>9.33333...</td>
</tr>
<tr>
<td>8</td>
<td>8.46780...</td>
</tr>
<tr>
<td>12</td>
<td>8.08644...</td>
</tr>
<tr>
<td>20</td>
<td>7.83921...</td>
</tr>
</tbody>
</table>

and $V(K) + V(K^*) = 8.37758...$. The infimum of $V(P) + V(P^*)$, extended over all convex polyhedra $P$ inscribed in $K$, remains unknown and is not attained by the cube or the regular octahedron. In place of the volume, various other functionals may be considered. A simple example is given by the mean width $M(C)$ of a convex body $C$ in
$E^d(d \geq 2)$, i.e. the mean value of the widths of $C$, taken over all possible directions in $E^d$. Let the origin $O$ be an interior point of a body $C$ which need not necessarily be a subset of $K$. W. Firey observed that $\frac{(C+C^*)}{2} \supset K$ (formula (1) in [2]). This implies that

$$M(C) + M(C^*) \geq 4$$

with equality only if $C = K$. However, if $O$ is not an interior point of $C$, than $C^*$ is unbounded.

References


