VELOCITY AND CORIOLIS QUADRATICS OF ROBOT-MANIPULATORS

Marie Kargerová

Department of Technical Mathematics, Czech Technical University, Horská 3, 128 03 Praha 2, Czech Republic

Herrn o. Univ. Prof. Dr. Hans Vogler zu seinem sechzigsten Geburtstag gewidmet

Received October 1994

MSC 1991: 53 A 17

Keywords: Robot-manipulator, velocity operator, Coriolis acceleration, velocity quadrics, Coriolis quadrics.

Abstract: The connection between the velocity and Coriolis acceleration of 3R robot-manipulators is described. It is shown that the velocity and Coriolis quadrics have the same axes but of different length. Classification of Coriolis quadratical surfaces is given.

1. Introduction

This paper is a continuation of [3], where the basic properties of the velocity and acceleration fields of 3R robot-manipulators have been described. At the beginning we shall briefly summarize denotations and basic properties of velocity and acceleration operators from [3].

The geometry of robot-manipulators with \( p \) degrees of freedom is determined by the product of \( p \)-revolutions or translations given by axes \( X_1, \ldots, X_p \). We suppose that axes \( X_1, \ldots, X_p \) are determined by their Plücker coordinates \( X_i = (\bar{x}_i; \bar{y}_i), i = 1, \ldots, p \).

Remark. For simplicity reasons we consider only rotational axes, for prismatic joints we have to change all formulas accordingly.

The motion of the end-effector of such a robot-manipulator is expressed by the matrix

\[
g(\varphi_1, \varphi_2, \ldots, \varphi_p) = r_1(\varphi_1) r_2(\varphi_2) \cdots r_p(\varphi_p),
\]
where \( r_i(\varphi_i) \) is the matrix of the revolution around \( X_i \). If \( \varphi_i = \varphi_i(t) \) are functions of time we obtain a one parametric motion of the end-effector, 
\[
g(t) = r_1(\varphi_1(t)) \cdot r_2(\varphi_2(t)) \cdots r_p(\varphi_p(t)).
\]
Trajectory of a point \( \vec{A} \) of the end-effector space is \( \vec{A}(t) = g(t) \vec{A} \). Let us denote \( \Omega(\Theta) \) the velocity (acceleration) operator of \( g(t) \), respectively. We have
\[
\Omega = g'g^{-1}, \quad \Theta = \Omega' + \Omega^2, \quad \Omega = \sum_{i=1}^{p} Y_i v_i,
\]
where \( Y_i \) is the instantaneous position of \( i \)-th axis and \( v_i \) is the angular velocity of \( r_i(\varphi_i(t)); v_i = \frac{d\varphi_i}{dt} \). For the derivative of \( \Omega \) we have
\[
\Omega' = \sum_{i=1}^{p} Y'_i v_i + \sum_{i=1}^{p} Y_i \frac{dv_i}{dt}.
\]
We can split the acceleration operator into three parts:

a) \( \Omega^2 \) is the centrifugal acceleration;

b) \( \sum_{i=1}^{p} Y_i \varepsilon_i \) is the Euler acceleration, where \( \varepsilon = \frac{d\varphi_i}{dt} \) is the angular acceleration of \( r_i(\varphi_i(t)) \);

c) \( \sum_{i=1}^{p} Y_i v_i = \sum_{i<j=1}^{p} Y_i \times Y_j v_i v_j \) is the Coriolis acceleration where \( Y_i \times Y_j \) denotes the cross product of Plücker coordinates of \( Y_i \) with \( Y_j \).

2. Velocity and Coriolis quadrics

In the following we shall concentrate on velocity and acceleration properties of 3R robot-manipulators. We shall show that both velocity and Coriolis acceleration operators are connected with quadratical surfaces. Let us have a 3R robot-manipulator determined by axes \( X_1, X_2, X_3 \). Let us consider an instantaneous position \( Y_1, Y_2, Y_3 \) of these axes. Then the velocity operator \( \Omega \) for this configuration is given by
\[
(1) \quad \Omega = \omega_1 Y_1 + \omega_2 Y_2 + \omega_3 Y_3,
\]
Coriolis acceleration \( C \) is given by the formula
\[
(2) \quad C = Y_1 \times Y_2 \omega_1 \omega_2 + Y_1 \times Y_3 \omega_1 \omega_3 + Y_2 \times Y_3 \omega_2 \omega_3,
\]
\( Y_i \times Y_j \) is the cross product of Plücker coordinates which is defined as
follows. Denote \( Y_i = (\vec{x}_i; \vec{y}_i), Y_j = (\vec{x}_j; \vec{y}_j) \). Then
\[
(\vec{x}_i; \vec{y}_i) \times (\vec{x}_j; \vec{y}_j) = (\vec{x}_i \times \vec{x}_j; \vec{y}_i \times \vec{y}_j + \vec{y}_i \times \vec{x}_j),
\]
see [2]. We can see that velocity operator is always a linear combination of \( Y_1, Y_2, Y_3 \). This shows that we have to work in the 6-dimensional vector space of screws. It is the vector space of all pairs \((\vec{x}; \vec{y})\) of ordinary vectors of the Euclidean space \( E_3 \). \( V_6 \) contains Plücker coordinates of all straight lines of \( E_3 \). Their image is called Klein's quadratical hypersurface \( K \).

Let us assume that the direction vectors \( \vec{x}_1, \vec{x}_2, \vec{x}_3 \) of \( Y_1, Y_2, Y_3 \) are independent. All velocity operators \( \Omega \) for the given configuration \( Y_1, Y_2, Y_3 \) generate a 3-dimensional subspace \( V_3 \) of \( V_6 \). The intersection of \( K \) with \( V_3 \) is a ruled hyperboloid \( Q_v \). (We know that it contains three straight lines with independent directions.) \( Q_v \) is connected with the velocity operator and it is uniquely determined by the instantaneous configuration \( Y_1, Y_2, Y_3 \) of axes \( X_1, X_2, X_3 \) of the robot-manipulator. We shall call it \textit{velocity quadrics}.

We have similar situation with Coriolis acceleration. According to (2) the Coriolis acceleration operator \( C \) for the given configuration \( Y_1, Y_2, Y_3 \) lies in the 3-dimensional space \( \mathcal{W}_3 \) generated by screws \( \vec{x}_i \times \vec{y}_1, \vec{x}_2 \times \vec{y}_1, \vec{x}_3 \times \vec{y}_1 \).

\textbf{Remark.} The difference between \( V_3 \) and \( \mathcal{W}_3 \) is that \( \mathcal{W}_3 \) need not contain any straight lines and it is also not true that any screw of \( \mathcal{W}_3 \) is a Coriolis acceleration. Let us denote \( Q_c \) the quadratical surface obtained as the intersection of \( \mathcal{W}_3 \) with \( K \), we shall call it \textit{Coriolis quadrics}.

\section*{3. Properties of velocity and Coriolis quadrics}

\textbf{Lemma.} The Coriolis quadrics \( Q_c \) is independent of the choice of screws \( Y_1, Y_2, Y_3 \) in the subspace \( \mathcal{W}_3 \).

\textbf{Proof.} \( \mathcal{W}_3 \) is generated by independent screws \( Y_1, Y_2, Y_3 \). Let us choose another basis \( \vec{Y}_1, \vec{Y}_2, \vec{Y}_3 \) of \( \mathcal{W}_3 \) by
\[
\vec{Y}_1 = a_{11} Y_1 + a_{12} Y_2 + a_{13} Y_3,
\vec{Y}_2 = a_{21} Y_1 + a_{22} Y_2 + a_{23} Y_3,
\vec{Y}_3 = a_{31} Y_1 + a_{32} Y_2 + a_{33} Y_3,
\]
where the determinant \( D = |a_{ij}| \neq 0 \). The Coriolis quadrics corresponding to \( \vec{Y}_1, \vec{Y}_2, \vec{Y}_3 \) is determined by screws \( \vec{Y}_1 \times \vec{Y}_2, \vec{Y}_1 \times \vec{Y}_3, \vec{Y}_2 \times \vec{Y}_3 \).
Computation yields
\[
\begin{align*}
\vec{Y}_1 \times \vec{Y}_2 &= b_{11} Y_1 \times Y_2 + b_{12} Y_1 \times Y_3 + b_{13} Y_2 \times Y_3 \\
\vec{Y}_1 \times \vec{Y}_3 &= b_{21} Y_1 \times Y_2 + b_{22} Y_1 \times Y_3 + b_{23} Y_2 \times Y_3 \\
\vec{Y}_2 \times \vec{Y}_3 &= b_{31} Y_1 \times Y_2 + b_{32} Y_1 \times Y_3 + b_{33} Y_2 \times Y_3,
\end{align*}
\]
where for instance \( b_{11} = a_{11} a_{22} - a_{12} a_{21} \) and similarly for the others. Computation shows that for the determinant \( D_1 = \det b_{ij} \) we have \( D_1 = -D^2 \) and therefore \( D_1 \neq 0 \). \( \diamond \)

From Lemma we see that the connection between \( Q_v \) and \( Q_c \) is independent of the choice of screws which determine them. This means that we can choose screws corresponding to axes of \( Q_v \). We can prove the following theorem.

**Theorem.** Let us suppose that \( Q_v \) has the equation
\[
v_1 x^2 + v_2 y^2 + v_3 z^2 + v_1 v_2 v_3 = 0
\]
in the canonical basis. Then \( Q_c \) has the equation
\[
(v_2 + v_3) x^2 + (v_3 + v_1) y^2 + (v_2 + v_1) z^2 + (v_2 + v_3)(v_3 + v_1)(v_2 + v_1) = 0
\]
in the same basis.

**Proof.** If \( Q_v \) is given by (3) in a Cartesian system of coordinates \( \{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\} \), then the corresponding subspace \( V_3 \) is determined by screws \( Y_i = (\vec{e}_i; v_i \vec{e}_i), \ i = 1, 2, 3 \). Then
\[
\begin{align*}
Y_1 \times Y_2 &= (\vec{e}_1; v_1 \vec{e}_1) \times (\vec{e}_2; v_2 \vec{e}_2) = (\vec{e}_3; (v_1 + v_2) \vec{c}_3) \\
Y_1 \times Y_3 &= (\vec{e}_1; v_1 \vec{e}_1) \times (\vec{e}_3; v_3 \vec{c}_3) = (\vec{e}_2; (v_1 + v_3) \vec{c}_2) \\
Y_2 \times Y_3 &= (\vec{e}_2; v_2 \vec{e}_2) \times (\vec{e}_3; v_3 \vec{c}_3) = (\vec{e}_1; (v_2 + v_3) \vec{c}_1).
\end{align*}
\]
We see that \( Q_c \) has the canonical form in the same system of Cartesian coordinates. \( \diamond \)

**Corollary.** \( Q_c \) and \( Q_v \) have the same axes. Length \( a, b, c \) of axes of these surfaces are not in general the same. Relation between those quantities is following:
\[
\begin{align*}
Q_v: \quad a &= \sqrt{|v_2 v_3|} \quad b = \sqrt{|v_1 v_3|} \quad c = \sqrt{|v_1 v_2|} \\
Q_c: \quad a &= \sqrt{|(v_3 + v_1)(v_2 + v_1)|} \quad b = \sqrt{|(v_2 + v_3)(v_1 + v_2)|} \\
&\quad c = \sqrt{|(v_2 + v_3)(v_3 + v_1)|}.
\end{align*}
\]
4. Classification of velocity and Coriolis quadrics

Let us consider $Q_v$ and $Q_c$ given by (3) and (4). We shall classify them according to different values of $v_1, v_2, v_3$.

1) $v_1v_2v_3 \neq 0$. $Q_v$ is a one sheet hyperboloid.
   a) $(v_1 + v_2)(v_2 + v_3)(v_1 + v_3) \neq 0$, $v_1 \geq v_2 > 0$, $v_3 < 0$. $Q_c$ is one sheet hyperboloid if $v_1 < -v_3 < v_2$ or $v_1 < v_2 < -v_3$. $Q_c$ is empty if $-v_3 < v_1 < v_2$.
   b) $(v_1 + v_2)(v_1 + v_3) \neq 0$, $v_2 + v_3 = 0$. $Q_c$ has the equation

   \[(v_1 - v_2)y_2^2 + (v_1 + v_2)z^2 = 0.\]

According to the sign of $v_1 + v_2$ and $v_1 - v_2$ we obtain two planes which can be different, coinciding or imaginary.

2) $v_1v_2 \neq 0$, $v_3 = 0$. We may suppose $v_1 > 0$, $v_2 < 0$ and $Q_v$ consists of two pencils of straight lines, two axes intersect. $Q_c$ has the equation

   \[v_2x^2 + v_1y^2 + (v_1 + v_2)z^2 + (v_1 + v_2)v_1v_2 = 0.\]

If $v_1 + v_2 \neq 0$, $Q_c$ is one sheet hyperboloid and if $v_1 + v_2 = 0$ we obtain two pencils of straight lines.

3) The case $v_2 = v_3 = 0$ was excluded.

References

