SEPARATION BY MONOTONIC FUNCTIONS

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Abstract: It is shown that real functions \( f \) and \( g \) defined on an arbitrary interval \( I \) can be separated by a monotonic function \( h \) if

\[
 f(tx + (1-t)y) \leq \max \{g(x), g(y)\}
\]

and

\[
 g(tx + (1-t)y) \geq \min \{f(x), f(y)\}
\]

for all \( x, y \in I \) and \( t \in [0,1] \). Some results on the existence of monotonic selections of multifunctions and on the Hyers-Ulam stability of the monotonicity are also presented.

1. Introduction

The aim of this note is to characterize real functions \( f, g \) defined on an interval \( I \subseteq \mathbb{R} \) which can be separated by a monotonic function.
This problem is connected with quasiconvex and quasiconcave functions and leads to functional inequalities
\[ f(tx + (1 - t)y) \leq \max \{ g(x), g(y) \} \]
and
\[ g(tx + (1 - t)y) \geq \min \{ f(x), f(y) \} \].
The first of them appeared previously in a paper of J. Smolarz [3] and is equivalent to the fact that there exist a quasiconvex function \( h : I \to \mathbb{R} \) such that \( f \leq h \leq g \). The results presented by us are related to a sandwich theorem obtained recently by K. Nikodem and Sz. Wąsowicz [2]. It states that there exists an affine function separating \( f \) and \( g \) iff
\[ f(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y) \]
and
\[ g(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y) \].
The first of the above inequalities implies that \( f \) and \( g \) can be separated by a convex function (cf. K. Baron, J. Matkowski and K. Nikodem [1]).

As an application of our separation theorem we obtain a result on the existence of monotonic selections of multifunctions whose values are compact intervals in \( \mathbb{R} \). We also get a stability result of Hyers-Ulam type for monotonic functions.

Let us recall that a function \( f : I \to \mathbb{R} \) is quasiconvex if
\[ f(tx + (1 - t)y) \leq \max \{ f(x), f(y) \}, \quad x, y \in I, \; t \in [0, 1]; \]
it is quasiconcave if
\[ f(tx + (1 - t)y) \geq \min \{ f(x), f(y) \}, \quad x, y \in I, \; t \in [0, 1]. \]
Obviously, a function \( f : I \to \mathbb{R} \) is monotonic iff it is quasiconvex and quasiconcave.

2. A separation theorem

Our main result reads as follows.

**Theorem.** Let \( I \subseteq \mathbb{R} \) be an arbitrary interval and \( f, g : I \to \mathbb{R} \) be given functions. The following properties are equivalent:

(a) there is a monotonic function \( h : I \to \mathbb{R} \) such that \( f \leq h \leq g \);
(b) there are functions \( h_1, h_2 : I \to \mathbb{R} \), \( h_1 \)-quasiconcave, \( h_2 \)-quasiconvex, such that \( f \leq h_1 \leq g \) and \( f \leq h_2 \leq g \);
(c) there are functions \( h_1, h_2 : I \to \mathbb{R} \), \( h_1 \)-quasiconcave, \( h_2 \)-quasiconvex, such that \( f \leq h_1 \leq h_2 \leq g \);
(d) for all \( x, y \in I \) and \( t \in [0, 1] \) the following inequalities hold.
It is evident that $f$ is nonincreasing. We will show that at least one of these functions separates the given functions.

Proof. Implication $(a) \implies (d)$ follows from the fact that every monotonic function is both quasiconvex and quasiconcave.

To prove that $(d)$ implies $(c)$ consider the functions $h_1, h_2 : I \to \mathbb{R}$ defined by

$$h_1(u) := \sup \{ \min \{f(x), f(y)\} : x \leq u, y, y \in I \}$$

$$h_2(u) := \inf \{ \max \{g(x), g(y)\} : x \leq u, y, y \in I \}. $$

By (1) the definitions are correct and

$$f(u) \leq h_2(u) \quad \text{and} \quad h_1(u) \leq g(u)$$

for every $u \in I$. Moreover, $f \leq h_1$ and $h_2 \leq g$. We will show that $h_1 \leq h_2$, $h_1$ is quasiconcave and $h_2$ is quasiconvex. Suppose, contrary to our claim, that there exist $w \in I$ such that $h_1(w) > h_2(w)$. Then there exist $u \leq w \leq v$ and $u \leq w \leq v$ such that

$$\min \{f(x), f(y)\} > \max \{g(u), g(v)\}. $$

If $u \leq v$, then from (2) and (5) it follows that

$$h_1(u) \geq \min \{f(x), f(y)\} > g(u)$$

which contradicts (4). If $u \leq v$, then from (5) and (3) we get

$$f(u) > \max \{g(u), g(v)\} \geq h_2(u),$$

contrary to (4). These contradictions show that $h_1 \leq h_2$.

Now we will prove that $h_2$ is quasiconvex (the quasiconcavity of $h_1$ follows similarly). Suppose that it is false. Then there exist $z \leq u \leq y$ such that

$$h_2(u) > \max \{h_2(x), h_2(y)\}. $$

By the definition of $h_2$ we can find $\alpha \leq x \leq \beta$ and $\gamma \leq y \leq \delta$ such that

$$h_2(u) > \max \{g(\alpha), g(\beta), g(\gamma), g(\delta)\}. $$

However $\alpha \leq u \leq \delta$, which implies that

$$h_2(u) \leq \max \{g(\alpha), g(\delta)\}. $$

This contradicts (6) and proves that $h_2$ is quasiconvex.

Implication $(c) \implies (b)$ is obvious.

To prove that $(b)$ implies $(a)$ assume first that $\sup \{f(z) : z \leq x\} < \infty$ and $\inf \{g(z) : z \leq x\} = -\infty$ for any $x \in I$. Define $m_1, m_2 : I \to \mathbb{R}$ by

$$m_1(x) := \sup \{f(z) : z \leq x\} $$

$$m_2(x) := \inf \{g(z) : z \leq x\}. $$

It is evident that $f \leq m_1$, $m_2 \leq g$, $m_1$ is nondecreasing and $m_2$ is nonincreasing. We will show that at least one of these functions sepa-
rates $f$ and $g$. Assume the contrary. Then there are $a, b \in I$ such that $m_1(a) > g(a)$ and $m_2(b) < f(b)$. Without loss of generality we may assume that $a \leq b$ (otherwise we change $f$ to $-f$ and $g$ to $-g$, which interchanges the roles of $h_1, h_2$ and $m_1, m_2$). Let

$$0 < \varepsilon < \frac{1}{2} \min \{m_1(a) - g(a), f(b) - m_2(b)\}.$$  

By the definition of $m_1$ and $m_2$ there are $x_1, x_2 \in I, x_1 \leq a, x_2 \leq b$ such that

$$f(x_1) > m_1(a) - \varepsilon \quad \text{and} \quad g(x_2) < m_2(b) + \varepsilon.$$  

If $x_1 \leq x_2$, then $x_2, a \in [x_1, b]$ and by (b), (7) and (8) we get

$$h_1(x_2) \leq g(x_2) < m_2(b) + \varepsilon \leq f(b) \leq h_1(b)$$  

which contradicts the quasiconcavity of $h_1$ on $[x_1, b]$. If $x_2 < x_1$, then $x_1 \in [x_2, a]$ and by (8), the definition of $m_2$ and (7) we obtain

$$h_2(x_2) \leq g(x_2) < m_2(b) + \varepsilon \leq g(a) + \varepsilon$$  

$$h_2(a) \leq g(a) < g(a) + \varepsilon$$  

which contradicts the quasiconvexity of $h_2$ on $[x_2, a]$.

Thus in any case we get a contradiction showing that at least one of the functions $m_1, m_2$ separates $f$ and $g$.

Now we will deal with the existence of the functions $m_1, m_2$. As $f$ is bounded from above by $h_0$ and therefore by $\max \{h_2(\alpha), h_2(\beta)\}$ on every compact interval $[\alpha, \beta] \subseteq I$ (and similarly $g$ by $h_1$ from below), the only possibility for the nonexistence of $m_1$ (or $m_2$) is given if $f$ (respectively $g$) is unbounded at the left border point of $I$. We need to consider the following three cases.

(i) Suppose that $\sup \{f(z) : z \leq x\} = \infty$ and $\inf \{g(z) : z \leq x\} = -\infty$ for some $x \in I$. Then, by the arguments given above, for any $y \leq x$ we have $\sup \{f(z) : z \leq y\} = \infty$ and $\inf \{g(z) : z \leq y\} = -\infty$. Thus there is a point $z_1 \leq x$ such that $f(z_1) > 0$. Furthermore, we can find a point $z_0 < z_1$ such that $g(z_0) < 0$, and a point $z_2 < z_1$ such that $f(z_2) > 0$. Since $f \leq h_1 \leq g$, this is a contradiction to the quasiconcavity of $h_1$.

(ii) Suppose that $\sup \{f(z) : z \leq x\} = \infty$ for some $x \in I$ and $m_2$ exists. Fix $x_0 \in I$ and choose $y_0 < x_0$ such that $f(y_0) > g(x_0)$. For any $y \in I, y \leq y_0$, we may restrict $m_2$ to the interval $[y, \infty] \cap I$ and define $m_1$ as previously on this interval. Because $y_0 < x_0$ and $f(y_0) > g(x_0)$,
the function $m_1$ cannot separate $f$ and $g$, so, only the restriction of $m_2$ works as a separating function on $[y, \infty] \cap I$. Since this is valid for any $y \in I$, $y \leq y_0$, we conclude that $m_2$ separates $f$ and $g$ on $I$.

(iii) The case where $m_1$ exists but $\inf \{g(z) : z \leq x\} = -\infty$ for some $x \in I$ can be treated like (ii).

This finishes the proof. ♦

Remark. The above theorem can be proved in a different manner. For instance, since the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (d) are obvious and (d) $\Rightarrow$ (b) follows by the result of Smoliarz [8], it is enough to show that (b) $\Rightarrow$ (a). The implication (c) $\Rightarrow$ (a) can be also obtained from the fact that quasiconvex and quasiconcave functions defined on an interval are either monotonic or unimodal (i.e. consist of two monotonic segments). Then it remains to prove that (d) $\Rightarrow$ (c), because the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (d) are evident. However, the proof presented by us is direct and it contains some ideas which may be interesting in themselves.

3. Applications

Let $\varepsilon$ be a nonnegative constant. We say that a function $f : I \to \mathbb{R}$ is $\varepsilon$-monotonic if

$$\min \{f(x), f(y)\} - \varepsilon \leq f(tx + (1 - t)y) \leq \max \{f(x), f(y)\} + \varepsilon$$

for all $x, y \in I, t \in [0, 1]$.

As an immediate consequence of our Theorem we obtain the following stability result of Hyers–Ulam type for monotonic functions.

Corollary 1. A function $f : I \to \mathbb{R}$ is $\varepsilon$-monotonic if and only if there exists a monotonic function $\varphi : I \to \mathbb{R}$ such that

$$|f(x) - \varphi(x)| \leq \frac{\varepsilon}{2}, \quad x \in I.$$ (9)

Proof. If $f$ is $\varepsilon$-monotonic, then the inequalities (1) hold with $g = f + \varepsilon$. By the Theorem there exists a monotonic function $h : I \to \mathbb{R}$ such that $f \leq h \leq f + \varepsilon$. Putting $\varphi = h - \frac{\varepsilon}{2}$ we get a monotonic function satisfying (9). Now, assume that $f$ satisfies (9) with a monotonic function $\varphi$. Then, for arbitrary $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \varphi(tx + (1 - t)y) + \frac{\varepsilon}{2} \leq \max \{\varphi(x), \varphi(y)\} + \frac{\varepsilon}{2} \leq \max \{f(x), f(y)\} + \varepsilon.$$

Similarly

$$f(tx + (1 - t)y) \geq \min \{f(x), f(y)\} - \varepsilon,$$

which ends the proof. ♦
Recall that a function \( f : I \to \mathbb{R} \) is the selection of a multifunction \( \Phi : I \to \mathfrak{n} (\mathbb{R}) \) (where \( \mathfrak{n} (\mathbb{R}) \) denotes the family of all non-empty subsets of \( \mathbb{R} \)) if \( f(x) \in \Phi(x) \) for all \( x \in I \). As a consequence of our Theorem we get also the following result on the existence of monotonic selections. Here \( \mathfrak{cc} (\mathbb{R}) \) denotes the family of all compact intervals in \( \mathbb{R} \) and \( \text{conv}(A) \) — the convex hull of a set \( A \).

**Corollary 2.** A multifunction \( \Phi : I \to \mathfrak{cc}(\mathbb{R}) \) has a monotonic selection if and only if

\[
\Phi(tx + (1-t)y) \cap \text{conv}(\Phi(x) \cup \Phi(y)) \neq \emptyset
\]

for all \( x, y \in I, t \in [0,1] \).

**Proof.** Let us put \( f(x) := \inf \Phi(x) \), and \( g(x) := \sup \Phi(x) \) for all \( x \in I \). Then

\[
\Phi(x) = [f(x), g(x)] \quad \text{and} \quad \text{conv}(\Phi(x) \cup \Phi(y)) = [\min\{f(x), f(y)\}, \max\{g(x), g(y)\}].
\]

Hence \( \Phi \) satisfies (10) iff \( f \) and \( g \) satisfy (1), and a function \( h : I \to \mathbb{R} \) is a selection of \( \Phi \) iff it separates \( f \) and \( g \). So, to finish the proof it is enough to apply the Theorem. \( \Box \)

**References**

