BOUNDED SOLUTIONS OF SCHILLING’S PROBLEM

Janusz Morawiec

Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, PL-40-007 Katowice, Poland

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Abstract: Let \( n \) be a positive integer, \( q_n \) be the unique \( x \in \left( \frac{1}{2}, \frac{3}{2} \right) \) with \( x^{n+1} - 3x + 1 = 0 \), and \( q \in (0, q_n] \). We found a set \( A_q \) of reals with the following property (P): Every solution \( f: \mathbb{R} \to \mathbb{R} \) of the functional equation

\[
f(qx) = \frac{1}{4q} \left[ f(x - 1) + f(x + 1) + 2f(x) \right]
\]

which vanishes outside of \( \left[ \frac{1}{2}, \frac{3}{2} \right] \) and is bounded in a neighbourhood of a point of that set vanishes everywhere. It is also observed that for \( q \in (0, \frac{2}{3}] \) the set \( \bigcup_{n=1}^{\infty} A^n_q \), which equals then

\[
\left\{ \sum_{n=1}^{\infty} \varepsilon(n)q^n : \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \right\},
\]

is the largest one with property (P).

Following R. Schilling [9] we consider solutions \( f: \mathbb{R} \to \mathbb{R} \) of the functional equation

\[
f(qx) = \frac{1}{4q} \left[ f(x - 1) + f(x + 1) + 2f(x) \right]
\]

such that

\[
f(x) = 0 \quad \text{for} \quad |x| > Q
\]

where \( q \) is a fixed number from the open interval \( (0, 1) \) and
In what follows any solution $f : \mathbb{R} \to \mathbb{R}$ of (1) satisfying (2) will be called a solution of Schilling’s problem.

If

$$3q \leq 1 - \sqrt{2} + \sqrt{4}$$

then according to [7] the zero function is the only solution of Schilling’s problem which is bounded in a neighbourhood of a point of the set

$$\left\{ \varepsilon \sum_{i=1}^{n} q^{i} : n \in \mathbb{N} \cup \{0, +\infty\}, \varepsilon \in \{-1, 1\} \right\}.$$  

This generalizes in particular [1; Th. 1]. It is the aim of the present paper to obtain such a result with the set (4) replaced by a larger one. However, we are not able to enlarge (4) for all $q$’s satisfying (3) but, on the other hand, for $q \leq \frac{1}{2}$ we succeeded in finding even the largest set to be put in the place of (4) (cf. Cor. 1).

Given a positive integer $n$ and $q \in (0, 1]$ consider the set $A_{n}^{0}$ of all the real numbers of the form

$$\sum_{i=1}^{L} (-1)^{i} \sum_{k=1}^{K_{i}} q^{\sum_{m=1}^{K_{i}} \nu(l, m) + \sum_{j=1}^{K_{j}} \sum_{m=1}^{K_{j}} \nu(j, m) + \varepsilon(-1)^{L} \sum_{m=1}^{M} q^{m},$$

where $\varepsilon \in \{-1, 1\}$, $M$, $L$ are non-negative integers, $K_{1}, \ldots, K_{L} \in \{1, \ldots, n\}$, and $\nu : \{1, \ldots, L\} \times \{1, \ldots, n\} \to \mathbb{N}$. Evidently, the set (4) is a subset of $\text{cl} A_{n}^{0}$. Let us observe also that for $l_{1}, l_{2} \in \{1, \ldots, L\}$, $k_{1} \in \{1, \ldots, K_{l_{1}}\}$, $k_{2} \in \{1, \ldots, K_{l_{2}}\}$, if $(l_{1}, k_{1}) \neq (l_{2}, k_{2})$ then

$$\sum_{m=k_{1}}^{K_{l_{1}}} \nu(l_{1}, m) + \sum_{j=l_{1}+1}^{L} \sum_{m=1}^{K_{j}} \nu(j, m) \neq \sum_{m=k_{2}}^{K_{l_{2}}} \nu(l_{2}, m) + \sum_{j=l_{2}+1}^{L} \sum_{m=1}^{K_{j}} \nu(j, m).$$

The proof of the following fact is left to the reader (cf. also [6; Th. 21(a), (d)]).

**Remark 1.** If $q \in (0, \frac{1}{2}]$ then

$$\text{cl} \bigcup_{n=1}^{\infty} A_{n}^{0} = \left\{ \sum_{n=1}^{\infty} \varepsilon(n)q^{n} : \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \right\},$$

and if $q \in [\frac{1}{2}, 1]$ then
\[
\left\{ \sum_{n=1}^{\infty} x(n)q^n : \quad x \in \{-1, 0, 1\}^\mathbb{N} \right\} = [-Q, Q].
\]

For every positive integer \( n \) let \( q_n \) denote the unique \( x \in \left( \frac{1}{3}, \frac{1}{2} \right) \) with
\[
x^{n+1} - 3x + 1 = 0,
\]
and observe that if \( q \in (0, \frac{1}{3}) \) then
\[
q \leq q_n \quad \text{iff} \quad q^{n+1} - 3q + 1 \geq 0.
\]
Our main result reads.

**Theorem 1.** If \( n \) is a positive integer and \( q \in (0, q_n] \) then the zero function is the only solution of Schilling’s problem which is bounded in a neighbourhood of a point of the set \( \mathcal{A}_q^n \).

The proof of this theorem is based on four lemmas. However, we start with the following simple remarks.

**Remark 2.** If \( f \) is a solution of Schilling’s problem then so is the function \( g : \mathbb{R} \to \mathbb{R} \) defined by the formula \( g(x) = f(-x) \).

**Remark 3.** Assume \( f \) is a solution of Schilling’s problem.

If \( q \neq \frac{1}{2} \) then \( f(-Q) = f(Q) = 0 \). If \( q < \frac{1}{3} \) then \( f(0) = 0 \).

**Lemma 1.** Assume \( q \in (0, \frac{1}{2}) \). If \( f \) is a solution of Schilling’s problem then
\[
f(q^{N+M}x + \sum_{m=1}^{M} q^m) = \left( \frac{1}{2q} \right)^{N+M} f(x)
\]
for all \( x \in (Q-1, 1-Q) \) (for all \( x \in [Q-1, 1-Q] \) if \( q \neq \frac{1}{2} \)), for all \( \varepsilon \in \{-1, 1\} \), and for all non-negative integers \( M \) and \( N \).

For \( x \in (Q-1, 1-Q) \) this was proved in [7] as Lemma 2. In the case of the closed interval \([Q-1, 1-Q]\) and \( q \neq \frac{1}{2} \) we argue similarly as in the proof of [7; Lemma 2] using also [7; Remarks 1 and 2(ii)].

**Lemma 2.** Let \( n \in \mathbb{N} \), \( q \in (0, q_n] \) and
\[
y = q^{N+\sum_{l=1}^{L} K_l} \sum_{k=1}^{K_l} \nu(l,k) x + \sum_{l=1}^{L} (-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=0}^{K_l} \nu(l,m) + \sum_{j=1}^{k} \sum_{m=1}^{K_j} \nu(j,m)},
\]
where \( N \) is a non-negative integer, \( L \) is a positive integer, \( K_1, \ldots, K_L \in \mathbb{N} \), \( \nu : \{1, \ldots, L\} \times \{1, \ldots, n\} \to \mathbb{N} \), and \( x \in [0, 1-Q] \) \( (x \in \mathbb{N} \) if \( q \neq \frac{1}{2} \).

If \( L \) is even then \( y \in (0, 1-Q) \).
If $L$ is odd then $y \in [Q - 1, 0]$ ($y \in (Q - 1, 0]$ if $q < \frac{1}{3}$).

**Proof.** Since $q \leq q_n < \frac{1}{3}$ we have

(9) \hspace{1cm} Q < 1.

Moreover, as $q_n$ is a solution of (7),

(10) \hspace{1cm} \sum_{i=1}^{n} q_i \leq \sum_{i=1}^{n} q_n^i = 1 - \frac{q_n}{1 - q_n} \leq 1 - \frac{q}{1 - q} = 1 - Q,

and

(11) \hspace{1cm} \text{if } q < \frac{1}{3} \text{ then } \sum_{i=1}^{n} q^i < Q < 1 - Q.

Observe also that

\[
y = q^{(L,K_L)} \left( q^N + \sum_{l=1}^{L-1} \sum_{i=1}^{K_l} \nu(l,k) \cdot q^{(L,L)} \right) + \\
+ \sum_{l=1}^{L-1} (-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=1}^{N_k} \nu(l,m) + \sum_{j=m+1}^{N_k} \nu(l,m) - \nu(L,K_L)} + \\
+ (-1)^L \sum_{k=1}^{K_L} q^{\sum_{m=1}^{N_k} \nu(L,m)} (-1)^L,
\]

(12)

\[
y = q^N \sum_{l=1}^{L} \sum_{i=1}^{K_l} q^{\nu(l,k)} (q^N - 1) - \\
- \sum_{k=2}^{K_L} q^{\sum_{m=1}^{N_k} \nu(1,m) + \sum_{j=m+1}^{N_k} \nu(j,m)} + \\
+ \sum_{l=2}^{L} (-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=1}^{N_k} \nu(l,m) + \sum_{j=m+1}^{N_k} \nu(l,m)},
\]

(13)

\[
y = q^N \sum_{l=1}^{L} \sum_{i=1}^{K_l} q^{\nu(l,k)} (q^N - 1) < 0
\]

and

\[
\sum_{k=1}^{K_L} q^{\sum_{m=1}^{N_k} \nu(L,m)} \leq \sum_{k=1}^{K_L} q^{\sum_{m=1}^{N_k} \nu(L,m)} \leq \sum_{k=1}^{n} q^k.
\]

(15)

Suppose first $L$ is even. Applying (13), (14), (6), (15), (9) and (10) we obtain
y \leq \sum_{l=2}^{L} (-1)^l \sum_{k=1}^{K} q^{N_{mk}} \mu(l,m) + \sum_{j=1}^{L} \sum_{m=j+1}^{K} q^{N_{jm}} \mu(j,m) \\
\leq \sum_{l=2}^{L-2} \sum_{k=1}^{K} q^{N_{mk}} \mu(l,m) + \sum_{j=1}^{L} \sum_{m=j+1}^{K} q^{N_{jm}} \mu(j,m) \\
- \sum_{k=1}^{K_{L-1}} q^{N_{mk}} \mu(L-1,m) + \sum_{j=1}^{L} \sum_{m=j+1}^{K} q^{N_{jm}} \mu(j,m) + \sum_{k=1}^{K} q^{N_{mk}} \mu(L,m) \\
\leq \sum_{i=1}^{n} q^{\mu(L-1,K_{L-1})} + \sum_{m=1}^{K} q^{N_{mk}} \mu(L,m) + \sum_{k=1}^{n} q^{h} \\
= q^{\mu(L-1,K_{L-1})} + \sum_{m=1}^{K} q^{N_{mk}} \mu(L,m) (Q - 1) + \sum_{k=1}^{n} q^{h} < \sum_{k=1}^{n} q^{h} \leq 1 - Q,

whereas (12), (6) and (9) give

\[ y \geq q^{\mu(L,K_{L})} \left( - \sum_{i=1}^{\infty} q^{i} + 1 \right) = q^{\mu(L,K_{L})} (-Q + 1) > 0. \]

Suppose now \( L \) is odd. If \( L = 1 \) then using the definition of \( y \), (15) and (10) we see that

\[ y \geq - \sum_{k=1}^{K} q^{N_{mk}} \mu(1,m) \geq - \sum_{k=1}^{n} q^{h} \geq Q - 1, \]

with the last inequality being strict if \( q < \frac{1}{2} \) (cf. (11)). If \( L \geq 3 \) then on account of the definition of \( y \), (6), (15), (9) and (10) we have

\[ y \geq - \sum_{l=1}^{L-2} \sum_{k=1}^{K} q^{N_{mk}} \mu(l,m) + \sum_{j=1}^{L} \sum_{m=j+1}^{K} q^{N_{jm}} \mu(j,m) + \sum_{k=1}^{K_{L-1}} q^{N_{mk}} \mu(L-1,m) + \sum_{j=1}^{L} \sum_{m=j+1}^{K} q^{N_{jm}} \mu(j,m) - \sum_{k=1}^{K} q^{N_{mk}} \mu(L,m) \]

\[ \geq - \sum_{i=1}^{n} q^{\mu(L-1,K_{L-1})} + \sum_{m=1}^{K} q^{N_{mk}} \mu(L,m) + \sum_{k=1}^{n} q^{h} \geq Q - 1. \]
Finally, if \( L \) is odd then taking into account (12) and (6) we obtain
\[
y \leq q^{p(L,K_L)}(x + \sum_{i=1}^{n} q^i - 1) \leq q^{p(L,K_L)}[(1 - Q) + Q - 1] = 0. \tag*{\Box}
\]

**Lemma 3.** Assume \( n \in \mathbb{N} \) and \( q \in (0, q_n] \). If \( f \) is a solution of Schilling’s problem then for every \( x \in [0, 1 - Q) \), for every non-negative integers \( M, L \) and \( N \), for every \( K_1, \ldots, K_L \in \{1, \ldots, n\} \), and for every \( \nu : \{1, \ldots, L\} \times \{1, \ldots, n\} \rightarrow \mathbb{N} \) we have
\[
f(q^{N + \sum_{i=1}^{L} \nu(i,k)} + M, x) + \sum_{i=1}^{L} (-1)^i \sum_{k=1}^{K_i} q^{\sum_{m=1}^{K_i} \nu(i,m) + \sum_{j=1}^{K_j} \nu(j,m) + M} + (-1)^L \sum_{m=1}^{M} q^m =
\]
\[
= \left( \frac{1}{2} \right)^{L} \sum_{i=1}^{L} K_i + M \left( \frac{1}{2 q} \right)^{N + \sum_{i=1}^{L} \sum_{k=1}^{K_i} \nu(i,k)} \right) f(x).
\]

**Proof.** According to Lemma 1, (16) holds for \( L = 0 \). Assume \( L \) is a positive integer.

Consider first the case \( M = 0 \).

Let \( L = 1 \). Equality (16) takes then the form
\[
f(q^{N + \sum_{i=1}^{K_1} \nu(1,k)} x - \sum_{k=1}^{K_1} q^{\sum_{m=1}^{K_1} \nu(1,m)}) =
\]
\[
= \left( \frac{1}{2} \right) K_1 \left( \frac{1}{2 q} \right)^{N + \sum_{i=1}^{K_1} \nu(1,k)} f(x),
\]
and making use of Lemma 1 we see that if \( K_1 = 0 \) then (17) holds for all \( x \in (Q - 1, 1 - Q) \) (for all \( x \in \{Q - 1, 1 - Q\} \) if \( q \neq \frac{1}{2} \)) and for every non-negative integer \( N \). Fix now a \( K_1 \in \{0, \ldots, n - 1\} \) and suppose that (17) is satisfied for every non-negative integer \( N \), for every
\[ \nu : \{1\} \times \{1, \ldots, n\} \rightarrow \mathbb{N}, \text{ and for all } x \in [0, 1 - Q] \text{ (for all } x \in [0, 1 - Q] \text{ if } q \neq \frac{1}{2}). \] Let \( N \in \mathbb{N} \cup \{0\}, \nu : \{1\} \times \{1, \ldots, n\} \rightarrow \mathbb{N} \text{ and } x \in [0, 1 - Q] \) (\( x \in [0, 1 - Q] \text{ if } q \neq \frac{1}{2}\)). Putting

\[ z = q^{N + \sum_{k=1}^{K_1} \nu(1,k)} x - \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1} \nu(1,m)} = 1 \]

we have

\[ z \leq x - 1 \]

and, according to Lemma 2, \( y := z \in [Q - 1, 0] \) (and \( y \in (Q - 1, 0] \) if \( q < \frac{1}{2}\)). This jointly with the definition of \( z \), Lemma 1, (1), (17), (2), Remark 3 and (17) gives

\[ f(q^{N + \sum_{k=1}^{K_1} +1} \nu(1,k)) x - \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1} +1} \nu(1,m)) = \]

\[ = f(q^{\nu(1,K_1) +1} y) = \left( \frac{1}{2q} \right)^{\nu(1,K_1) +1} f(y) = \]

\[ = \left( \frac{1}{2q} \right)^{\nu(1,K_1) +1} \frac{1}{2} f(z + 1) = \]

\[ = \left( \frac{1}{2q} \right)^{\nu(1,K_1) +1} \frac{1}{2} \left( \frac{1}{2} \right)^{N + \sum_{k=1}^{K_1} \nu(1,k)} f(x) = \]

\[ = \left( \frac{1}{2q} \right)^{\nu(1,K_1) +1} \left( \frac{1}{2} \right)^{N + \sum_{k=1}^{K_1} \nu(1,k)} f(x). \]

Hence (17) holds for every \( K_1 \in \{1, \ldots, n\} \), for every non-negative integer \( N \), for every \( \nu : \{1\} \times \{1, \ldots, n\} \rightarrow \mathbb{N}, \) and for all \( x \in [0, 1 - Q] \) (for all \( x \in [0, 1 - Q] \) if \( q \neq \frac{1}{2}\)). Consequently, taking into account Remark 2 we have also

\[ f(q^{N + \sum_{k=1}^{K_1} \nu(1,k)} x + \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1} \nu(1,m)} = \frac{K_1}{2} \left( \frac{1}{2q} \right)^{N + \sum_{k=1}^{K_1} \nu(1,k)} f(x) \]

for every \( K_1 \in \{1, \ldots, n\} \), for every non-negative integer \( N \), \( \nu : \{1\} \times \{1, \ldots, n\} \rightarrow \mathbb{N}, \) and for all \( x \in (Q - 1, 0] \) (for all \( x \in [Q - 1, 0] \) if \( q \neq \frac{1}{2}\)).
Fix now a positive integer \( L \) and suppose that (16) holds with \( M \equiv 0 \) for every \( K_1, \ldots, K_L \in \{1, \ldots, n\} \), for every non-negative integer \( N, \nu : \{1, \ldots, L\} \times \{1, \ldots, n\} \to \mathbb{N} \), and for all \( x \in [0, 1 - Q] \) (for all \( x \in [0, 1 - Q] \) if \( q \neq \frac{1}{2} \)). Defining \( y \) as in Lemma 2 and making use of Lemma 2, (17) and (19) with \( x \) replaced by \( y \) and (16) with \( M \equiv 0 \) we obtain

\[
\begin{align*}
&f\left(q^{N+\sum_{i=1}^{L+1} \sum_{k_{i,1}} \nu(l,k_{i,1})} \right) + \\
&\quad \sum_{k_{i,1}} (-1)^i \sum_{k_{j,1}} q^{\sum_{i=1}^{L+1} \nu(l,m) + \sum_{j=1}^{L+1} \sum_{k_{j,1}} \nu(j,m)} = \\
&= f\left(q^{N+\sum_{i=1}^{L+1} \sum_{k_{i,1}} \nu(l+1,k_{i,1})} \right) + (-1)^{L+1} \sum_{k_{i,1}} q^{\sum_{i=1}^{L+1} \sum_{k_{i,1}} \nu(L+1,m)} = \\
&= \left(\frac{1}{2}\right)^{L+1} \frac{1}{2q} \sum_{k_{i,1}} f(y) = \\
&= \left(\frac{1}{2}\right)^{L+1} \frac{1}{2q} \sum_{k_{i,1}} f(x) = \\
&= \left(\frac{1}{2}\right)^{L+1} K_1 \frac{1}{2q} N^{L+1} \sum_{k_{i,1}} f(x).
\end{align*}
\]

This ends the proof of (16) in the case where \( M \equiv 0 \).

If \( M \) is a positive integer then defining once more \( y \) as in Lemma 2 and making use of this lemma, (8) with \( N \equiv 0 \) and \( x \) replaced by \( y \), and (16) with \( M \equiv 0 \) we get

\[
\begin{align*}
&f\left(q^{N+\sum_{i=1}^{L} \sum_{k_{i,1}} \nu(l,k_{i,1})+M} \right) + \\
&\quad \sum_{k_{i,1}} (-1)^i \sum_{k_{j,1}} q^{\sum_{i=1}^{L} \nu(l,m)+\sum_{j=1}^{L} \sum_{k_{j,1}} \nu(j,m)+M} = \\
&= f\left(q^M \right) + (-1)^L \sum_{m=1}^{M} q^m = \left(\frac{1}{2}\right)^{M} \left(\frac{1}{2q}\right) f(y) = \\
&= \left(\frac{1}{2}\right)^{L+1} K_1 + M \frac{1}{2q} \sum_{k_{i,1}} f(x). \quad \diamond
\end{align*}
\]
The fourth lemma is just [7; Lemma 1].

**Lemma 4.** Assume $q \in (0, \frac{1}{2})$. If a solution of Schilling’s problem vanishes either on the interval $(-q, 0)$ or on the interval $(0, q)$ then it vanishes everywhere.

**Proof of Theorem 1.** Suppose $f$ is a solution of Schilling’s problem bounded in a neighbourhood of a point $x_0 \in \text{cl} A_q^0$. We may (and we do) assume that $x_0$ is of the form (5), where $\varepsilon \in \{-1, 1\}$, $M, L$ are non-negative integers, $K_1, \ldots, K_L \in \{1, \ldots, n\}$, and $\nu : \{1, \ldots, L\} \times \{1, \ldots, n\} \to \mathbb{N}$. Moreover, according to Remark 2, we may (and we do) assume $\varepsilon = 1$.

If $x \in [0, 1 - Q]$ is fixed then the left-hand side of (16) is bounded with respect to $N$ whereas the right-hand side is bounded if $f(x) = 0$. This shows that $f$ vanishes on $[0, 1 - Q]$. Hence and from (10) it follows that $f$ vanishes, in particular, on $[0, q]$ which jointly with Lemma 4 proves that $f$ vanishes everywhere. \hfill \Box

To formulate a corollary accept the following definition.

**Definition 1.** Let $q \in (0, 1)$ and $x \in [-Q, Q]$. We say that $x \in B_q$ (resp. $x \in C_q$) if and only if the zero function is the only solution of Schilling’s problem which is bounded in a neighbourhood of $x$ (resp. continuous at $x$).

We will use also the following result of W. Förg-Rob; cf. [6; Theorems 20, 21, 23–26 and 28] and Remark 1.

If $q \in (0, 1)$ and $f$ is a solution of Schilling’s problem then

$$\text{supp } f \subset \left\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n : \varepsilon \in \{-1, 0, 1\}^{[N]} \right\},$$

and for every $q \in (0, \frac{1}{2}]$ the Schilling’s problem has a nonzero solution.

**Corollary 1.** If $q \in (0, \frac{1}{2}]$ then

$$B_q = C_q = \left\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n : \varepsilon \in \{-1, 0, 1\}^{[N]} \right\}.$$
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\[ \left\{ \sum_{n=1}^{\infty} \varepsilon(n)q^n : \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \right\} \subseteq B_0. \]

Applying Lemma 3 (formula (16) with \( x = 0 \) and Remark 2) and Remark 3 we obtain also the following result.

**Theorem 2.** If \( n \) is a positive integer and \( q \in (0, q_0] \) then any solution of Schilling's problem vanishes on the set \( A^n_0 \).


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**References**


