PSEUDO-ELLIPTIC INTEGRALS
AND THE VALUES OF THE
WEIERSTRASS $\zeta$-FUNCTION
AT TORSION POINTS

Daniel Mall

Mathematik Departement der ETH Zürich, ETH Zentrum, CH-8092 Zürich, Switzerland

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Abstract: A connection is established between early results of Abel and Tchebycheff on pseudo-elliptic integrals and a result of Baker concerning the values of the Weierstrass $\zeta$-function at torsion points.

1. Introduction

Let $E$ be an elliptic curve defined over a number field $K \subset \mathbb{C}$ and $\wp(z)$, $\wp'(z)$ the corresponding Weierstrass functions. We assume that $E$ is given through the usual parameterization

$\text{ParaE} : \quad \mathbb{C} \longrightarrow E$

$z \longmapsto (\wp(z), \wp'(z))$

such that

(1) $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$

where $g_2, g_3 \in K$. We denote the periods of $E$ with $\omega_1, \omega_2$ and the lattice generated by the periods with $\Lambda$.

Assume that there is a point $z_0 \in \mathbb{C}$ such that $\wp(z_0) \in E(K)$. Then it follows easily from the relation (1) that the values of all derivatives of $\wp(z)$ at $z_0$ lie in the field $K$. This is not correct for $\zeta(z_0)$, the
value of the Weierstrass $\zeta$-function, the primitive of $-\wp(z)$, at $z_0$. The value $\zeta(z_0)$ is transcendental by the theorem of Schneider (cf. [9]). However A. Baker proved the following fact (cf. [2], lemma 5): let $E$ be an elliptic curve defined over the number field $K \subset \mathbb{C}$ and let $z_0 = r_1 \omega_1 + + r_2 \omega_2$, $r_1, r_2 \in \mathbb{Q}$ be a torsion point of this curve, then the value $\zeta(z_0) = (r_1 \eta_1 + r_2 \eta_2)$ where $\eta_i = 2\zeta(\omega_i/2)$, $i = 1, 2$, denote the quasi-periods of $E$, is an element of $K$. In particular, there is a canonical splitting $\zeta(z_0) = T(z_0) + A(z_0)$ into a transcendental and an algebraic part.

The purpose of this note is to show how this splitting can be expressed in terms of a pseudo-elliptic integral and to derive an algorithm which allows the computation of $A(z_0)$ by the means of a continued fraction expansion of an appropriate algebraic function. Our exposition is structured as follows. Section 2 assembles some known results and describes the pseudo-elliptic integrals involved. Section 3 states the precise connection to Weierstrass' $\zeta$-function. In Section 4 the connection proved by Baker is re-established in the context of pseudo-elliptic integrals using an early result by Abel. In Section 5 we present a computed example.

2. The integrals

Let $\wp(z)$, $\zeta(z)$, $\sigma(z)$ be the Weierstrass elliptic functions belonging to the lattice $\Lambda$ generated by $\omega_1, \omega_2 \in \mathbb{C}$. Let us recall the following relations (cf. [6], p. 239):

\begin{equation}
2 \zeta(u + v) - \zeta(u) - \zeta(v) = \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)},
\end{equation}

\begin{equation}
(\log \sigma(u))' = \frac{\sigma'(u)}{\sigma(u)} = \zeta(u).
\end{equation}

Let an elliptic curve be given by the equation

\begin{equation}
y^2 = 4x^3 - g_2x - g_3,
\end{equation}

together with a point $z_0 \in \mathbb{C}$ such that $(x_0, y_0) = (\wp(z_0), \wp'(z_0)) \in E(K)$, i.e., is a $K$-rational point.

By means of the rational transformation
\[ \xi = \frac{1}{2} \frac{y - y_0}{x - x_0}, \quad \eta = 2x + x_0 - \frac{1}{4} \left( \frac{y - y_0}{x - x_0} \right)^2 \]

one obtains a singular model \( S_K \) defined over the field \( K \). We say this model is generated by the point \( z_0 \) and write sometimes \( S_K(z_0) \). This model is parameterized by the function

\[
\text{Para} S : \quad \mathbb{C} \rightarrow S_K(z_0) \\
z \mapsto (P(z), P'(z))
\]

where

\[
P(z) = \frac{1}{2} \frac{\varphi'(z) - \varphi'(z_0)}{\varphi(z) - \varphi(z_0)}.
\]

\( S_K(z_0) \) is the zero set of the equation

\[
\eta^2 = R_{z_0}(\xi) = \xi^4 + c_2 \xi^2 + c_3 \xi + c_4
\]

where

\[ c_2 = -6x_0, \quad c_3 = 4y_0, \quad c_4 = g_2 - 3x_0^2. \]

The singular locus of \( S_K \) is the image of the points \( 0, z_0 \in \mathbb{C} \) by \( \text{Para} S \), the points at infinity.

Elliptic integrals of the form

\[
\text{Int} = \int \frac{\xi + A}{\sqrt{R_{z_0}(\xi)}} \, d\xi
\]

which are defined on \( S_K \) were extensively studied in the last century (cf. [10], [11]): Tchebycheff was able to reduce the problem of integration in finite terms (cf. [4], [7]) for elliptic integrals to the question of pseudo-ellipticity of integrals of the form (6): \( \text{Int} \) is pseudo-elliptic if there exist \( p(\xi), q(\xi) \in \mathbb{C}[\xi] \) such that

\[
\text{Int} = \frac{1}{\lambda} \log \frac{p(\xi) - q(\xi) \sqrt{R_{z_0}(\xi)}}{p(\xi) + q(\xi) \sqrt{R_{z_0}(\xi)}}
\]

for some \( \lambda \in \mathbb{Z} \) (cf. [3]).

It is easy to see that for a given \( R_{z_0}(\xi) \) at most one value \( A \) exists such that (7) holds (cf. [10], p. 2).

We recall some facts about continued fraction expansion (cf. [5], p.84). Let \( \alpha_0 = \sum_{m \geq m_0} \gamma_m t^{-m} \) be the Laurent expansion of \( \sqrt{R_{z_0}} \) at a point \( p \) at infinity. One puts \( a_1 := [\alpha_0] := \sum_{0 \geq m \geq m_0} \gamma_m t^{-m} \) and \( a_i := [\alpha_{i-1}] \) with \( \alpha_i \) the Laurent expansion of \( \frac{1}{\alpha_{i-1} - a_{i-1}} \) for \( i \geq 1 \). The sequence \( \{a_i\}_{i=1}^{\infty} \) is called the continued fraction expansion of \( \alpha_0 \) at the point \( p \). One puts as usual
\[ P_0 = 1, \quad P_1 = a_1, \quad P_i = a_i P_{i-1} + P_{i-2} \]
\[ Q_0 = 0, \quad Q_1 = 1, \quad Q_i = a_i Q_{i-1} + Q_{i-2}. \]

The continued fraction expansion \( \{a_i\}_{i=1}^{\infty} \) is called pseudo-periodic if there exists a \( k \in \mathbb{N}^* \) such that
\[ P_k^2 - Q_k^2 R_{z_0} = c \in K. \]
The smallest \( k \in \mathbb{N}^* \) with this property is called the pseudo-period.

The following proposition summarizes various known results (cf. [8], p.296; [11], p.105; [5], p.90).

**Proposition 2.1.** The following conditions for \( \text{Int} = \int \frac{\xi+A}{\sqrt{R_{z_0}(\xi)}} d\xi \)
are equivalent

a) the continued fraction expansion of \( \sqrt{R_{z_0}} \) at one of the points at infinity (hence on both) is pseudo-periodic with pseudo-period \( l - 1 \);

b) there exists a value \( A \) such that (7) holds for appropriate \( p(\xi), q(\xi) \in K[\xi] \) of \( \deg p = l \) and \( \deg q = l - 2 \), with \( \lambda = 2l \);

c) \( z_0 \) is a torsion point of order \( l \).

**Remark 2.2.** If \( \text{Int} \) is pseudo-elliptic and \( z_0 \) is a torsion point of order \( l \) then \( p(\xi) = P_{l-1}(\xi) \) and \( q(\xi) = Q_{l-1}(\xi) \).

3. The splitting

**Proposition 3.1.** Let \( S_K \) be generated by the torsion point \( z_0 \) of order \( l \). If \( l z_0 = n_1 \omega_1 + n_2 \omega_2 \in \Lambda \), then \( \zeta(z_0) = T(z_0) + A(z_0) \) where \( l T(z_0) = n_1 \eta_1 + n_2 \eta_2 \), and \( A(z_0) = A \) where \( A \) is the unique value such that \( \int \frac{\xi+A}{\sqrt{R_{z_0}(\xi)}} d\xi \) is pseudo-elliptic.

**Proof.** Applying (2) we obtain by elementary calculations:
\[
\text{Int} = \int \left( \frac{1}{2} \frac{\varphi'(z)}{\varphi(z)} - \frac{\varphi'(z_0)}{\varphi(z_0)} + A \right) dz =
\]
\[
= \int \left( \zeta(z + z_0) - \zeta(z) - \zeta(z_0) + A \right) dz =
\]
\[
= \int \left( \zeta(z + z_0) - \zeta(z) \right) dz + (A - \zeta(z_0)) z.
\]
Hence putting
\( \Sigma(z) = \exp(2l \cdot \text{Int}) \),

(3) implies that

\[
\Sigma(z) = \exp(2l \cdot \int (\zeta(z + z_0) - \zeta(z))dz) \cdot \exp(2l \cdot (A - \zeta(z_0))z) = \\
= \frac{\sigma(z + z_0)^{2l}}{\sigma(z)^{2l}} \cdot \exp(2l \cdot (A - \zeta(z_0))z).
\]

Since \( z_0 \) is a torsion point, \( \Sigma(z) \) must be a doubly periodic function of \( z \) by Prop. 2.1: \( \Sigma(z + \omega) = \Sigma(z) \). This imposes a condition on the expression \( A - \zeta(z_0) \):

We have \( \sigma(z + \omega) = \sigma(z)e^{\pi z + c} \), where \( c \in \mathbb{C} \) (cf. [6]). Now

\[
\Sigma(z+\omega) = \frac{\sigma(z + \omega + z_0)^{2l}}{\sigma(z + \omega)^{2l}} \cdot \exp(2l \cdot (A - \zeta(z_0))(z + \omega)) = \\
= \frac{\sigma(z + z_0)^{2l} \exp(2l\eta(z + z_0) + 2l\omega)}{\sigma(z)^{2l} \exp(2l\eta z + 2l\omega)} \cdot \exp(2l \cdot (A - \zeta(z_0))(z + \omega)) = \\
= \Sigma(z) \cdot N
\]

where

\[ N = \exp(2l \cdot \eta \cdot z_0 + 2l \cdot (A - \zeta(z_0)) \cdot \omega). \]

This implies that \( N = 1 \). Hence there is a value \( m(\omega) \in \mathbb{Z} \) such that

(8) \[ \zeta(z_0) - A = \frac{l \cdot z_0 \cdot \eta - \pi \eta \cdot m(\omega)}{l \omega}. \]

We determine the number \( m(\omega) \): the expression \( \zeta(z_0) - A \) is independent of \( \omega \) and \( \eta \). We substitute \( \omega_1, \eta_1 \) and then \( \omega_2, \eta_2 \) in equation (8):

\[ \frac{l\omega_0 \eta_1 - \pi \eta \cdot m(\omega_1)}{l \omega_1} = \frac{l\omega_0 \eta_2 - \pi \eta \cdot m(\omega_2)}{l \omega_2}. \]

Applying the Legendre relation \( \eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i \) (cf. [6], p.241) we obtain

\[ 2l\omega_0 = m(\omega_1)\omega_2 - m(\omega_2)\omega_1. \]

This implies that \( n_1 = -m(\omega_2)/2, n_2 = m(\omega_1)/2 \)

and that

\[ \zeta(z_0) - A = \frac{(n_1 \omega_1 + n_2 \omega_2)\eta_1 - 2\pi i n_2}{l \omega_1} = \frac{n_1 \eta_1 + n_2 \eta_2}{l} = T(z_0). \]

\textbf{Example.} Let \( z_0 = \omega_1/2 \). Hence \( l = 2 \), and \( n_1 = 1, n_2 = 0 \). This implies that \( \zeta(\omega_1/2) - A = \eta_1/2 \). By definition \( \eta_1 = 2\zeta(\omega_1/2) \) and hence \( A = 0 \). This implies that the integral \( \int_{R_{\omega_1/2}(\xi)}^{\xi} \frac{d\xi}{\sqrt{R_{\omega_1/2}(\xi)}} \) on \( S_K(\omega_1/2) \) is pseudo-elliptic.
4. The algebraic part

The following proposition reveals the algebraic nature of $A(z_0)$ and re-establishes Baker's result.

**Proposition 4.1.** Let $E/K$ be an elliptic curve defined over $K \subset \mathbb{C}$ and $p_0 = (\varphi(z_0), \varphi'(z_0))$ a $K$-rational torsion point of $E$. Then $A(z_0) \in K$.

**Proof.** We expand $\sqrt{R_{z_0}(\xi)}$ at one of the places at infinity in a Laurent series. This series is an element of $K((t))$, where $t$ is a uniformizing parameter, since the coefficient of $\xi^4$ equals 1. We calculate the continued fraction expansion $\{a_i\}_{i=1}^\infty$ of the Laurent series. Assume that $z_0$ is a torsion point of order $l$. By Prop. 2.1 the continued fraction expansion is pseudo-periodic with pseudo-period $l - 1$. Hence

$$P_{l-1}^2 - Q_{l-1}^2 R_{z_0}$$

is a constant. Applying the following result of Abel (cf. [1], p. 106) about the connection between $P_{l-1}, Q_{l-1}$ and the nominator of the integrand we obtain finally

$$x + A = 2(P_{l-1}Q_{l-1} - Q_{l-1}P'_{l-1})R_{z_0} + P_{l-1}Q_{l-1}R'_{z_0} \in K[x],$$

and hence $A \in K$. \hfill \Box

**Remark 4.2.** By the theorem of Schneider mentioned in the introduction it follows now that the expression $T(z_0)$ is transcendental.

5. Examples

Baker's formula (cf. [2] p. 148) expresses $A$ in terms of $\varphi(mz_0)$, $\varphi'(mz_0)$, $m = 2, \ldots l - 1$, which can be computed from $g_2, g_3, \varphi(z_0)$. The proof of Prop. 4.1 yields a more local algorithm to compute the value $A(z_0)$ from the data $g_2, g_3, \varphi(z_0)$. We used this algorithm to calculate in the following example for some torsion points the corresponding pseudo-elliptic integrals, i.e., $c_2, c_3, c_4$ and $A(z_0)$.

Let the following elliptic curve over $\mathbb{Q}$ be given $E: y^2 = 4x^3 - 172x + 664$. Its rational Mordell-Weil group has a subgroup of order 7. Hence we find 6 torsion points $z_0 = (x, y) \neq \infty$ defined over $\mathbb{Q}$. The following table gives the coordinates of the torsion points, the coefficient of the corresponding equation of degree 4 for the singular model $S_K(z_0)$ and the algebraic part $A(z_0)$. 


\[ \begin{array}{|c|c|c|c|c|c|c|} \hline (x, y) & (3, 16) & (-5, -32) & (11, -64) & (11, 64) & (-5, 32) & (3, -16) \\ \hline c_2 & -18 & 30 & -66 & -66 & 30 & -18 \\ \hline c_3 & 64 & -128 & -256 & 256 & 128 & -64 \\ \hline c_4 & 145 & 97 & -191 & -191 & 97 & 145 \\ \hline A(z_0) & 1/7 & -5/7 & 17/7 & -17/7 & 5/7 & -1/7 \\ \hline \end{array} \]

Computations were performed using the symbolic computer algebra systems Mathematica and Maple.

References

[1] ABEL, N. H.: Sur l'intégration de la formule différentielle \( \rho dx/\sqrt{R} \), \( R \) et \( \rho \) étant des fonctions entières, in "Oeuvres complètes", Christiania 1881, p.104-144.


