AUTOMORPHISMS OF FREE NILPOTENT ASSOCIATIVE ALGEBRAS

Vesselin Drensky

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria

Dimitar Stefanov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria

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Abstract: In this paper we study the automorphism group of the free nilpotent algebra \( F_m(N_c) \) of class \( c \) and \( m \) generators over a field \( K \) of characteristic 0. Under some restrictions on \( m \) and \( c \), we have found three automorphisms which, together with the general linear group \( GL_m(K) \) generate the whole group of automorphisms. A similar problem has been considered by Drensky and C.K. Gupta for free nilpotent Lie algebras and free nilpotent metabelian Lie algebras.

Let \( K \) be a field of characteristic 0. Drensky and C.K. Gupta [3] have shown that the automorphism group of the free metabelian and nilpotent of class \( c \) Lie algebra \( L_m(N_c \cap A^2) \) of rank \( m \) is generated by

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the automorphisms induced by the action of the general linear group $GL_m(K)$ and by one more automorphism $\delta_1$ defined by $\delta_1(x_1) = x_1 + [x_1, x_2], \delta_1(x_j) = x_j, j > 1$. (Another proof of this result based on different techniques is given by Papistas [6].) As a consequence in [3] it has been established that for $m \geq c$ the automorphism group of the free nilpotent Lie algebra $L_m(N_c)$ is generated by $GL_m(K)$ and the same automorphism $\delta_1$. The approach in [3] is based on the representation theory of $GL_m(K)$. Using the same approach and the result of Anick [1] that the tame automorphisms of the polynomial algebra $K[x_1, \ldots, x_m]$ form a dense subgroup of the whole automorphism group with respect to the formal power series topology, one can show that the automorphism group of the free nilpotent of class $c$ commutative and associative algebra $F_m(A \cap N_c)$ is generated by $GL_m(K)$ and one more automorphism $\delta_2$ defined by $\delta_2(x_1) = x_1 + x_1 x_2, \delta_2(x_j) = x_j, j > 1$. In this paper we establish the following main result.

**Theorem.** Let $F_m(N_c)$ be the free nilpotent of class $c$ associative algebra freely generated by $x_1, \ldots, x_m$ over a field $K$ of characteristic 0, $m \geq 2$. Let $\delta_i, i = 1, 2, 3,$ be the automorphisms of $F_m(N_c)$ defined by

$$
\delta_1(x_1) = x_1 + [x_1, x_2], \delta_2(x_1) = x_1 + x_1 x_2, \delta_3(x_1) = x_1 + x_1^2 x_2, \delta_i(x_j) = x_j, j > 1.
$$

If $c = 3$ and $m$ is arbitrary, or if $m = 2$, $c \leq 7$ or if $m$ and $c$ satisfy the inequality $m(m-4) \geq c-2$, then Aut $F_m(N_c)$ is generated by $GL_m(K)$ and the automorphisms $\delta_1, \delta_2, \delta_3$.

Our proof is based on the same idea as in [3]. Some factors of the subgroup of Aut $F_m(N_c)$ generated by $GL_m(K)$ and $\delta_1, \delta_2, \delta_3$ have a natural structure of $GL_m(K)$-modules and we show that these factors are isomorphic to the corresponding factors of the whole automorphism group. The main difference of our approach from that in [3] is the following. The essential difficulty in [3] is to handle the case of the automorphisms of the free nilpotent metabelian Lie algebra. It turns out that the number of the irreducible $GL_m(K)$-submodules in all factors in consideration is bounded. In our case this number is not bounded and we have further developed the technique from [3].

The paper is organized as follows. In Section 1 we present necessary preliminaries, fix the notation and consider the simplest case of nilpotent of class 3 algebras. Section 2 is devoted to the automorphism group of the free nilpotent algebras in two generators and of class $\leq 7$. Finally, in Section 3 we handle the case $m > 2$ and $m(m-4) \geq c-2$. 

The results of our paper have been announced in [4]. Due to a technical error, in [4] we have imprecisely stated that $GL_m(K)$ and the automorphisms $\delta_j$, $j = 1, 2, 3$, generate Aut $F_m(N_c)$ for $m^2 \geq c - 1$. We still do not know whether Aut $F_m(N_c)$ is generated by $GL_m(K)$ and $\delta_j$, $j = 1, 2, 3$, for any $m$ and $c$.

1. Preliminaries

Throughout the paper we fix a field $K$ of characteristic 0 and all algebras, vector spaces and tensor products are over $K$. We also fix the set $\{x_1, \ldots, x_m\}$, $m \geq 2$, as a set of free generators of the free algebras.

We start with the necessary background on the representation theory of the general linear group $GL_m = GL_m(K)$. We follow the exposition from [3] and [2]. Let $V_m$ be the vector space with basis $x_1, \ldots, x_m$ and let $GL_m$ act canonically from the left on $V_m$. We consider the elements from $GL_m$ as invertible $m \times m$ matrices with entries from $K$. For an $s$-dimensional vector space $W$, also with a fixed basis, a homomorphism

$$\phi : GL_m \rightarrow GL_s = GL(W)$$

is called a polynomial representation of $GL_m$ if the entries $\phi_{pq}(g)$ of the $s \times s$ matrix $\phi(g)$ are polynomial functions of the entries $a_{ij}$ of the matrix $g = (a_{ij}) \in GL_m$. Similarly, if $\phi_{pq}(g)$ are rational functions of $a_{ij}$, then $\phi$ is called a rational representation. If the functions $\phi_{pq}$ are homogeneous of the same degree, the representation $\phi$ and the corresponding $GL_m$-module $W$ are called homogeneous. Let

$$D_m = \{ d \in GL_m \mid d = (z_1, \ldots, z_m) = z_1e_{11} + \ldots + z_me_{mm} \}$$

be the subgroup of the diagonal matrices of $GL_m$. For any degree sequence $\alpha = (\alpha_1, \ldots, \alpha_m)$ of length $m$ we define the $\alpha$-homogeneous component of the $GL_m$-module $W$ by

$$W^\alpha = \{ w \in W \mid d(z_1, \ldots, z_m)w = z_1^{\alpha_1} \ldots z_m^{\alpha_m}w, d(z_1, \ldots, z_m) \in D_m \}.$$

We shall use the following well known statements for the finite dimensional rational representation $\phi : GL_m \rightarrow GL(W)$.

1. The $GL_m$-module $W$ is completely reducible and is a direct sum of its homogeneous submodules.
2. As a $K$-vector space, $W$ is a direct sum of its homogeneous components $W^\alpha$.
3. The Hilbert series of $W$

$$H(W) = H(W, t_1, \ldots, t_m) = \sum (\dim_K W^\alpha) t_1^{\alpha_1} \ldots t_m^{\alpha_m}$$

is a symmetric function in $t_1, \ldots, t_m$. 
4. If \((\det)^n, n \in \mathbb{Z} \setminus \{0\}\) is the one-dimensional representation of \(GL_m\) defined by
\[
(\det)^n(g) = (\det g)^n,
\]
then every rational \(GL_m\)-module has the form \(W\) or \((\det)^{-n} \otimes W\), where \(n \in \mathbb{N}\) and \(W\) is a polynomial \(GL_m\)-module; \((\det)^{-n} \otimes W\) is irreducible if and only if \(W\) is irreducible. The Hilbert series of \(\det^n\) is
\[
H(\det^n) = \left( \prod_{i=1}^{m} t_i \right)^n.
\]

The irreducible polynomial representations of \(GL_m\) are described by partitions and Young diagrams (see e.g. [5, 7]). For a partition \(\lambda = (\lambda_1, \ldots, \lambda_m), \lambda_1 \geq \ldots \geq \lambda_m \geq 0, \lambda_1 + \ldots + \lambda_m = n\), we consider the corresponding Young diagram \([\lambda]\) and the related \(GL_m\)-module \(W_m(\lambda)\). A \(\lambda\)-tableau with content \(\alpha = (\alpha_1, \ldots, \alpha_m)\) is a diagram \([\lambda]\) whose boxes are filled in with \(\alpha_1\) numbers \(1, \ldots, \alpha_m\) numbers \(m\). A tableau is semistandard if its entries do not decrease from left to right in the rows and increase from top to bottom in the columns. The Hilbert series of the module \(W_m(\lambda)\) is equal to the Schur function \(S_\lambda(t_1, \ldots, t_m)\),
\[
H(W_m(\lambda), t_1, \ldots, t_m) = S_\lambda(t_1, \ldots, t_m) = \sum \dim_K(W_m(\lambda)) \alpha_1^{\alpha_1} \cdots \alpha_m^{\alpha_m},
\]
where the coefficient \(a_\alpha = \dim_K(W_m(\lambda))\) is equal to the number of the semistandard \(\lambda\)-tableaux of content \(\alpha = (\alpha_1, \ldots, \alpha_m)\).

Let \(N_c\) be the variety of all nilpotent of class \(\leq c\) associative algebras. By definition, this is the class of all algebras satisfying the polynomial identity \(x_1 \ldots x_c = 0\). We fix the notation
\[
F = F_m(N_c)
\]
for the free nilpotent of class \(c\) algebra of rank \(m\). If \(A_m = K\langle x_1, \ldots, x_m \rangle\) is the free associative nonunitary algebra of rank \(m\), then \(F \cong A_m/\langle x_1 \ldots x_m \rangle\). Every map \(\varepsilon : x_s \to F\), \(s = 1, \ldots, m\), is uniquely extended to an endomorphism \(\phi \in \text{End } F\) of the algebra \(F\). Since \(F\) is nilpotent, \(\phi\) is an automorphism if and only if it induces an invertible linear map on the vector space \(F/F^2\). Therefore, an endomorphism \(\phi \in \text{End } F\) is an automorphism when there exists a \(g \in GL_m\) (considered as an automorphism of \(F\)) such that \(g^{-1} \circ \phi(x_s) = x_s + u_s, u_s \in F^2, s = 1, \ldots, m\). We consider the descending series of normal subgroups of \(\text{Aut } F\)
\[
I_k A = \{ \phi \in \text{Aut } F \mid \phi(x_s) = x_s + u_s, u_s \in F^k, s = 1, \ldots, m \}.
\]
Clearly
\[ I_2A > I_3A > \ldots > I_cA = \langle \text{id} \rangle. \]

The factor groups \( I_kA/I_{k+1}A \) are abelian and \( \text{Aut } F = GL_m.I_2A \), i.e. \( \text{Aut } F \) is a split extension of \( GL_m \) and \( I_2A \). Let \( F^{(k)} \) be the homogeneous component of degree \( k \) of the algebra \( F \). The group \( GL_m \) acts diagonally on \( F^{(k)} \) by

\[ g(a(x_1, \ldots, x_m)) = a(g(x_1), \ldots, g(x_m)), \quad g \in GL_m, a \in F^{(k)}. \]

We define a map

\[ \theta_k : I_kA/I_{k+1}A \to (F^{(k)})^\oplus m = F^{(k)} \oplus \ldots \oplus F^{(k)} \]

in the following way. If \( \phi \in I_kA \) and \( \bar{\phi} = \phi I_{k+1}A \), where \( \phi(x_s) = x_s + a_s + b_s, \quad a_s, b_s \in F^{(k)}, \quad b_s \in F^{k+1}, \quad s = 1, \ldots, m \), then

\[ \bar{\phi} = \theta_k(\phi) = (a_1, \ldots, a_m) \in (F^{(k)})^\oplus m. \]

We do not write the index \( k \) of \( \theta_k \) if it is clear from the context. By [3] \( \theta \) is an isomorphism of the multiplicative group \( I_kA/I_{k+1}A \) and the additive group \( (F^{(k)})^\oplus m \) and, if \( \bar{\phi} \in I_kA/I_{k+1}A \) we write \( \bar{\phi}(x_s) = x_s + a_s, \quad s = 1, \ldots, m \). The group \( I_kA/I_{k+1}A \) admits also the multiplication with elements from the base field \( K \). If \( \bar{\phi}(x_s) = x_s + a_s, \quad s = 1, \ldots, m \), and \( \nu \in K \), then

\[ (\nu \bar{\phi})(x_s) = x_s + \nu a_s, \quad s = 1, \ldots, m, \]

and in this way \( I_kA/I_{k+1}A \) has a natural structure of a \( K \)-vector space. The addition is replaced by the composition of automorphisms and the multiplication by scalars as defined above. Since \( \theta(\nu \bar{\phi}) = \nu(\theta(\bar{\phi})) \), the group isomorphism \( \theta \) is also an isomorphism of vector spaces and very often we shall identify \( I_kA/I_{k+1}A \) and \( (F^{(k)})^\oplus m \) via \( \theta \). The groups \( I_kA \) are normal and they are invariant under the conjugation by elements from \( GL_m \). It turns out that the action

\[ g \circ \bar{\phi} = (g \phi g^{-1})I_{k+1}A, \quad g \in GL_m, \phi \in I_kA, \]

equips the vector space \( I_kA/I_{k+1}A \) with the structure of a \( GL_m \)-module. This action induces a \( GL_m \)-module structure also on \( (F^{(k)})^\oplus m \) described as follows.

**Proposition 1.1** [3]. Let \( \bar{\phi} \in I_kA/I_{k+1}A \), \( \theta(\bar{\phi}) = (a_1, \ldots, a_m) \in (F^{(k)})^\oplus m \). Considering \( \theta(\bar{\phi}) \) and \( g \in GL_m \) as a row vector and an \( m \times m \) matrix, respectively, it holds

\[ g \circ (a_1, \ldots, a_m) = (g(a_1), \ldots, g(a_m))(b_{ij}), \]

where \( (b_{ij}) = g^{-1} \).

**Example 1.2.** Let \( u = x_{p_1} \ldots x_{p_k} \), \( \deg x_j u = \alpha_j, j = 1, \ldots, m \), and let \( \phi \in I_kA \) be defined by \( \phi(x_s) = x_s + u, \phi(x_i) = x_i, i \neq s \). If \( d = d(z_1, \ldots, z_m) \) is a diagonal matrix from \( GL_m \), then
\[ d \circ \tilde{\phi} = z_1^{\alpha_1} \ldots z_m^{\alpha_m} z_s^{-1} \tilde{\phi}, \]
i.e. the element \( \theta(\tilde{\phi}) \) of the \( GL_m \)-module \( \theta(I_k A/I_{k+1} A) = (F^{(k)})^{\otimes m} \) is homogeneous of degree \((\alpha_1, \ldots, \alpha_{s-1}, \alpha_s-1, \alpha_{s+1}, \ldots, \alpha_m)\).

Similarly, the element \((f_1, \ldots, f_m) \in (F^{(k)})^{\otimes m}\) is homogeneous of degree \( \alpha = (\alpha_1, \ldots, \alpha_m) \) if every polynomial \( f_i \in F^{(k)}, \ i = 1, \ldots, m, \)
is homogeneous of degree \( \alpha_i + 1 \) in \( x_i \) and \( \alpha_j \) in the other variables \( x_j, j \neq i. \) For example, for \( m = k = 3, \) the elements \((x_1^2 x_2, 0, 0)\) and \((x_1 x_2 x_1, x_1 x_2^2, x_1 x_3 x_2)\) are homogeneous of degree \((1, 1, 0)\).

One of the key observations in \([3, 2]\) is the \( GL_m \)-module isomorphism
\[ I_k A/I_{k+1} A \cong \det^{-1} \otimes W_m(1^{m-1}) \otimes F^{(k)}. \]
The \( GL_m \)-module decomposition of the homogeneous component of degree \( k < c \) of the free nilpotent of class \( c \) algebra is the same as that of the homogeneous component of the free associative algebra (or of the \( k \)-th tensor power of \( V_m \)), i.e.
\[ F^{(k)} \cong \sum d_{\lambda} W_m(\lambda), \]
where the summation runs on all partitions \( \lambda = (\lambda_1, \ldots, \lambda_m) \) of \( k. \)
Since for \( \lambda = (\lambda_1, \ldots, \lambda_m), \lambda_m \geq 1, \)
\[ \det^{-1} \otimes W_m(\lambda) \cong W_m(\lambda_1 - 1, \ldots, \lambda_m - 1), \]
as in \([3]\) we can use the Young rule in order to calculate the multiplicities of the irreducible components of \( I_k A/I_{k+1} A. \) In particular, we obtain

**Corollary 1.3.** The \( GL_m \)-module \( I_k A/I_{k+1} A \) is isomorphic to a direct sum of irreducible \( GL_m \)-modules of the following kind:

(i) \( W_m(\mu), \) where \( \mu = (\mu_1, \ldots, \mu_m) \) is a partition of \( k-1; \)
(ii) \( \det^{-1} \otimes W_m(\mu'), \) where \( \mu' = (\mu'_1, \ldots, \mu'_{m-1}, 0) \) is a partition of \( k + m - 1. \)

Let \( \delta_i, i \in I, \) be automorphisms of \( F, \) where \( I \) is a set of indices, and let
\[ G = \langle GL_m, \delta_i \mid i \in I \rangle \]
be the subgroup of \( \text{Aut} F_m(N_c) \) generated by \( GL_m \) and \( \delta_i, i \in I. \)

**Proposition 1.4** \([3]\). Let \( G \subset (F^{(c-1)})^{\otimes m} \) be the image of \((G \cap \cap I_{c-1} A)/(G \cap I_c A) \) \((\cong G \cap I_{c-1} A \text{ because } I_c A = \langle \text{id} \rangle) \) under the isomorphism \( \theta. \) Then \( G \) is a \( GL_m \)-submodule of \((F^{(c-1)})^{\otimes m}.\)

The following assertion follows easily by induction as in the proof of \([3, \text{ Th. } 3.7]\). Since we consider nilpotent algebras, it is equivalent to \([2, \text{ Cor. } 2.9]\).

**Corollary 1.5** \([3, 2]\). Let \( G = \langle GL_m, \delta_i \mid i \in I \rangle \subseteq \text{Aut} F_m(N_c) \) and let
\[(G \cap I_k A) / (G \cap I_{k+1} A) = I_k A / I_{k+1} A, \quad k = 2, \ldots, c - 1.\]

Then the groups \(G\) and \(\text{Aut} F_m(N_c)\) coincide.

We shall use the following technical assertion which is a consequence of Prop. 1.4 and the fact that every rational \(GL_m\)-module is a direct sum of its homogeneous \(K\)-vector spaces.

**Corollary 1.6.** In the notation of Prop. 1.4 let \(\bar{\phi} = \bar{\phi}_1 + \ldots + \bar{\phi}_s \in \bar{\mathcal{G}},\)
where \(\bar{\phi}_j, j = 1, \ldots, s,\) are the homogeneous components of \(\bar{\phi}\) in the \(GL_m\)-module \(\bar{\mathcal{G}}.\) Then \(\bar{\phi} \in \bar{\mathcal{G}}, j = 1, \ldots, s.\)

**Definition 1.7.** Let \(W_i, i \in I,\) be the irreducible components of the rational \(GL_m\)-module \(W.\) We call the content \(\alpha = (\alpha_1, \ldots, \alpha_m)\) essential if all homogeneous components \(W^\alpha_i, i \in I,\) are nonzero.

Clearly, if \(\alpha\) is an essential content for the \(GL_m\)-module \(W\) and \(W_1\) is a submodule of \(W,\) then the equality
\[
\dim_K W_1^\alpha = \dim_K W^\alpha
\]
implies that \(W_1 = W.\)

**Lemma 1.8.** Let \(\lambda = (\lambda_1, \ldots, \lambda_m)\) be a partition of \(s, s = pm + r, 0 \leq r < m.\) Then the content \(\alpha = (p + \epsilon_1, \ldots, p + \epsilon_m)\) is essential for the \(GL_m\)-module \(W_m(\lambda),\) where \(\epsilon_1 = \ldots = \epsilon_r = 1, \epsilon_{r+1} = \ldots = \epsilon_m = 0.\)

**Proof.** It is sufficient to show that there exists a semistandard \(\lambda\)-tableau of content \(\alpha.\) We use induction on \(s\) and \(m.\) The base of the induction \(m = 1, s\) any and \(s = 1\) and \(m\) any is obviously true. We assume that the statement holds for all \(m_0 \leq m\) and \(s_0 < s.\)

**Case 1.** Let \(\lambda_m \neq 0.\) If \(T\) is a semistandard \(\lambda\)-tableau, then each of the first \(\lambda_m\) columns of the diagram \([\lambda]\) is filled in with the integers \(1, \ldots, m.\) We delete these columns and reduce the problem to the existence of a semistandard \(\nu\)-tableau of content \(\beta,\)

\[\nu = (\lambda_1 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0), \quad \beta = (\alpha_1 - \lambda_m, \ldots, \alpha_m - \lambda_m).\]

Since \(\nu\) is a partition of \(s - m\lambda_m < s,\) we use the inductive assumption and obtain that \(\alpha\) is an essential content for \(W_m(\lambda).\)

**Case 2.** Let \(\lambda_m = 0.\) We consider all the boxes of the diagram \([\lambda]\) which are in the bottoms of the columns of \([\lambda]\). The number of these boxes is \(\lambda_1 \geq p\) and we fill in \(p\) of them with \(m.\) We delete these \(p\) boxes and obtain a new diagram \([\nu].\) In order to make the inductive step we have to construct a semistandard \(\nu\)-diagram of content \(\beta = (\alpha_1, \ldots, \alpha_{m-1}, 0).\) Since \(\nu = (\nu_1, \ldots, \nu_{m-1}, 0)\) is a partition of \(s - p < s,\) and \(s - p = p(m - 1) + r\) if \(r < m - 1,\) or \(s - p = (p + 1)(m - 1)\) if \(r = m - 1,\) by the inductive arguments the content \((\alpha_1, \ldots, \alpha_{m-1})\) is essential for the \(GL_{m-1}\) module \(W_{m-1}(\nu_1, \ldots, \nu_{m-1}).\) \(\Diamond\)
Corollary 1.9. Let $k - 1 = pm + r$, $0 \leq r < m$. Then the content
\[ \alpha = (p + \varepsilon_1, \ldots, p + \varepsilon_m) \]
is essential for the $GL_m$-module $I_k A/I_{k+1} A$, where $\varepsilon_1 = \ldots = \varepsilon_r = 1$, $\varepsilon_{r+1} = \ldots = \varepsilon_m = 0$.

Proof. By Cor. 1.3 the $GL_m$-module $I_k A/I_{k+1} A$ is a direct sum of irreducible submodules $W_m(\mu)$, where $\mu = (\mu_1, \ldots, \mu_m)$ is a partition of $k - 1$, and $\det^{-1} \otimes W_m(\mu')$, where $\mu' = (\mu'_1, \ldots, \mu'_{m-1}, 0)$ is a partition of $k + m - 1$. The submodules $W_m(\mu)$ are covered directly by Lemma 1.8. In order to prove that the homogeneous component of degree $\alpha$ of $W = \det^{-1} \otimes W_m(\mu')$ is not zero, it is sufficient to show that the coefficient $a_\alpha$ in the Hilbert series of $W$

\[ H(W) = \sum a_\gamma t_1^{\gamma_1} \ldots t_m^{\gamma_m} \]
is different from 0. If

\[ H(W_m(\mu')) = \sum b_\beta t_1^{\beta_1} \ldots t_m^{\beta_m} \]
is the Hilbert series of $W(\mu')$, then

\[ H(W) = (t_1 \ldots t_m)^{-1} H(W_m(\mu')). \]

Therefore, $a_\alpha = b_{\alpha'}$, where $\alpha' = (\alpha_1 + 1, \ldots, \alpha_m + 1)$. Again, the proof follows directly from Lemma 1.8.

Remark 1.10. Let $\sigma \in S_m$ be an element from the symmetric group $S_m$ of degree $m$ and let $g_\sigma \in GL_m$ be defined by $g_\sigma(x_i) = x_{\sigma(i)}$, $i = 1, \ldots, m$. If $w \in W = (F^{(k)})^\otimes m$ and $w = (f_1, \ldots, f_m) \in W^\alpha$, then direct verification shows that $g_\sigma \circ w \in W^\beta$, where $\beta = (\beta_1, \ldots, \beta_m)$ is obtained by permuting of $\alpha = (\alpha_1, \ldots, \alpha_m)$ by $\sigma^{-1}$. We shall use this statement only in the case when $\sigma$ is a transposition.

Proposition 1.11. The automorphism group of the free nilpotent of class 3 algebra $F_m(N_3)$ of any rank $m$ is generated by $GL_m$ and the automorphisms $\delta_1$ and $\delta_2$ defined by

\[ \delta_1(x_1) = x_1 + x_1^2, \quad \delta_2(x_1) = x_1 + [x_1, x_2], \]

\[ \delta_j(x_i) = x_i, \quad i = 2, \ldots, m, \quad j = 1, 2. \]

Proof. Let $G$ be the subgroup of $Aut F_m(N_3)$ generated by $GL_m$, $\delta_1$ and $\delta_2$, and let $\tilde{G}$ be the image under $\theta$ of $(G \cap I_2 A)/(G \cap I_3 A) \cong G \cap I_2 A$. If $W = \theta(I_2 A/I_3 A) = (F^{(2)})^\otimes m$, by Corollaries 1.5 and 1.9, it is sufficient to show that $\tilde{G}^\alpha = W^\alpha$ for $\alpha = (1, 0, \ldots, 0)$. The homogeneous component $W^\alpha$ has a basis

\[ \{(x_1^2, 0, \ldots, 0), (0, \ldots, 0, x_1 x_i, 0, \ldots, 0), \]

\[ (0, \ldots, 0, x_i x_1, 0, \ldots, 0) \mid i = 2, \ldots, m, \}

where $x_1 x_i$ and $x_i x_1$ are in the $i$-th position. We shall complete the
proof in two steps.

Step 1. Clearly, \( \tilde{\delta}_1 \in \tilde{G} \). Let \( g_1 \in GL_m \) be defined by
\[
g_1(x_1) = x_1 + x_i, \quad g_1(x_s) = x_s, \; s \neq 1.
\]
Then \( g_1^{-1}(x_1) = x_1 - x_i, \quad g_1^{-1}(x_s) = x_s, \; s \neq 1 \), and direct calculations show that
\[
(g_1 \circ \delta_1)(x_1) = g_1 \delta_1 g_1^{-1}(x_1) = x_1 + (x_1 + x_i)^2, \quad (g_1 \circ \delta_1)(x_s) = x_s, \; s \neq 1.
\]
By Cor. 1.6 the homogeneous components of \( g_1 \circ \delta_1 \) also belong to \( \tilde{G} \).
Considering the component of degree \((0, \ldots, 0, 1, 0, \ldots, 0)\), where 1 is in the \( i \)-th position, we obtain that \( \tilde{\psi} = (x_1 \circ x_i, \ldots, 0) \in \tilde{G} \), where by definition \( x_1 \circ x_i = x_1 x_i + x_i x_1 \). Let \( \sigma = (1i) \in S_m \) and let \( g_\sigma \in GL_m \) be as in Remark 1.10. Then
\[
\tilde{\eta}_i = g_\sigma \circ \tilde{\psi} = (0, \ldots, 0, x_1 \circ x_i, 0, \ldots, 0) \in \tilde{G}.
\]

Step 2. We start with \( \tilde{\delta}_2 = ([x_1, x_2], 0, \ldots, 0) \in \tilde{G} \). For \( \tau = (12) \in S_m \), \( g_\tau \in GL_m \), we obtain
\[
g_\tau \circ \delta_2(x_2) = x_2 - [x_1, x_2], \quad g_\tau \circ \delta_2(x_s) = x_s, \; s \neq 2,
\]
\( \tilde{\pi} = g_\tau \circ \tilde{\delta}_2 = -(0, [x_1, x_2], 0, \ldots, 0) \in \tilde{G} \).
For \( \rho = (2i) \in S_m \), \( g_\rho \in GL_m \) we calculate
\[
\tilde{\zeta}_i = g_\rho \circ \tilde{\pi} = -(0, \ldots, 0, [x_1, x_i], 0, \ldots, 0) \in \tilde{G}.
\]
Since \( \tilde{\delta}_1, \tilde{\eta}_i \) and \( \tilde{\zeta}_i, i = 2, \ldots, m \), belong to \( \tilde{G} \) and form a basis for the whole homogeneous component of degree \( \alpha = (1, 0, \ldots, 0) \) of \( W \), we obtain that \( \tilde{G}^\alpha = W^\alpha \) and this implies that \( \tilde{G} = W \). \( \diamond \)

The following technical lemma is useful in the further computations. It is a restatement of [2, Lemma 2.3, Remark 2.4].

**Lemma 1.12.** Let \( \phi \in I_k A \) and \( \psi \in I_l A \) be automorphisms of \( F = F_m(N_c), \; k + l \leq c \). Then the group commutator \( (\phi, \psi) = \phi^{-1} \psi^{-1} \phi \psi \) is in \( I_{k+l-1} A \). If
\[
\tilde{\phi} = (f_1, \ldots, f_m) \in (F^k)^{\otimes m}, \quad \tilde{\psi} = (g_1, \ldots, g_m) \in (F^l)^{\otimes m},
\]
then \( \theta((\phi, \psi)I_{k+l} A) = (h_1, \ldots, h_m) \in (F^{(k+l-1)})^{\otimes m} \) satisfies
\[
h_i = g_i(x_1 + f_1, \ldots, x_m + f_m)(k+l-1) + f_i(x_1 - g_1, \ldots, x_m - g_m)(k+l-1),
\]
where we denote by \( u(s) \) the homogeneous component of degree \( s \) of \( u \in F \).

For simplicity of notation, when \( k \) and \( l \) for \( \phi \in I_k A \) and \( \psi \in I_l A \) are clear from the context, we shall write \( (\phi, \psi) \) instead of \( \theta_{k+l-1}((\phi, \psi)I_{k+l} A) \).

**Remark 1.13.** In the notation of Lemma 1.12, let \( \Delta_{\phi} \) be the derivation of \( F \) defined by \( \Delta_{\phi} x_i = f_i, \; i = 1, \ldots, m \); we define \( \Delta_\psi \) in a similar way. Then
\[(\bar{\phi}, \bar{\psi}) = (\Delta_\phi g_1 - \Delta_\psi f_1, \ldots, \Delta_\phi g_m - \Delta_\psi f_m).\]

**Example 1.14.** Let \( m = 2, c > 3, \) and let \( \phi, \psi \in I_2 A \) be such that 
\[\bar{\phi} = (x_1^2, 0), \bar{\psi} = (x_1 \circ x_2, 0) \in F^{(2)} \oplus F^{(2)}.
\]
Then, modulo \( F^4 \),
\[(\phi, \psi)(x_1) = x_1 - ((x_1 \circ x_2)x_1 + x_1(x_1 \circ x_2)) + x_1 \circ x_2 = x_1 - 2x_1x_2x_1, (\phi, \psi)(x_2) = x_2.
\]
Hence
\[(\bar{\phi}, \bar{\psi}) = (2x_1x_2x_1, 0) \in F^{(3)} \oplus F^{(3)}.
\]
Similarly, for \( \bar{\phi} = (x_1^2, 0), \bar{\psi} = (0, x_1^2) \),
\[(\bar{\phi}, \bar{\psi}) = (x_2 \circ x_1^2, -x_1 \circ x_2^2).
\]

2. Nilpotent algebras with two generators

Till the end of the paper we fix the notation \( \delta_1, \delta_2, \delta_3 \) for the automorphisms of the algebra \( F = F_m(N_c) \) defined by
\[
\delta_1(x_1) = x_1 + x_1^2, \quad \delta_3(x_1) = x_1 + x_1^2 x_2, \quad \delta_2(x_1) = x_1 + [x_1, x_2],
\]
\[
\delta_j(x_i) = x_i, \quad i \neq 1, \quad j = 1, 2, 3.
\]
Especially in this section we denote by \( G = \langle GL_2, \delta_1, \delta_2, \delta_3 \rangle \) the subgroup of \( \text{Aut} F \) generated by \( GL_2 \) and \( \delta_j, j = 1, 2, 3 \). Here \( F = F_2(N_c), \)
\[
c = 4, 5, 6, 7.
\]

**Proposition 2.1.** The group \( \text{Aut} F_2(N_4) \) is generated by \( GL_2 \) and \( \delta_j, j = 1, 2, 3 \).

**Proof.** Let \( W = \theta_3(I_3 A / I_4 A) = F^{(3)} \oplus F^{(3)} \) and let \( \tilde{G} = \theta_3(G \cap I_3 A) / (G \cap I_4 A) \subseteq W \). In virtue of Cor. 1.9 the proof of the proposition will be completed if we establish that \( \tilde{G}^\alpha = W^\alpha \) for the essential content \( \alpha = (1, 1) \). The following elements form a \( K \)-basis for the homogeneous component \( W^{(1,1)} \)
\[
e_1 = (x_1 x_2 x_1, 0), \quad e_2 = (x_1^2 x_2, 0), \quad e_3 = (x_2 x_1^2, 0),
\]
\[
e_4 = (0, x_2 x_1 x_2), \quad e_5 = (0, x_2^2 x_1), \quad e_6 = (0, x_1 x_2^2).
\]
If \( \sigma = (12) \in S_2 \), then the element \( g_\sigma \in GL_2 \) defined in Remark 1.10 satisfies
\[
g_\sigma \circ e_i = e_{i+3}, \quad i = 1, 2, 3.
\]
We shall show that \( e_1, e_2, e_3 \in \tilde{G} \) in several steps.

**Step 1.** Let \( \phi = (x_1^2, 0), \psi = (x_1 \circ x_2, 0) \in F^{(2)} \oplus F^{(2)} \). By Prop. 1.11 there exist automorphisms \( \phi \) and \( \psi \) in \( G \) such that their images in \( \theta : I_2 A / I_3 A \to F^{(2)} \oplus F^{(2)} \) coincide respectively with \( \phi \)
and \(\tilde{\psi}\). In Ex. 1.14 we have seen that the image \((\tilde{\phi}, \tilde{\psi})\) in \(W\) of group commutator \((\phi, \psi) = 2(x_1 x_2 x_1, 0)\), i.e. \(e_1 \in \tilde{G}\). Therefore, \(e_4\) also belong to \(\tilde{G}\).

**Step 2.** Let \(g \in GL_2, g(x_1) = -2x_1, g(x_2) = x_2\). We calculate for \(\varepsilon = g \circ \delta_1\) that
\[
\varepsilon(x_1) = x_1 - 2x_1^2, \quad \delta_1^2(x_1) = x_1 + 2x_1^2 + 2x_1^3, \quad \varepsilon(x_2) = x_2, \quad \delta_1^2(x_2) = x_2,
\]
\[
(\varepsilon \circ \delta_1^2)(x_1) = x_1 - 6x_1^3, \quad (\varepsilon \circ \delta_1^2)(x_2) = x_2,
\]
i.e. \((x_1^3, 0) \in \tilde{G}\).

Now, let \(g_1 \in GL_2, g_1(x_1) = x_1 + x_2, g_1(x_2) = x_2\). Then
\[
g_1 \circ (x_1^3, 0) = ((x_1 + x_2)^3, 0) \in \tilde{G}.
\]
Considering the \((1, 1)\)-homogeneous component of this element, which is also in \(\tilde{G}\), we obtain \(e_1 + e_2 + e_3 \in \tilde{G}\). Since \(e_1, e_2 \in \tilde{G}\), this gives that \(e_3 \in \tilde{G}\). In this way, all basis elements \(e_j, j = 1, \ldots, 6\), of the \(K\)-vector space \(W^{(1, 1)}\) are in \(\tilde{G}^{(1, 1)}\) and this implies that \(\tilde{G} = W\). ∎

In the following considerations \(m\) and \(c\) are arbitrary and \(F = F_m(N_c)\). Let \(*\) be the involution of \(F\) given by
\[
(x_{i_1} \ldots x_{i_k})^* = x_{i_k} \ldots x_{i_1}.
\]
For an automorphism \(\phi\) of \(F\) we denote by \(\phi^*\) the automorphism defined by \(\phi^*(x_j) = (\phi(x_j))^*\). Clearly, \(*\) induces a \(K\)-vector space automorphism of \(F^{(k)}, k = 1, 2, \ldots, c - 1\), and if \(\phi \in I_k A\), then \(\phi^* \in I_k A\).

**Lemma 2.2.** If \(\phi \in I_k A, \psi \in I_l A\), then for the group commutators of \(\phi\) and \(\psi\) it holds
\[
\theta_{k+l-1}((\phi, \psi)^*) = \theta_{k+l-1}(\phi^*, \psi^*).
\]

**Proof.** Let \(\zeta = (\phi, \psi) \in I_{k+l-1} A,\)
\[
\tilde{\phi} = \theta_k(\phi) = (f_1, \ldots, f_m) \in (F^{(k)})^{\otimes m},
\]
\[
\tilde{\psi} = \theta_l(\phi) = (g_1, \ldots, g_m) \in (F^{(l)})^{\otimes m}.
\]
By Lemma 1.12, the coordinates \(h_i\) of
\[
\tilde{\zeta} = \theta_{k+l-1}(\zeta) = (h_1, \ldots, h_m) \in (F^{(k+l-1)})^{\otimes m}
\]
are equal to the homogeneous components of degree \(k + l - 1\) of
\[
g_i(x_1 + f_1, \ldots, x_m + f_m) + \tilde{f}_i(x_1 - g_1, \ldots, x_m - g_m).
\]
Since \(*\) is an involution of \(F\), we obtain
\[
g_i^*((x_1 + f_1)^*, \ldots, (x_m + f_m)^*) + \tilde{f}_i^*((x_1 - g_1)^*, \ldots, (x_m - g_m)^*) = h_i^*;
\]
and this completes the proof. ∎

**Proposition 2.3.** The group \(\text{Aut} F_2(N_3)\) is generated by \(GL_2\) and \(\delta_j, j = 1, 2, 3\).
**Proof.** We repeat the scheme and the notation of the proof of Prop. 2.1. By Cor. 1.9, the content \( \alpha = (2,1) \) is essential for the \( GL_2 \)-module \( W = \theta_4(I_4A/I_5A) = F^{(4)} \oplus F^{(4)} \) and it is sufficient to show that \( \tilde{G}^\alpha = W^\alpha \), where \( \tilde{G} \) is the image of \( G \) in \( W \). The basis of the \( K \)-vector space \( W^\alpha \) consists of

\[
\begin{align*}
  e_1 &= (x_1^2 x_2^2, 0), \\
  e_2 &= (x_1 x_2^2 x_1, 0), \\
  e_3 &= (x_2^3 x_1, 0), \\
  e_4 &= ((x_1 x_2)^2, 0), \\
  e_5 &= ((x_2 x_1)^2, 0), \\
  e_6 &= (x_2 x_1^2 x_2, 0), \\
  e_7 &= (0, x_1 x_2^3), \\
  e_8 &= (0, x_2 x_1 x_2^2), \\
  e_9 &= (0, x_2^2 x_1 x_2), \\
  e_{10} &= (0, x_2^3 x_1).
\end{align*}
\]

We make use of Prop. 2.1 and assume that for any \( (f_1, f_2) \in F^{(k)} \oplus F^{(k)}, k = 2, 3 \), there exists a \( \phi \in G \) such that \( \theta_k(\phi I_k A) = (f_1, f_2) \). We shall complete the proof in several steps. In the following calculations \( \phi, \psi \in G \).

**Step 1.** Let \( \tilde{\phi} = (0, x_2^2), \tilde{\psi} = (x_1^2 x_2, 0) \). Then we calculate that

\[
(\tilde{\phi}, \tilde{\psi}) = (x_1^2 x_2^2, 0) = e_1 \in \tilde{G}.
\]

By Lemma 2.2 we obtain that \( e_3 \in \tilde{G} \).

**Step 2.** Let \( \phi = (x_2 x_1, 0), \psi = (x_1^2 x_2, 0) \). Then \( (\phi, \psi) = ((x_1 x_2)^2, 0) = e_4 \in \tilde{G}, e_4^* = e_5 \in \tilde{G} \).

**Step 3.** Let \( \tilde{\phi} = (0, x_2^2), \tilde{\psi} = (x_1 x_2 x_1, 0) \). Then \( (\phi, \psi) = (x_1 x_2^2 x_1, 0), e_2 \in \tilde{G} \).

**Step 4.** Let \( \tilde{\phi} = (x_1^2, 0), \tilde{\psi} = (x_2 x_1 x_2, 0) \). Then \( (\phi, \psi) = (x_2 x_1^2 x_2 - (x_1 x_2)^2 - (x_2 x_1)^2, 0) = e_6 - e_4 - e_5 \in \tilde{G}, e_6 \in \tilde{G} \).

In this way all vectors \( (f, 0) \in W^\alpha \) are in \( \tilde{G}^\alpha \) and in the rest of the calculations we are not interested in the first coordinates of the elements of \( \tilde{G}^\alpha \): if \( (f_1, f_2) \in \tilde{G}^\alpha \), then \( (0, f_2) \in \tilde{G}^\alpha \).

**Step 5.** If \( \tilde{\phi} = (x_1 x_2, 0), \tilde{\psi} = (0, x_1 x_2^2) \), then for some \( f \in F^{(2,2)} \), \( (\phi, \psi) = (f, x_1 x_2^3) \in \tilde{G}, e_7, e_{10} = e_7^* \in \tilde{G} \).

**Step 6.** If \( \tilde{\phi} = (x_2 x_1, 0), \tilde{\psi} = (0, x_1 x_2^2) \), then for some \( f \in F^{(2,2)} \), \( (\phi, \psi) = (f, x_2 x_1 x_2^2) \in \tilde{G}, e_8, e_9 = e_8^* \in \tilde{G} \).

In this way, all \( e_i \in \tilde{G} \), \( i = 1, \ldots, 10 \).

\( \diamond \)

**Proposition 2.4.** The group \( Aut F_2(N_6) \) is generated by \( GL_2 \) and \( \delta_j \), \( j = 1, 2, 3 \).

**Proof.** We repeat the arguments from the proof of Prop. 2.3. The content \( \alpha = (2,2) \) is essential for the \( GL_2 \)-module \( W = \theta_5(I_5A/I_6A) \). The subspace of \( W^\alpha \), consisting of all vectors \( (f, 0) \) has a basis.
\[ e_1 = (x_1^2x_2x_1x_2, 0), \quad e_2 = (x_1x_2x_1^2x_2, 0), \quad e_3 = ((x_1x_2)^2x_1, 0), \]
\[ e_4 = (x_2x_1^3x_2, 0), \quad e_5 = (x_2x_1^2x_2x_1, 0), \quad e_6 = (x_2x_1x_2x_1^2, 0), \]
\[ e_7 = (x_2^2x_1^3, 0), \quad e_8 = (x_1x_2^2x_1^2, 0), \quad e_9 = (x_1^2x_2x_1, 0), \]
\[ e_{10} = (x_1^3x_2, 0). \]

The basis of the subspace of \( W^\alpha \) consisting of all vectors \((0, f)\) has a basis

\[ \{g_\sigma \circ e_k \mid k = 1, \ldots, 10\}, \]

where \( \sigma = (12) \in S_2 \) and \( g_\sigma \in GL_2 \) was defined in Remark 1.10. It is sufficient to show that \( e_i \in \tilde{G}, \quad i = 1, \ldots, 10. \)

**Step 1.** The elements \( e_7, \ldots, e_{10} \) are of the form \((x_1^{a_1}x_2x_2^{a_2}x_1^{a_3}, 0)\). Starting with \( \phi, \psi \in G \) such that

\[ \tilde{\phi} = \theta_{k}(\phi I_{k+1}A) = (0, x_2^{a_2}), \quad \tilde{\psi} = \theta_{l}(\psi I_{l+1}A) = (x_1^{a_1}x_2x_1^{a_3}, 0), \]

\( k = a_2, \quad l = a_1 + a_3 + 1, \) we obtain

\[ (\tilde{\phi}, \tilde{\psi}) = \theta_{k+l-1}((\phi, \psi) I_{k+l}A) = (x_1^{a_1}x_2x_2^{a_2}x_1^{a_3}, 0) \in \tilde{G} \]

and \( e_7, \ldots, e_{10} \in \tilde{G}. \)

**Step 2.** For \( \tilde{\phi} = (x_1x_2, 0), \tilde{\psi} = (x_1x_2x_1^2, 0), \)

\[ (\tilde{\phi}, \tilde{\psi}) = (x_1x_2^2x_1^2 + (x_1x_2)^2x_1, 0) = e_3 + e_8 \in \tilde{G}, \quad e_3 \in \tilde{G}. \]

**Step 3.** For \( \tilde{\phi} = (x_1^2, 0), \tilde{\psi} = ((x_1x_2)^2, 0), \)

\[ (\tilde{\phi}, \tilde{\psi}) = (x_1x_2x_1^2x_2 - (x_1x_2)^2x_1, 0) = e_2 - e_3 \in \tilde{G}. \]

Hence \( e_2 \in \tilde{G} \) and by Lemma 2.2, \( e_5 \in \tilde{G}. \)

**Step 4.** For \( \tilde{\phi} = (x_2^2x_1, 0), \tilde{\psi} = (x_1x_2x_1, 0), \)

\[ (\tilde{\phi}, \tilde{\psi}) = (x_1^2x_2^2x_1 - x_1^3x_2x_1x_2, 0) = e_9 - e_1 \in \tilde{G}. \]

Hence \( e_1 \in \tilde{G} \) and by Lemma 2.2, \( e_6 \in \tilde{G}. \)

**Step 5.** For \( \tilde{\phi} = (x_1, 0), \tilde{\psi} = (x_2x_1^2x_2, 0), \)

\[ (\tilde{\phi}, \tilde{\psi}) = (2x_2x_1^3x_2 - x_1x_2x_2^2x_2 - x_2x_1^2x_2x_1, 0) = 2e_4 - e_2 - e_5 \in \tilde{G}, \quad e_4 \in \tilde{G}. \]

In this way \( \tilde{G}^\alpha = W^\alpha \) and this gives that \( \text{Aut} \ F = \tilde{G} = \langle GL_2, \delta_1, \delta_2, \delta_3 \rangle \) and this completes the proof. \( \diamond \)

**Proposition 2.5.** The group \( \text{Aut} \ F_2(N_7) \) is generated by \( GL_2 \) and \( \delta_j, \) \( j = 1, 2, 3. \)

**Proof.** Again, we repeat the arguments from the proof of Prop. 2.3. The content \( \alpha = (3, 2) \) is essential for the \( GL_2 \)-module \( W = \theta_6(I_6A/ \langle I_7A \rangle). \) The subspace of \( W^\alpha \) consisting of all vectors \((f, 0)\) has a basis
\[ e_1 = (x_1^3 x_2 x_1 x_2, 0), \quad e_2 = (x_1^2 x_2 x_1^2 x_2, 0), \quad e_3 = (x_1^2 (x_2 x_1)^2, 0), \]
\[ e_4 = (x_1 x_2 x_1^2 x_2 x_1, 0), \quad e_5 = (x_1 x_2 x_1^3 x_2, 0), \quad e_6 = ((x_1 x_2)^2 x_1^2, 0), \]
\[ e_7 = (x_2 x_1^4 x_2, 0), \quad e_8 = (x_2 x_3 x_2 x_1 x_2, 0), \quad e_9 = (x_2 x_1^2 x_2 x_1^2, 0), \]
\[ e_{10} = (x_2 x_1 x_2 x_1^3, 0), \quad e_{11} = (x_2^2 x_1^4, 0), \quad e_{12} = (x_1 x_2^2 x_1^3, 0), \]
\[ e_{13} = (x_1^2 x_2^2 x_1^2, 0), \quad e_{14} = (x_1^3 x_2^2 x_1, 0), \quad e_{15} = (x_1^4 x_2^2, 0). \]

First we shall prove that all \((f, 0) \in W^\alpha\) belong to \(\tilde{G}\). The elements \(e_{11}, \ldots, e_{15}\) are of the form \((x_1^a x_2 x_2^a x_1^a, 0)\). As in the first step of the proof of Prop. 2.4 these basis elements are in \(\tilde{G}\). The next calculations show that all elements \((f, 0) \in W^\alpha\) are also in \(\tilde{G}^\alpha\).

For \(\phi, \psi \in G\) such that \(\tilde{\phi} = (x_1 x_2 x_1, 0), \quad \tilde{\psi} = (x_1^2 x_2 x_1, 0)\) we obtain that
\[ (\tilde{\phi}, \tilde{\psi}) = (x_1^2 x_2 x_1 x_2 x_1, 0) = e_3 \in \tilde{G}. \]

Hence, by Lemma 2.2, \(e_6 \in \tilde{G}\). Similarly, we choose properly \(\phi, \psi \in G\) and compute \((\tilde{\phi}, \tilde{\psi})\).
\[ \tilde{\phi} = (x_1^2 x_2, 0), \quad \tilde{\psi} = (x_1^3 x_2, 0), \quad (\tilde{\phi}, \tilde{\psi}) = (x_1^2 x_2 x_1^2 x_2, 0) = e_2 \in \tilde{G}; \quad e_9 \in \tilde{G}. \]
\[ \tilde{\phi} = (x_1^1, 0), \quad \tilde{\psi} = (x_1^2 x_2 x_1, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_1 + e_2 - e_3 \in \tilde{G}; \quad e_1, e_{10} \in \tilde{G}. \]
\[ \tilde{\phi} = (x_1 x_2, 0), \quad \tilde{\psi} = (x_1^3 x_2 x_1, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_3 + e_4 + e_{14} \in \tilde{G}; \quad e_4 \in \tilde{G}. \]
\[ \tilde{\phi} = (x_1^1, 0), \quad \tilde{\psi} = (x_1^2 x_2 x_1^2 x_2, 0), \quad (\tilde{\phi}, \tilde{\psi}) = -e_4 + 2e_5 \in \tilde{G}; \quad e_5, e_8 \in \tilde{G}. \]
\[ \tilde{\phi} = (x_1^1, 0), \quad \tilde{\psi} = (x_2 x_1^3 x_2, 0), \quad (\tilde{\phi}, \tilde{\psi}) = -e_5 + 3e_7 - e_8 \in \tilde{G}; \quad e_7 \in \tilde{G}. \]

Now we shall establish that all \((f, 0) \in W^\beta\) are also in \(\tilde{G}^\beta\) for \(\beta = (2, 3)\). Then, by Remark 1.10 for \(\sigma = (12) \in S_2\), this will imply that all \((0, f) \in W^\alpha\) are in \(\tilde{G}^\alpha\) and this will complete the proof. All the polynomials \(f\) for the vector space \{\((f, 0) \in W^\beta\)\} are homogeneous and \(\deg x_1 f = \deg x_2 f = 3\). Hence this vector space is of dimension \((^6_3) = 20\). It has a basis \(e_i = (f_i, 0), i = 1, \ldots, 20,\) where
\[ f_1 = x_1^3 x_2^3, \quad f_2 = x_1^2 x_2 x_1 x_2^2, \quad f_3 = x_1^2 x_2^2 x_1 x_2, \quad f_4 = x_1^2 x_2^3 x_1, \]
\[ f_5 = x_1 x_2 x_1^2 x_2^2, \quad f_6 = (x_1 x_2)^3, \quad f_7 = x_1 x_2 x_1 x_2^2 x_1, \quad f_8 = x_1 x_2^2 x_1^3 x_2, \]
\[ f_9 = f_7^*, \quad f_{10} = f_4^*, \quad f_{11} = x_2 x_1^3 x_2, \quad f_{12} = x_2 x_1^2 x_2 x_1 x_2, \]
\[ f_{13} = f_8^*, \quad f_{14} = f_{12}^*, \quad f_{15} = f_6^*, \quad f_{16} = f_3^*, \]
\[ f_{17} = f_{11}^*, \quad f_{18} = f_5^*, \quad f_{19} = f_2^*, \quad f_{20} = f_1^*. \]

where the involution \(*\) is defined above Lemma 2.2. Four of the ba-
sics elements, \(e_1, e_4, e_{10}, e_{20}\) belong to \(\tilde{G}\) because they are of the form \((x_1^{a_1} x_2 x_3^{a_2} x_4^{a_3}, 0)\). For the other elements we work with \(\phi, \psi \in G\) and compute \((\tilde{\phi}, \tilde{\psi})\):

\[
\tilde{\phi} = (x_1 x_2, 0), \quad \tilde{\psi} = (x_1 x_2 x_3^{a_2}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_9 + e_{10} \in \tilde{G}; \quad e_9, e_{10} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_1^2, 0), \quad \tilde{\psi} = (x_1 x_2 x_3^{a_2}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_8 - e_9 \in \tilde{G}; \quad e_8, e_{13} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_1^2, 0), \quad \tilde{\psi} = (x_1 x_2 x_3^{a_2} x_4^{a_3}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_5 - e_7 \in \tilde{G}; \quad e_5, e_{18} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_2 x_1, 0), \quad \tilde{\psi} = (x_1^3 x_2^{a_2}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_2 + e_5 \in \tilde{G}; \quad e_2, e_{19} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_1 x_2, 0), \quad \tilde{\psi} = (x_1 x_2 x_3^{a_2} x_2^{a_3}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_6 + e_8 \in \tilde{G}; \quad e_6, e_{15} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_1 x_2, 0), \quad \tilde{\psi} = (x_1 x_2 x_3^{a_2} x_4^{a_3}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = e_3 + e_6 \in \tilde{G}; \quad e_3, e_{16} \in \tilde{G}.
\]

\[
\tilde{\phi} = (x_1^2, 0), \quad \tilde{\psi} = (x_2 x_3^{a_2} x_4^{a_3}, 0), \quad (\tilde{\phi}, \tilde{\psi}) = -e_5 + 2e_{11} + e_{13} \in \tilde{G};
\]

\[
\tilde{\phi} = (x_1 x_2 x_1, 0), \quad \tilde{\psi} = (x_2 x_1 x_2, 0), \quad (\tilde{\phi}, \tilde{\psi}) = -e_7 + e_{12} - e_{15} \in \tilde{G};
\]

\[
e_{11}, e_{17} \in \tilde{G}.
\]

In this way all \((f, 0) \in W^\beta\) are also in \(\tilde{G}^\beta\). ◊

3. Nilpotent algebras with more than two generators

In this section we prove our theorem in the general case. We fix positive integers \(m\) and \(c\) satisfying the condition \(m(m - 4) \geq c - 2\). Since it is sufficient to consider the case \(c > 3\), this implies that \(m \geq 5\). As in Section 2 we consider the automorphisms \(\delta_1, \delta_2, \delta_3\) of the algebra \(F = F_m(N_c)\) and denote by \(G = \langle GL_2, \delta_1, \delta_2, \delta_3 \rangle\) the subgroup of \(\text{Aut } F\) generated by \(GL_m\) and \(\delta_j, j = 1, 2, 3\). We also use the other conventions from Section 2. For example, if \(\phi \in G \cap I_k A\) and \(\psi \in G \cap I_l A\), and the values of \(k\) and \(l\) are clear from the context, we denote \(\theta_k(\phi I_{k+1} A)\) and \(\theta_l(\phi I_{l+1} A)\), respectively by \(\tilde{\phi}\) and \(\tilde{\psi}\) and by \((\tilde{\phi}, \tilde{\psi})\) the image \(\theta_{k+l-1}(\tilde{\phi}, \tilde{\psi}) I_{k+l} A\) of the group commutator \((\phi, \psi)\). The main result of Section 3 is the following.

**Theorem 3.1.** For \(m(m - 4) \geq c - 2\)

\[
\text{Aut } F_m(N_c) = G = \langle GL_m, \delta_1, \delta_2, \delta_3 \rangle.
\]

We shall prove this theorem by induction on \(c\). The base of the induction \(c = 3\) is considered in Section 1 and we assume that the state-
ment of Th. 3.1 holds for all algebras \( F_m(N_k) \), \( k < c \). Using Cor. 1.5, we shall establish the case \( k = c \). By Cor. 1.9, it is sufficient to show that the homogeneous component of degree \( \alpha = (p + \varepsilon_1, \ldots, p + \varepsilon_m) \) of the \( GL_m \)-module \( W = (F^{(c-1)})^{\otimes m} \) coincides with the homogeneous component of \( \tilde{G} = \theta_{c-1}(G \cap I_{c-1}A)/(G \cap I_cA) \), where \( c - 2 = pm + r, \leq r < m \) and \( \varepsilon_1 = \ldots = \varepsilon_r = 1, \varepsilon_{r+1} = \ldots = \varepsilon_m = 0 \). Till the end of the section we fix \( \alpha \) and the integers \( p \) and \( r \) as above. Finally, let \( f \in F^{(k)}, k \leq 6 \), and let \( f \) depend on \( x_1, x_2 \) only. By the results of Section 2, the automorphism \( \phi \) of \( F_2(N_{k+1}) \) defined by
\[
\phi(x_1) = x_1 + f, \phi(x_2) = x_2,
\]
belongs to the group generated by \( GL_2 \) and \( \delta_1, \delta_2, \delta_3 \). If we consider \( (GL_2, \delta_1, \delta_2, \delta_3) \subseteq \text{Aut} F_2(N_{k+1}) \) canonically embedded into \( \text{Aut} F_m(N_{k+1}) \) (fixing the other variables \( x_3, \ldots, x_m \)) we obtain that \( \phi \in G \subseteq \text{Aut} F_m(N_{k+1}) \).

We start the proof of Th. 3.1 with several lemmas.

Lemma 3.2. Let \( f = v_1uv_2 \) be a monomial from \( F^{(c-1)} \) such that \( \deg u \geq 2, \deg v_1 + \deg v_2 \geq 1, u \) does not contain the variable \( x_1 \) and the subwords \( v_1 \) and \( v_2 \) of the word \( f \) do not contain \( x_k \) for some \( k > 1 \). Then \( (f, 0, \ldots, 0) \in \tilde{G} \).

Proof. By inductive arguments, there exist automorphisms \( \phi, \psi \in G \) such that
\[
\tilde{\phi} = (0, \ldots, 0, u, 0, \ldots, 0), \tilde{\psi} = (v_1x_kv_2, 0, \ldots, 0),
\]
\( (u \) is the \( k \)-th coordinate of \( \tilde{\phi} \). Then by Lemma 1.12 we compute that
\[
(\tilde{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) \in \tilde{G}.
\]

Lemma 3.3. Let \( f = f(x_1, \ldots, x_m) \in F^{(c-1)} \) be a monomial of degree \( \lambda_i \) with respect to \( x_i \), where \( \lambda_i \geq 0 \) and \( |\lambda_i - \lambda_j| \leq 2, 1 \leq i, j \leq m \). If \( \lambda_m = 1 \) and \( \lambda_t = 0 \) for some \( t < m \), then \( (f, 0, \ldots, 0) \in \tilde{G} \).

Proof. If \( f = f(x_2, \ldots, x_m) \), then the statement is true by Lemma 3.2. If \( f = f(x_1, x_m) \), then \( \deg f \leq 4 \) and we may work in \( \text{Aut} F_2(N_c) \) for \( c \leq 5 \) applying the results of Section 2. Therefore, without loss of generality we may assume that \( f \) depends on \( x_2 \) and does not depend on \( x_m \). If \( f \) contains a subword \( u = x_ix_m \) or \( u = x_mx_i, l > 1 \), we apply Lemma 3.2 for \( k = m \). Hence the only cases to consider are \( f = v_1x_mx_1v_2 \) and \( f = v_1x_1x_mv_2 \). Let \( f = v_1x_mx_1v_2 \) and let
\[
\tilde{\phi} = (0, \ldots, 0, x_mx_1, 0), \tilde{\psi} = (v_1x_{m-1}v_2, 0, \ldots, 0).
\]
We obtain
\[
(\tilde{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) - b_{m-1} \in \tilde{G}, b_{m-1} = (0, \ldots, 0, x_mv_1x_{m-1}v_2, 0).
\]
For $\sigma = (1, m - 1)$, we define the linear automorphism $g_\sigma$ and obtain

$$g_\sigma \circ b_{m-1} = b_1 = (x_m \bar{v}_1 x_1 \bar{v}_2, 0, \ldots, 0),$$

where $x_1$ does not participate in $\bar{v}_1$ and $\bar{v}_2$. Applying Lemma 3.2, it is sufficient to consider the case $b_1 = (x_m x_1 x_2, 0, \ldots, 0)$. Let

$$\tilde{\phi} = (0, \ldots, 0, x_m x_1), \quad \tilde{\psi} = (x_m x_2, 0, \ldots, 0).$$

Direct calculations show that

$$(\tilde{\phi}, \tilde{\psi}) = b_1 + (0, \ldots, 0, x_m^2 x_2) \in \tilde{G},$$

$$(0, \ldots, 0, x_m^2 x_2) = g_\sigma \circ \delta_3 \in \tilde{G}, \quad \sigma = (m) \in S_m,$$

and $(f, 0, \ldots, 0) \in \tilde{G}$. The case $f = v_1 x_1 x_m v_2$ follows applying Lemma 2.2 to $f^*$. 

Now we generalize Lemma 3.3 in the following way.

**Lemma 3.4.** Let $f = f(x_1, \ldots, x_m) \in F^{(c-1)}$ be a monomial of degree $\lambda_i$ with respect to $x_i$, where $\lambda_i \geq 0$ and $|\lambda_i - \lambda_j| \leq 2$, $1 \leq i, j \leq m$. If $\lambda_k = 1, k \neq 1$, then $(f, 0, \ldots, 0) \in \tilde{G}$.

**Proof.** Without loss of generality we may assume that $k = m$. Clearly, $\lambda_1$ may be equal to $0, 1, 2$ or $3$ and, by Lemma 3.3 it is sufficient to consider the case when $\lambda_i > 0$ for all $i = 1, \ldots, m$. We consider several cases.

**Case 1.** The monomial $f$ is of the form $f = v_1 x_m x_1 v_2$ or $f = v_1 x_1 x_2 x_m v_2$ and $l \neq 1$. Then we apply Lemma 3.2.

**Case 2.** $f$ is of the form $f = u_1 x_1^\gamma u_2$, where $u_1$ and $u_2$ do not depend on $x_1$. Excluding the monomials covered by Case 1 we may consider the symmetric cases $f = x_m x_1^\gamma u_2$ and $f = u_1 x_1^\gamma x_m$. Since $m \geq 5$, we apply twice Lemma 3.2 in order to obtain consequently

$$(x_m x_1^\gamma u_2, 0, \ldots, 0), (u_1 x_1^\gamma x_m, 0, \ldots, 0) \in \tilde{G}, \quad (f, 0, \ldots, 0) \in \tilde{G}.$$

**Case 3.** Let $f = u_1 x_1^\gamma u_2 x_1^\gamma u_3$, where $u_1$, $u_2$ and $u_3$ do not contain $x_1$. Considering the monomials not covered by the previous cases we assume that $x_m$ coincides with one of $u_1, u_2, u_3$. For example, let $f = u_1 x_1^\gamma x_m x_1^\gamma u_3$, the other cases are analogous. Since $m \geq 5$ and $f$ depends on all the variables, $\deg u_1 + \deg u_3 \geq 3$. Let some of the variables $x_2$ or $x_3$ participate in one of the monomials $u_1$ and $u_3$ only, for example $x_2$ is in $u_1$. If $u_1 = x_2$, then we apply Lemma 3.2 for $x_3$ and $u_3$, because $\deg u_3 > 1$. If $\deg u_1 > 1$, then we apply Lemma 3.2 for $x_2$ and $u_1$. Now, let both $x_2$ and $x_3$ participate in $u_1$ and $u_3$. Let $u_1 = w_1 x_i w_2, i_1 \in \{2, 3\}$ and let $w_1$ do not contain $x_2, x_3$. We consider
\[ \bar{\phi} = (0, x_2w_2, 0, \ldots, 0), \text{ if } i_1 = 2, \quad \text{and} \]
\[ \tilde{\phi} = (0, 0, x_3w_2, 0, \ldots, 0), \text{ if } i_1 = 3, \]
\[ \tilde{\psi} = (w_1x_i, x_1^{\gamma_1}x_mx_1^{\gamma_2}u_3, 0, \ldots, 0), \]
\[ (\bar{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) + (w_1x_i, x_1^{\gamma_1}x_mx_1^{\gamma_2}u_3, 0, \ldots, 0) \in \bar{G}, \]
where \( \bar{u}_3 = \Delta u_3 \) and \( \Delta \) is the derivation of \( F \) defined by \( \Delta x_{i_1} = x_{i_1}w_2, \]
\( \Delta x_j = 0, j \neq i_1 \). The variable \( x_{i_2} = x_2, x_3, x_{i_2} \neq x_{i_1}, \) participates in \( \bar{u}_3 \) only and we apply Lemma 3.2. In this way, \( (f, 0, \ldots, 0) \in \bar{G} \).

**Case 4.** Let \( f = u_1x_1u_3x_1u_4, \) where \( x_1 \) does not participate in the monomials \( u_i, i = 1, 2, 3, 4 \). Since the other possibilities are covered by the previous considerations, it is sufficient to assume that \( x_m \) coincides with some of the monomials \( u_i \) and the monomials \( u_2, u_3 \) are different from 1. We have to handle two subcases:

(i) \( u_1 = x_m \) (or, by symmetry, \( u_4 = x_m \)), i.e.
\[ f = x_mx_1u_2x_1u_3x_1u_4. \]
Let
\[ \bar{\phi} = (0, \ldots, 0, x_mx_1), \tilde{\psi} = (x_mx_1u_3x_1u_4, 0, \ldots, 0). \]
We calculate that
\[ (\bar{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) - b_m \in \bar{G}, \]
\[ b_m = (0, \ldots, 0, x_m^2x_1u_2x_1u_3x_1u_4). \]
By conjugation of \( b_m \) by \( g_{\sigma}, \sigma = (1m) \) we obtain
\[ b_1 = g_{\sigma} \circ g_m = (x_1^2x_mx_1u_2x_3x_3u_4, 0, \ldots, 0) \]
which belongs to \( \bar{G} \) by Lemma 3.2.

(ii) \( u_2 = x_m \) (or \( u_3 = x_m \)), i.e.
\[ f = u_1x_1x_mx_1u_3x_1u_4, \] \( x_1 \) does not participate in the monomials \( u_i \) and \( u_3 \neq 1 \). For
\[ \bar{\phi} = (x_1x_m, 0, \ldots, 0), \tilde{\psi} = (u_1x_1^2u_3x_1u_4, 0, \ldots, 0) \]
we obtain
\[ (\bar{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) + v \in \bar{G}, \]
where \( v \) is a sum of elements of \( W \) considered in Case 3. This completes all the possible cases and gives the proof of the lemma. \( \diamond \)

**Lemma 3.5.** Let \( f = f(x_1, \ldots, x_m) \in F^{(c-1)} \) be a monomial of degree \( \lambda_i \) with respect to \( x_i \), where \( \lambda_i \geq 0 \) and \( |\lambda_i - \lambda_j| \leq 2, 1 \leq i, j \leq m \) and \( f \) be presented in the form
\[ f = w_0x_1^{\gamma_1}w_1x_1^{\gamma_2} \ldots x_1^{\gamma_s}w_s, \]
where \( \gamma_i > 0, i = 1, \ldots, s, w_i, i = 0, 1, \ldots, s, \) are monomials which do not depend on \( x_1 \) and \( \gamma_i > 0, i = 1, \ldots, s \). Let there exist two sets with the same number of elements
\[ M_1 = \{x_{i_1}, \ldots, x_{i_t}\}, \quad M_2 = \{j_1, \ldots, j_t\} \]

with the following property: In the above presentation of \( f \) the variables \( x_{i_k} \in M_1 \) participate only in the monomials \( w_{j_i} \) with indices \( j_i \) from the set \( M_2 \). Then \( (f, 0, \ldots, 0) \in \tilde{G} \).

**Proof.** We shall prove the statement by induction of the number \( t \) of the elements of \( M_1 \) and \( M_2 \). If \( t = 1 \), then there exists a variable \( x_k \), \( k \neq 1 \), which participate in one monomial \( w_i \) only. Then \( f = v_1 w_i v_2 \), where \( v_1, v_2 \) do not depend on \( x_k \). If \( \lambda_k = \deg x_k f = 1 \), we apply Lemma 3.4. If \( \lambda_k > 1 \), since \( \deg v_1 + \deg v_2 \geq 1 \) and \( \deg w_j \geq 2 \), we apply Lemma 3.2. In this way we complete the case \( t = 1 \). Now we assume that the statement is true for sets \( M_1 \) and \( M_2 \) with less than \( t \) elements and shall prove it for \( |M_1| = |M_2| = t \). Without loss of generality we may assume that \( M_1 = \{x_2, \ldots, x_{t+1}\} \). Let \( j_1 \) be the smallest index in \( M_2 \) with the property that \( w_{j_1} \) does depend on some \( x_k \) from \( M_1 \). Let, for example, \( k = 2 \). If \( w_{j_1} = u_1 x_2 u_2 \) and \( u_1, u_2 \) do not involve variables from \( M_1 \), then the other variables \( x_3, \ldots, x_{t+1} \) participate in the monomials \( w_{j_2}, \ldots, w_{j_t} \) and we apply the inductive arguments for \( M_1 = \{x_3, \ldots, x_{t+1}\} \) and \( M_2 = \{j_2, \ldots, j_t\} \). Let \( w_{j_1} = u_1 x_2 u_2 x_1 u_3 \), where \( x_i \in M_1 \) and \( u_1, u_3 \) do not depend on elements from \( M_1 \). We consider
\[
\tilde{\phi} = (0, x_2 u_2 x_1, 0, \ldots, 0), \quad \tilde{\psi} = (w_0 \ldots x_1^{\gamma_{j_1}} u_1 x_2 u_3 x_1^{\gamma_{j_1+1}} \ldots w_s, 0, \ldots, 0)
\]
and calculate
\[
(\tilde{\phi}, \tilde{\psi}) = (f, 0, \ldots, 0) + (g, 0, \ldots, 0) \in \tilde{G},
\]
where \( \Delta \) is the derivation of \( F \) defined by
\[
\Delta x_2 = x_2 u_2 x_1, \quad \Delta x_i = 0, \quad i \neq 2.
\]
If we write \( g \) as a linear combination of monomials we obtain that each summand is of the form
\[
g_i = z_0 x_1^{\mu_1} z_1 x_1^{\mu_2} \ldots z_{s-1} x_1^{\mu_s} z_s,
\]
and the variables \( x_3, \ldots, x_{t+1} \) participate in \( z_{j_2}, \ldots, z_{j_t} \) only. Again, we apply the inductive assumption and obtain that \( (g_i, 0, \ldots, 0) \in \tilde{G} \).

Hence \( (f, 0, \ldots, 0) \) also belongs to \( \tilde{G} \). \( \diamond \)

Now we are ready to prove Th. 3.1 and, in this way to complete the proof of the main result of our paper.

**Proof of Theorem 3.1.** It is sufficient to establish that the homogeneous component \( W^\alpha \) is contained in \( \tilde{G} \), where \( \alpha = (p+\varepsilon_1, \ldots, p+\varepsilon_m) \), \( c - 2 = pm + r, \ 0 \leq r < m \) and \( \varepsilon_1 = \ldots = \varepsilon_r = 1, \ \varepsilon_{r+1} = \ldots = \varepsilon_m = 0 \). First we consider the elements \( (f, 0, \ldots, 0) \in W^\alpha \). As in Ex. 1.2, \( \deg_{x_1} f = p + 1 \), if \( r = 0 \) and \( \deg_{x_1} f = p + 2 \), if \( r > 0 \). Let
where the $w_i$'s do not depend on $x_1$. The condition $m(m - 4) \geq c - 2$ gives that $p \leq m - 4$, i.e. $p + 3 \leq m - 1$. The number of the words $w_i$ does not exceed $p + 3 \leq m - 1$. Applying Lemma 3.5 for $M_1 = \{x_2, \ldots, x_m\}$, we obtain that $(f, 0, \ldots, 0) \in G$. Now, let $b_k = (0, \ldots, 0, g, 0, \ldots, 0) \in W^\alpha$, where $g$ is the $k$-th coordinate of $b_k$, $k > 1$. Clearly, $\deg_{x_1} g = \alpha_k + 1$. Conjugating $b_k$ with $g_\sigma$ for $\sigma = (1k)$, we obtain that $b_1 = g_\sigma \circ b_k$ is of the form $(f, 0, \ldots, 0)$. If $\varepsilon_1 = \varepsilon_k$, then $b_1 \in W^\alpha$. We have already shown that $b_1 \in \tilde{G}$, and hence $b_k$ also belongs to $\tilde{G}$. If $\varepsilon_1 = 1, \varepsilon_k = 0$, then the monomial $f$ which is the first coordinate of $b_1$ is of degree $\alpha_k + 1 = p + 1$ with respect to $x_1$. Again we can apply Lemma 3.5 and obtain that $b_1 \in \tilde{G}$. This shows that $W^\alpha \subset \tilde{G}$ and completes the proof of the theorem. 

References


