A SEPARATION THEOREM FOR M_\phi-CONVEX FUNCTIONS

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Abstract: Some theorems about separation of two real functions by the function which is convex or affine with respect to the weighted quasi-arithmetic means are presented.

Introduction

It is shown in [2] that every real functions \( f \) and \( g \), defined on an interval \( I \subset \mathbb{R} \) and satisfying the inequality

\[
 f(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y),
\]

for all \( x, y \in I \), and \( t \in (0,1) \), can be separated by a convex function (cf. Th. A). Applying a Helly’s theorem, the authors of [5] proved that if, besides the above inequality, the functions \( f, g \) satisfy the reverse inequality with \( f \) and \( g \) interchanged, then there exists an affine function which separates these functions (cf. Th. B).

In Section 1 we quote these results and we show that Th. B is a consequence of Th. A. Moreover, we discuss the inequality
\[ f(tx + (1-t)y) \leq tg_1(x) + (1-t)g_2(y) \]
with three functions defined again on a real interval \( I \) and we show that, in general, there is no a separating convex function between \( f \) and \( \min(g_1, g_2) \).

The main results of this paper are given in Section 3 and 4 where we transfer the Ths. A and B to the class \( M_\phi \)-convex and \( M_\phi \)-affine functions (\( M_\phi \) denotes the family of the weighted quasi-arithmetic means of the generator \( \phi \)).

1. Remarks on separation theorems for convex and affine functions

We begin with recalling the following

**Theorem A** ([2]). *Real functions \( f \) and \( g \), defined on a real interval \( I \), satisfy the inequality*

\[ f(tx + (1-t)y) \leq tg(x) + (1-t)g(y), \]

*for all \( x, y \in I \) and \( t \in (0,1) \) if, and only if, there exists a convex function \( h : I \subset \mathbb{R} \) such that \( f \leq h \leq g \).*

As a simple consequence we obtain

**Corollary 1.** Let \( I \subset \mathbb{R} \) be an interval. If \( f, g_1, g_2 : I \subset \mathbb{R} \) satisfy the inequality

\[ f(tx + (1-t)y) \leq tg_1(x) + (1-t)g_2(y), \quad x, y \in I, \quad t \in (0,1), \]

*then there exists a convex function \( h : I \mapsto \mathbb{R} \) such that \( f \leq h \leq \max(g_1, g_2) \).*

**Remark 1.** If \( f, g_1, g_2 : I \subset \mathbb{R} \) satisfy the inequality (1), then obviously that \( f \leq \min(g_1, g_2) \). In this connection a question arises whether there exists a convex function \( h : I \mapsto \mathbb{R} \) such that \( f \leq h \leq \min(g_1, g_2) \).

Taking \( I = \mathbb{R} \), \( g_1, g_2 : \mathbb{R} \mapsto \mathbb{R} \), \( g_1(x) = x^2 \), \( g_2(x) = (x-1)^2 \), \( x > 0 \), and \( f = \min(g_1, g_2) \), it is easy to see that the answer is negative.

However, we can prove the following

**Proposition.** Let \( I \subset \mathbb{R} \) be an interval, and suppose that the functions \( f, g_1, \ldots, g_n : I \mapsto \mathbb{R} \) satisfy the inequality

\[ f \left( \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} t_i g_i(x_i), \quad \sum_{i=1}^{n} t_i = 1, \quad t_i \geq 0, \quad x_i \in I. \]

If \( g_n \leq \min(g_1, \ldots, g_{n-1}) \), then there exists a convex function \( h : I \mapsto \mathbb{R} \) such that \( f \leq h \leq \min(g_1, \ldots, g_{n-1}) \). If moreover, \( g_n = \min(g_1, \ldots, g_{n-1}) \) then the converse implication also holds true.
Proof. Take arbitrary \( x, y \in I, \ t \in [0, 1], \) and \( i = 1, \ldots, n - 1. \) Setting \( t_i = t, \ t_n = 1 - t, \) and \( t_j = 0, \ j = 1, \ldots, n - 1, \ j \neq i; \ x_i = x, \ x_n = y, \) we get
\[
f(tx + (1 - t)y) \leq tg_i(x) + (1 - t)g_n(y), \quad i = 1, \ldots, n - 1.
\]
It follows that
\[
f(tx + (1 - t)y) \leq tg(x) + (1 - t)g_n(y) \leq tg(x) + (1 - t)g(y),
\]
where \( g = \min(g_1, \ldots, g_{n-1}). \) Now Th. A completes the proof. \( \Box \)

Applying Helly's theorem on the existence a straight line intersecting a family of parallel compact segment in \( \mathbb{R}^2, \) the authors of [5] proved the following

**Theorem B.** Let \( I \subset \mathbb{R} \) be an interval. The functions \( f, g : I \to \mathbb{R} \) satisfy the system of inequalities
\[
\begin{cases}
  f(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y) \\
  g(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)
\end{cases},
\]
x, y \in I, \ t \in (0, 1),
if, and only if, there exists an affine function \( h : I \to \mathbb{R} \) such that \( f \leq h \leq g. \)

It turns out that Th. B is a consequence of Th. A. In fact, applying Th. A to the first of the inequalities we get a convex function \( h_1 : I \to \mathbb{R} \) such that \( f \leq h_1 \leq g. \) Writing the second inequality in the equivalent form
\[
(-g)(tx + (1 - t)y) \leq t(-f)(x) + (1 - t)(-f)(y), \quad x, y \in I, \ t \in (0, 1),
\]
and applying again Th. A we obtain a concave function \( h_2 : I \to \mathbb{R} \)
such that \( f \leq h_2 \leq g. \)

Now there are three possible cases: either the graphs of \( h_1 \) and \( h_2 \) have two different common points or they have only one common point or there is no points of intersection of the graphs of \( h_1 \) and \( h_2. \)

Taking in the first case the straight line through the both common points; in the second case a straight line through the common point which lies between the graphs of \( h_1 \) and \( h_2, \) and, in the third case, any straight line between the graphs of \( h_1 \) and \( h_2, \) we get the desired affine function \( h. \)

2. Definitions and some properties of \( M_\phi \)-convex functions

Let \( I \subset \mathbb{R} \) be an interval. For a fixed continuous and strictly monotonic function \( \phi : I \to \mathbb{R} \) and for any fixed \( t \in (0, 1), \) we define \( M_{\phi, t} : I^2 \to I \) by the formula
\[(2) \quad M_{\phi,t}(x,y) = \phi^{-1}(t\phi(x) + (1-t)\phi(y)), \quad x, y \in I.\]

The function \(M_{\phi,t}\) is a mean in \(I\) i.e., for all \(x, y \in I\),
\[
\min(x,y) \leq M_{\phi,t}(x,y) \leq \max(x,y),
\]
and it is called a \textit{weighted quasi-arithmetic mean} (cf. [1], p. 287 and [3], p. 189). Note that, for any interval \(J \subset I\),
\[
M_{\phi,t}(J \times J) \subset J, \quad t \in (0,1).
\]
This property allows us to introduce the following

**Definition 1.** Let a subinterval \(J\) of \(I\) and \(t \in (0,1)\) be fixed. A function \(w : J \rightarrow I\) is said to be

(i) \(M_{\phi,t}\)-\textit{convex} if \(w(M_{\phi,t}(x,y)) \leq M_{\phi,t}(w(x),w(y)), x, y \in J\);

(ii) \(M_{\phi,t}\)-\textit{concave} if \(w(M_{\phi,t}(x,y)) \geq M_{\phi,t}(w(x),w(y)), x, y \in J\);

(iii) \(M_{\phi,t}\)-\textit{affine} if \(w(M_{\phi,t}(x,y)) = M_{\phi,t}(w(x),w(y)), x, y \in J\).

**Definition 2.** A function \(w : J \rightarrow I\) is called \(M_{\phi}\)-\textit{convex} if for every \(t \in (0,1)\) it is \(M_{\phi,t}\)-convex. Analogously we define \(M_{\phi}\)-\textit{concave} and \(M_{\phi}\)-\textit{affine} functions.

**Remark 2.** Let \(I = \mathbb{R}\) and let \(\phi : I \rightarrow \mathbb{R}\) be given by
\[
\phi(u) = au + b, \quad u \in I,
\]
where \(a, b \in \mathbb{R}, a \neq 0\), are fixed. It is easy to see that \(M_{\phi,\frac{1}{2}}\)-convexity of a function \(w\) is equivalent to the Jensen convexity of \(w\), and, for every fixed \(t \in (0,1)\), the \(M_{\phi,t}\)-convexity of \(w\) reduces to its \(t\)-convexity (cf. [4]). Moreover, \(M_{\phi}\)-convexity of a function coincides with its classical convexity. Thus the notion of the \(M_{\phi}\)-convexity generalizes the classical convexity.

In the sequel the following criterion of the \(M_{\phi}\)-convexity will be useful.

**Lemma 1.** Let \(\phi : J \rightarrow \mathbb{R}\) be continuous and strictly decreasing. Then \(u : \phi(J) \rightarrow J\) is concave if, and only if, the function \(\phi^{-1} \circ u \circ \phi\) is \(M_{\phi}\)-convex on \(J\).

**Proof.** By the concavity of \(u\) we have
\[
u(tr + (1-t)s) \geq tu(r) + (1-t)u(s), \quad r, s \in \phi(I), \quad t \in (0,1).
\]
Setting here \(r = \phi(x), s = \phi(y)\), for \(x, y \in J\), and applying the decreasing monotonicity of \(\phi\), we get
\[
w \left( \phi^{-1}(t\phi(x) + (1-t)\phi(y)) \right) \leq \phi^{-1}(t\phi(w(x)) + (1-t)\phi(w(y))),
\]
for all \(x, y \in J, t \in (0,1)\), where \(w := \phi^{-1} \circ u \circ \phi\). This shows that \(w\) is \(M_{\phi}\)-convex on \(J\). The converse implication is obvious. \(\diamondsuit\)

Similarly we prove the following

**Lemma 2.** Let \(\phi : J \rightarrow \mathbb{R}\) be continuous and strictly increasing. Then \(u : \phi(J) \rightarrow J\) is convex if, and only if, the function \(\phi^{-1} \circ u \circ \phi\) is \(M_{\phi}\)-convex on \(J\).
3. Separation theorem for $M_\phi$-convex functions

The main result of this section reads as follows:

**Theorem 1.** Let $I$ and $J$ be intervals such that $J \subset I$ and suppose that $\phi : J \rightarrow \mathbb{R}$ is continuous and strictly monotonic. Then $f, g : J \rightarrow I$ satisfy the inequality

\[ f(M_\phi t(x, y)) \leq M_\phi t(g(x), g(y)), \quad x, y \in J, \quad t \in (0, 1), \]

if, and only if, there exists an $M_\phi$-convex function $h : J \rightarrow I$ such that

\[ f(x) \leq h(x) \leq g(x), \quad x \in J. \]

**Proof.** Assume that (3) holds true. First consider the case when $\phi$ is strictly decreasing. From (2) and (3), for all $x, y \in J$, and $t \in (0, 1)$, we obtain

\[ f(\phi^{-1}(t\phi(x) + (1-t)\phi(y))) \leq \phi^{-1}(t\phi(g(x)) + (1-t)\phi(g(y))). \]

Choose arbitrary $r, s \in \phi(J)$. Substituting here $x = \phi^{-1}(r)$ and $y = \phi^{-1}(s)$ and making use of the decreasing monotonicity of $\phi$ we get

\[ (\phi \circ f \circ \phi^{-1})(tr + (1-t)s) \geq t(\phi \circ g \circ \phi^{-1})(r) + (1-t)(\phi \circ g \circ \phi^{-1})(s) \]

for all $r, s \in \phi(J)$ and $t \in (0, 1)$. Define $\bar{f}, \bar{g} : \phi(J) \rightarrow J$ by

\[ \bar{f} = \phi \circ f \circ \phi^{-1}, \quad \bar{g} = \phi \circ g \circ \phi^{-1}. \]

In view of (5) we have

\[ \bar{f}(tr + (1-t)s) \geq t\bar{g}(r) + (1-t)\bar{g}(s), \quad r, s \in \phi(J), \quad t \in (0, 1). \]

Now, applying Th. A, we infer that there exists a concave function $\bar{h} : \phi(J) \rightarrow J$ such that

\[ \bar{f}(r) \geq \bar{h}(r) \geq \bar{g}(r), \quad r \in \phi(J). \]

Putting here $r = \phi(x), x \in J$, and making use of the decreasing monotonicity of $\phi$, we get

\[ f(x) \leq (\phi^{-1} \circ \bar{h} \circ \phi)(x) \leq g(x), \quad x \in J. \]

In view of Lemma 1, the function $h : J \rightarrow I$ is defined by

\[ h = \phi^{-1} \circ \bar{h} \circ \phi \]

is the desired $M_\phi$-convex function.

Now consider the remaining case when $\phi$ is strictly increasing. A similar reasoning as in the previous part of the proof shows that

\[ (\phi \circ f \circ \phi^{-1})(tr + (1-t)s) \leq t(\phi \circ g \circ \phi^{-1})(r) + (1-t)(\phi \circ g \circ \phi^{-1})(s) \]

for all $r, s \in \phi(J)$, and $t \in (0, 1)$, which means that

\[ \bar{f}(tr + (1-t)s) \leq t\bar{g}(r) + (1-t)\bar{g}(s), \quad r, s \in \phi(J), \quad t \in (0, 1), \]

where $\bar{f}, \bar{g} : \phi(J) \rightarrow J$ are defined by (6). Applying again the Th. A gives the existence of convex function $h : \phi(J) \rightarrow J$ such that
\[ \bar{f}(r) \leq \bar{h}(r) \leq \bar{g}(r), \quad r \in \phi(J). \]

Putting here \( r = \phi(x), \ x \in J, \) and making use of the increasing monotonicity of \( \phi \) we obtain (4) with \( h : J \rightarrow I \) defined by formula \( h = \phi^{-1} \circ \bar{h} \circ \phi. \) By Lemma 2, \( h \) is the desired \( M_\phi \)-convex function.

The converse implication is an easy consequence of the fact that the weighted quasi-arithmetic mean is strictly monotonic with respect to each variable. \( \Diamond \)

**Remark 3.** Applying Th. 1 with \( \phi : J \rightarrow \mathbb{R} \) defined by \( \phi(u) = au + b, \ u \in J, \) where \( a, \ b \in \mathbb{R}, \ a \neq 0, \) are fixed, we get the result obtained in [2].

Recall that a function \( h : J \rightarrow (0, \infty) \) is *geometrically convex* if

\[ h(x^ty^{1-t}) \leq (h(x))^t (h(y))^{1-t}, \quad x, y \in J, \ t \in (0,1). \]

Taking \( I = (0, \infty), \) and \( \phi(t) = \log t \ (t > 0) \) in Th. 1 we obtain the following

**Corollary 2.** Let \( J \subset (0, \infty) \) be an interval and suppose that \( f, g : J \rightarrow (0, \infty). \) Then

\[ f(x^ty^{1-t}) \leq (f(x))^t (f(y))^{1-t}, \quad x, y \in J, \ t \in (0,1), \]

if, and only if, there exists a geometrically convex function \( h : J \rightarrow (0, \infty) \) such that

\[ f(x) \leq h(x) \leq g(x), \quad x \in J. \]

4. **Separation theorem for \( M_\phi \)-affine functions**

In this section we prove the following

**Theorem 2.** Let \( I, J \) be intervals such that \( J \subset I. \) Suppose that \( \phi : J \rightarrow \mathbb{R} \) is a continuous and strictly monotonic, and \( f, g : J \rightarrow I. \) Then the following conditions are equivalent:

(i) there exists an \( M_\phi \)-affine function \( h : J \rightarrow I \) such that

\[ f(x) \leq h(x) \leq g(x), \quad x \in J; \]

(ii) there exist an \( M_\phi \)-convex function \( h_1 : J \rightarrow I \) and an \( M_\phi \)-concave function \( h_2 : J \rightarrow I \) such that

\[ f(x) \leq h_1(x) \leq g(x), \quad x \in J, \quad f(x) \leq h_2(x) \leq g(x), \quad x \in J; \]

(iii) the functions \( f \) and \( g \) satisfy the system of inequalities:

\[
\begin{align*}
    f(M_{\phi,t}(x,y)) &\leq M_{\phi,t}(g(x),g(y)) \\
    g(M_{\phi,t}(x,y)) &\geq M_{\phi,t}(f(x),f(y))
\end{align*}
\]

\[ x, y \in I, \quad t \in (0,1). \]

**Proof.** Implication (i) \( \Rightarrow \) (ii) is a consequence of the fact that every affine function is both convex and concave.
The increasing monotonicity of the weighted quasi-arithmetic mean $M_{\phi,t}$ with respect to each variable yields the implication (ii) $\implies$ (iii).

To show the implication (iii) $\implies$ (i) first assume that $\phi$ is strictly decreasing. Taking $\tilde{f}, \tilde{g} : \phi(J) \mapsto J$ defined by (6) we can write the system (iii) in the form
\[
\begin{align*}
\tilde{f}(tr + (1-t)s) & \geq t\tilde{g}(r) + (1-t)\tilde{g}(s), & r, s \in \phi(J), & t \in (0, 1). \\
\tilde{g}(tr + (1-t)s) & \leq t\tilde{f}(r) + (1-t)\tilde{f}(s) & r, s \in \phi(J), & t \in (0, 1).
\end{align*}
\]
Applying Th. B we infer that there exists an affine function $\bar{h} : \phi(J) \mapsto J$ such that
\[
\tilde{g}(r) \leq \bar{h}(r) \leq \tilde{f}(r), \quad r \in \phi(J).
\]
Putting here $r = \phi(x)$, $x \in J$, and making use of the decreasing monotonicity of $\phi$ we get
\[
f(x) \leq h(x) \leq g(x), \quad x \in J,
\]
where $h : J \mapsto I$ is given by the formula $h = \phi^{-1} \circ \bar{h} \circ \phi$. Clearly, $h$ is the desired $M_{\phi}$-affine function.

Assume now that $\phi$ is strictly increasing. Similarly as in the previous case, the function $\tilde{f}, \tilde{g} : \phi(J) \mapsto J$ defined by (6) satisfy the system of inequalities
\[
\begin{align*}
\tilde{f}(tr + (1-t)s) & \leq t\tilde{g}(r) + (1-t)\tilde{g}(s), & r, s \in \phi(J), & t \in (0, 1). \\
\tilde{g}(tr + (1-t)s) & \geq t\tilde{f}(r) + (1-t)\tilde{f}(s) & r, s \in \phi(J), & t \in (0, 1).
\end{align*}
\]
The existence of the affine function $\bar{h} : \phi(J) \mapsto J$ such that
\[
\tilde{f}(r) \leq \bar{h}(r) \leq \tilde{g}(r), \quad r \in \phi(J).
\]
is again a consequence of theorem B. Now it is easy to check that $h : J \mapsto I$ given by $h = \phi^{-1} \circ \bar{h} \circ \phi$ satisfies the condition (i). ◊

References