TOLERANCES ON SEMILATTICES

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Abstract: The aim of this note is to prove that the tolerance lattice of semilattice is a p-algebra. An example shows that this p-algebra fails to be a relative p-algebra.

1. Preliminaries

A p-algebra (or pseudocomplemented lattice) is a universal algebra (L; V, A, *, 0, 1) of type (2, 2, 1, 0, 0) in which the deletion of the unary operation * yields a bounded lattice and * is the operation of pseudocomplementation that is

x ≤ a* if and only if a ∨ x = 0.

The class Bω of all distributive p-algebras is equational. K. B. Lee [4] has shown that the lattice of all equational subclasses of Bω forms a chain

B-1 ⊂ B0 ⊂ B1 ⊂ ... ⊂ Bn ⊂ ... ⊂ Bω,

of type ω + 1 where B-1 denotes the class of all trivial p-algebras, B0 is the class of all Boolean algebras and for n ≥ 1 the class Bn consists of all distributive p-algebras satisfying identity

(Ln)(x1 ∧ x2 ∧ ... ∧ xn)* ∨ (x1* ∧ x2 ∧ ... ∧ xn)* ∨ ... ∨ (x1 ∧ x2 ∧ ... ∧ xn*)* = 1.

We call the elements of B1 the Stone algebras. For n ≥ 2 the elements of Bn are called (Ln)-lattices. Distributive p-algebra in which for some n ≥ 1 every subinterval is an (Ln)-lattice is called a relative (Ln)-lattice.
Proposition 1.1 ([1]; Th. 1). Let $L$ be a distributive lattice with 1. The following conditions are equivalent:

(i) $L$ is a relative $(L_n)$-lattice,

(ii) for every $a \in L$, $[a, 1]$ is an $(L_n)$-lattice.

If we give up the distributivity we can study the following classes of $p$-algebras

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_n \subset \ldots \subset \mathcal{P}_\omega,$$

where $\mathcal{P}_\omega$ denotes the class of all $p$-algebras, $L \in \mathcal{P}_n$ if and only if $L$ is a $p$-algebra satisfying the identity $(L_n)$ for $1 \leq n < \omega$ and the elements of the class $\mathcal{P}_0$ are uniquely determined by the identity

$$(L_0) \quad (x \land y)^* = x^* \lor y^*.$$

(For distributive $p$-algebras the identities $(L_0)$ and $(L_1)$ are equivalent.)

Let $S$ be a $\land$-semilattice. A tolerance on a semilattice $S$ is a reflexive and symmetric binary relation $T$ on $S$ which has the substitution property with respect to $\land$, i.e.

$$(a, b) \in T, \quad (c, d) \in T \quad \text{implies} \quad (a \land c, b \land d) \in T.$$ 

The set of all tolerances on $S$ forms an algebraic lattice $\text{Tol}(S)$ with respect to the set inclusion and with $\Delta, \nabla$ the least and greatest elements, respectively (see [2]). The meet in this lattice corresponds with the intersection, i.e.

$$A \land B = A \cap B$$

and

$$A \lor B = T(A \cup B),$$

for any tolerances $A, B$ on $S$, where $T(M)$ denotes the least tolerance containing the set $M \subseteq S \times S$. It is called the tolerance generated by $M$. If $M = \{(a, b)\}$ then we denote $T(M) = T(a, b)$ and we call it a principal tolerance.

The following properties are easy to verify:

1. Let $M \subseteq S \times S$ be arbitrary set. Then $(x, y) \in T(M)$ if and only if $x = x_1 \land x_2 \land \ldots \land x_r, \ y = y_1 \land y_2 \land \ldots \land y_r$ and $(x_i, y_i) \in M$ or $(y_i, x_i) \in M$ or $x_i = y_i$, for $i = 1, 2, \ldots, r$.

2. $(x, y) \in T(a, b)$ if and only if $x = y$ or $x = a \land r$, $y = b \land r$ or $x = b \land r$, $y = a \land r$ for some $r \in S$.

3. $A \lor B = A \cup B \cup \{(x_1, x_2, y_1, y_2) : (x_1, y_1) \in A, (x_2, y_2) \in B\}$, for any $A, B \in \text{Tol}(S)$.

From these properties we immediately obtain next simple statement.
Lemma 1.2. Let $S$ be a $\land$-semilattice, $a, b \in S$, $a \neq b$ and $T \in \text{Tol} = (S)$. Then $T \land T(a, b) = \Delta$ if and only if $a \land r = c$, $b \land r = d$ implies $a \land r = b \land r$, for any $r \in S$ and $(c, d) \in T$.

In particular if $a \neq b$, $c \neq d$ then $T(a, b) \land T(c, d) = \Delta$ if and only if $a \land r = c \land s$, $b \land r = d \land s$ or $b \land r = c \land s$, $a \land r = d \land s$ implies $a \land r = b \land r$, for any $r, s \in S$.

2. Tolerance distributive semilattices

The following theorem is a connection of [6; Cor. 1.1] and [3; Th. 7].

Theorem 2.1. Let $S$ be a $\land$-semilattice. The following conditions are equivalent:

(a) $\text{Tol}(S)$ is modular,

(b) $\text{Tol}(S)$ is distributive,

(c) $S$ is a chain or $S$ contains a maximal chain $S_0$ and an element $z \in S_0$ such that each element of $S \setminus S_0$ covers $z$.

Since $\text{Tol}(S)$ is an algebraic lattice the condition (c) characterizes all $\land$-semilattics whose tolerance lattices are distributive relative $p$-algebras. From [7] follows that $\text{Tol}(S) \in B_0$ if and only if $S$ is a trivial semilattice or a two-element chain. In this section we will prove that for tolerance-distributive semilattice $S$ the tolerance lattice $\text{Tol}(S)$ is a relative $(L_2)$-lattice.

Let $S$ be a tolerance distributive semilattice and $T, U \in \text{Tol}(S)$, $T \leq U$. We denote $U \ast T$ the relative pseudocomplement of $U$ in $[T, \lor]$.

It is easy computation to verify that

$$U \ast T = T \lor \bigvee \{T(a, b) \colon (T(a, b) \lor T) \land U = T\}.$$ 

Lemma 2.2. Let $S$ be a $\land$-semilattice. If $S$ is a chain then $\text{Tol}(S)$ is a relative Stone algebra.

Proof. Take arbitrary $T, U \in \text{Tol}(S)$, $T \leq U$. We will prove that $U \ast T \cup (U \ast T) \ast T = \lor$. On the contrary suppose that $(a, b) \notin U \ast T \cup (U \ast T) \ast T$ for some $a, b \in S$, $a < b$. It follows that $(a, c) \notin T$ and $(a, d) \in U \ast T$, $(a, d) \notin T$ for some $c, d \in S$, $a < c, d \leq b$. Hence $(a, c \land d) \in U \land U \ast T = T$ which is a contradiction with $(a, c), (a, d) \notin T$. Therefore $U \ast T \lor (U \ast T) \ast T \supseteq U \ast T \cup (U \ast T) \ast T = S \times S = \lor$ and $[T, \lor] \in B_1$. From Prop. 1.1 we obtain that $\text{Tol}(S)$ is a relative Stone algebra. ◊
Lemma 2.3. Let $S$ be a tolerance-distributive $\wedge$-semilattice and $S$ is not a chain. Then $\text{Tol}(S)$ is a relative $(L_2)$-lattice but it is not a Stone algebra.

Proof. Let us denote $S_0$ the maximal chain in $S$ and $z \in S_0$ the element which is covered with every element from $S \setminus S_0$. Firstly we will show that $\text{Tol}(S)$ is not a Stone algebra.

Let $x, y \in S$ and $x||y$. Then $x \wedge y = z$ and we can assume that $y \in S \setminus S_0$. Let $T = T(x, z)$. Clearly $T(y, z) = \{(y, z), (z, y)\} \cup \Delta$ and $T \wedge T(y, z) = \Delta$. Obviously $(y, z) \in T(y, z) \subseteq T^*$ and $(x, z) \in T^{**}$. Hence $(x, y) \notin T^* \cup T^{**}$. Since $x, y$ are both $\wedge$-irreducible elements $(x, y) \notin T^* \vee T^{**}$ and $\text{Tol}(S) \notin B_1$. It remains to prove that $[T, \nabla] \in B_2$ for arbitrary $T \in \text{Tol}(S)$.

Let $U, V \in [T, \nabla]$. We denote $T_1 = U \wedge V, T_2 = U \ast T \wedge V, T_3 = U \wedge V \ast T$. Clearly $T_i \wedge T_j = T$ for $i, j \in \{1, 2, 3\}, i \neq j$.

Let $(x, y) \in S \times S$, $(x, y) \notin T$. Three possibilities can occur.

(a) $x, y \in S_0$, $x < y$. Then $T(x, y) = \{(x, t), (t, x) : x < t \leq y\} \cup \Delta$. Since $T_i \wedge T_j = T$ for $i \neq j$ there exists $j \in \{1, 2, 3\}$ such that $T_j$ contains no element $(x, u)$ such that $x < u$ and $(x, u) \notin T$. Therefore $(T \vee T(x, y)) \wedge T_j = (T \wedge T_j) \vee (T(x, y) \wedge T_j) = T$ and $(x, y) \in T_j \ast T$.

(b) $x, y \in S \setminus S_0$. Then $T(x, y) = \{(x, y), (y, x), (x, z), (z, x), (y, z), (z, y)\} \cup \Delta$. Since $T_i \wedge T_j = T$ for $i \neq j$ we can find again $j \in \{1, 2, 3\}$ such that $T_j$ does not contain neither $(x, z)$ neither $(y, z)$ if $(x, z), (y, z) \notin T$. Again $(T \vee T(x, y)) \wedge T_j = T$ and $(x, y) \in T_j \ast T$.

(c) $x \in S_0$, $y \in S \setminus S_0$. In this case $T(x, y) = \{(x, y), (y, x), (x \wedge y, y), (y, x \wedge y), (x \wedge y = y, t), (t, x \wedge y) : x \wedge y < t \leq x\}$. Repeating similar considerations as in (a) and (b) one can easily verify that there exists $j \in \{1, 2, 3\}$ for which the tolerance $T_j$ does not contain neither any element $(x \wedge y, s)$ such that $x \wedge y < s$ and $(x \wedge y, s) \notin T$ neither element $(x \wedge y, y)$ if $(x \wedge y, y) \notin T$. So again $(T \vee T(x, y)) \wedge T_j = (T \wedge T_j) \vee (T(x, y) \wedge T_j) = T$ and $(x, y) \in T_j \ast T$.

We can conclude that $T_1 \ast T \vee T_2 \ast T \vee T_3 \ast T \supseteq T_1 \ast T \cup T_2 \ast T \cup T_3 \ast T = S \times S = \nabla$. It means that $[T, \nabla] \in B_2$ and $\text{Tol}(S)$ is a relative $(L_2)$-lattice. $\Diamond$
3. Non-distributive case

Our aim in this section is to prove that Tol(S) is a p-algebra even for tolerance non-distributive semilattices. The following lemma plays the key role in our next considerations.

**Lemma 3.1.** Let \( S \) be a \( \land \)-semilattice, \( a, b, c_1, d_i \in S \), \( a \neq b, c_i \neq d_i \), \( i = 1, 2 \). If \( T(c_i, d_i) \land T(a, b) = \Delta \), \( i = 1, 2 \) then \( (T(c_1, d_1) \lor T(c_2, d_2)) \land T(a, b) = \Delta \).

**Proof.** Let \( T = T(c_1, d_1) \lor T(c_2, d_2) \). From (3) we obtain
\[
T = (T(c_1, d_1) \cup T(c_2, d_2) \cup \{ (c_1 \land c_2 \land r, d_1 \land d_2 \land r), (d_1 \land d_2 \land r, c_1 \land c_2 \land r), (c_1 \land c_2 \land r, d_1 \land d_2 \land r) : r \in S \}).
\]
Assume that \( T \land T(a, b) \neq \Delta \), i.e. \( c_1 \land c_2 \land r = a \land s \) and \( d_1 \land d_2 \land r = b \land s \) for some \( r, s \in S \) and \( a \land s \neq b \land s \). (Next three possibilities can be solved the same way only interchanging the letters \( c_i, d_j \).) Then \( (a \land s, b \land s) \land (c_1 \land c_2 \land d_2) = (c_1 \land c_2 \land r, d_1 \land d_2 \land r) \land (c_1 \land c_2 \land c_2 \land d_2) = (c_1 \land c_2 \land d_2 \land r, c_1 \land c_2 \land d_1 \land d_2 \land r) \in T(a, b) \). But since \( (c_1 \land r, d_1 \land r) \land (c_1 \land c_2 \land d_2) = (c_1 \land c_2 \land d_2 \land r, c_1 \land c_2 \land d_1 \land r = d_2 \land r) \in T(c_1, d_1) \), and \( T(a, b) \land T(c_1, d_1) = \Delta \), we obtain that \( c_1 \land c_2 \land d_2 \land r = c_1 \land c_2 \land d_1 \land r = d_2 \land r \). Clearly \( (c_2 \land r, d_2 \land r) \land (c_1 \land c_2) = (c_1 \land c_2 \land r, c_1 \land c_2 \land d_2 \land r) \in T(c_2, d_2) \). But again \( (a \land s, b \land s) \land (c_1 \land c_2) = (c_1 \land c_2 \land r, d_1 \land d_2 \land r) \land (c_1 \land c_2) = (c_1 \land c_2 \land c_2 \land r, c_1 \land c_2 \land d_1 \land d_2 \land r) \in T(a, b) \). Since \( T(a, b) \land T(c_2, d_2) = \Delta \), we obtain \( c_1 \land c_2 \land r = c_1 \land c_2 \land d_1 \land d_2 \land r \). The same way can be proved that also \( d_1 \land d_2 \land r = c_1 \land c_2 \land d_1 \land d_2 \land r \). But this is a contradiction with the assumption \( a \land s = c_1 \land c_2 \land r \neq d_1 \land d_2 \land r = b \land s \). Therefore \( (T(c_1, d_1) \lor T(c_2, d_2)) \land T(a, b) = \Delta \). \( \square \)

The property (1) enables us to generalize the previous statement for arbitrary set of principal tolerances disjoint with \( T(a, b) \).

**Lemma 3.2.** Let \( S \) be a \( \land \)-semilattice, \( a, b, c_i, d_i \in S \) for \( i \in I \) and \( a \neq b \). Let \( T(c_i, d_i) \land T(a, b) = \Delta \) for \( i \in I \). Then
\[
\bigvee_{i \in I} (T(c_i, d_i)) \land T(a, b) = \Delta.
\]

**Proof.** Let \( (e, f) \in \bigvee_{i \in I} (T(c_i, d_i)) \land T(a, b) \). From (1) follows that \( (e, f) \in \bigvee_{i \in J} (T(c_i, d_i)) \land T(a, b) \), for some finite \( J \subseteq I \). So it is enough to prove our statement only for finite index set \( I \).

Let \( T(c_i, d_i) \land T(a, b) = \Delta \) for \( i = 1, 2, \ldots, n \) and \( (e, f) \in \bigvee_{i = 1}^{n} (T(c_i, d_i)) \land T(a, b) \).
\( \wedge T(a, b) \). The previous lemma implies that \( e = f \) for \( n = 2 \). Assume that our statement is true for arbitrary \( n \leq k \) and that \( (e, f) \in \bigvee_{i=1}^{k+1} (T(c_i, d_i)) \wedge T(a, b) \). From (1) we obtain that
\[
eq x_1 \wedge x_2 \wedge \ldots \wedge x_m \wedge r, \quad f = y_1 \wedge y_2 \wedge \ldots \wedge y_m \wedge r,
\]
for \( r \in S \) and \( x_i = c_{j_i}, y_i = d_{j_i} \) or \( x_i = d_{j_i}, y_i = c_{j_i} \) for \( i = 1, 2, \ldots, m \). If \( j_i \leq k \) for all \( i = 1, 2 \ldots m \) then \( (e, f) \in \bigvee_{i=1}^{k} (T(c_i, d_i)) \wedge T(a, b) \) and \( e = f \). Assume that
\[
eq x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1} \wedge c_{k+1} \wedge r
\]
and
\[
f = y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1} \wedge d_{k+1} \wedge r.
\]
Then \( (e, f) \in T(x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1}, y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1}) \wedge \bigvee_{i=1}^{k} (T(c_i, d_i)) \) and \( (e, f) \in T(a, b) \). But \( T(x_1 \wedge x_2 \wedge \ldots \wedge x_{m-1}, y_1 \wedge y_2 \wedge \ldots \wedge y_{m-1}) \subseteq \bigvee_{i=1}^{k} (T(c_i, d_i)) \) and \( \bigvee_{i=1}^{k} (T(c_i, d_i)) \wedge T(a, b) = T(c_{k+1}, d_{k+1}) \wedge T(a, b) = \Delta \). Using Lemma 2.1 we obtain that \( \bigvee_{i=1}^{n} (T(c_i, d_i)) \wedge T(a, b) = \Delta \). \( \diamond \)

**Lemma 3.3.** Let \( S \) be a \( \wedge \)-semilattice. Let \( a, b \in S \) and \( a \neq b \). Then
\[
T^*(a, b) = \bigvee (T(c, d) : T(c, d) \wedge T(a, b) = \Delta).
\]

**Proof.** Let us denote the righ-hand tolerance \( T \), i.e. \( T = \bigvee (T(c, d) : T(c, d) \wedge T(a, b) = \Delta) \). We have already proved that \( T \wedge T(a, b) = \Delta \).

Let \( U \in \text{Tol}(S) \) and \( U \wedge T(a, b) = \Delta \). Clearly \( T(e, f) \subseteq U \) and \( T(e, f) \wedge T(a, b) = \Delta \) for every \( (e, f) \in U \). Therefore \( U = \bigvee (T(e, f) : (e, f) \in U) \subseteq \bigvee (T(c, d) : T(c, d) \wedge T(a, b) = \Delta) = T \) and \( T^*(a, b) = T \). \( \diamond \)

**Theorem 3.4.** Let \( S \) be a \( \wedge \)-semilattice. The lattice \( \text{Tol}(S) \) of tolerances on \( S \) is a \( p \)-algebra. More precisely
\[
T^* = \bigwedge (T^*(c, d) : (c, d) \in T)
\]
for arbitrary tolerance \( T \in \text{Tol}(S) \).

**Proof.** First we will prove that \( T \wedge \bigwedge (T^*(c, d) : (c, d) \in T) = \Delta \). Let \( (e, f) \in T \wedge \bigwedge (T^*(c, d) : (c, d) \in T) \). Then \( T(e, f) \subseteq T \) and \( (e, f) \in T(e, f) \wedge \bigwedge (T^*(c, d) : (c, d) \in T) \subseteq T(e, f) \wedge T^*(e, f) = \Delta \). Suppose that \( U \in \text{Tol}(S) \) and \( T \wedge U = \Delta \). Let \( (c, d) \in T \). Then \( U \wedge T(c, d) \subseteq U \wedge T = \Delta \), i.e. \( U \subseteq T^*(c, d) \) for any \( (c, d) \in T \). Since \( \text{Tol}(S) \) is an algebraic lattice \( U \subseteq \bigwedge (T^*(c, d) : (c, d) \in T) \) and \( \bigwedge (T^*(c, d) : (c, d) \in T) = T^* \). \( \diamond \)
The previous result reminds of results of Dona Papert. She proved [5] that congruences on semilattice form a $p$-algebra. Moreover she showed that for any two comparable congruences $\theta, \varphi$ on $S$ such that $\theta \leq \varphi$ we can define a congruence $\varphi \ast \theta$ for which $\varphi \land (\varphi \ast \theta) = \theta$ and which is the greatest congruence satisfying this equation.

Since toleration is a generalization of congruence a natural question arises whether we can analogously define a tolerance $U \ast T$ for any two comparable tolerances $T \leq U$. The following example shows that this is not possible in general.

**Example 1.** Let $S$ be a semilattice shown in Fig. 1.

Fig. 1

Let $T = T(a, b)$ and $U = T\{(a, b), (c, d)\}$. Clearly $T \subseteq U$. We will show that tolerance $U \ast T$ does not exist in $\text{Tol}(S)$. On the contrary suppose that $U \ast T$ exists. Then undoubtly $U \ast T \supseteq \bigvee (T(c, d)) : (T(c, d) \lor T) \land U = T) \lor T$. It does not take a long time to verify that $T(c_i, d_i) = \{c_i, d_i\}$, $(c_i \land a, 0), (c_i \land b, 0), (d_i \land b, 0), (d_i \land c_1 \land c_2, 0), (c_i \land c_1 \land c_2, 0), (c_i \land d_1 \land d_2, 0), (c_i \land d_1 \land d_2, 0), (c_i \land d_1 \land d_2, 0)$ and $T \lor T(c_i, d_i) = T \lor T(c_i, d_i) \lor \{a \land c_i, b \land d_i\}$, $(b \land d_i, 0), (0, d_1 \land d_2, 0), (0, d_1 \land d_2, 0), (0, d_1 \land d_2, 0), (0, d_1 \land d_2, 0)$ for $i = 1, 2$.

Therefore $(T \lor T(c_i, d_i)) \land U = T$, for $i = 1, 2$. But $(T \lor T(c_1, d_1) \lor T(c_2, d_2) \supseteq T \lor T(c_1, d_1) \lor T(c_2, d_2) \lor \{(a \land c_1 \land c_2, 0, \lor d_1 \land d_2, 0, a \land c_1 \land c_2, 0) = T \lor T(c_1, d_1) \lor T(c_2, d_2) \lor \{(c, d), (d, c)\}$ and so $(T \lor T(c_1, d_1) \lor T(c_2, d_2)) \land U = U \supseteq T$ which is a contradiction. So we can conclude that $U \ast T$ does not exist.

In Section 2, we proved that the identity $(L_2)$ is satisfied in $\text{Tol}(S)$ for every tolerance distributive semilattice. Asking which is the smallest $n$ for which the identity $(L_n)$ is satisfied in a tolerance non-distributive semilattice we obtain a much more motley answer.
Lemma 3.5. For arbitrary $n = 1, 2, 3, \ldots$ there exists a finite $\land$-semilattice $S_n$ such that $\text{Tol}(S_n) \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$.

Proof. Let $S_1$ denotes the $\land$-semilattice from Fig. 2.

\[ \begin{array}{c}
 a_1 \\
 \downarrow \\
 0 \\
 \uparrow \\
 a_2 \\
 \end{array} \]

Fig. 2.

Then $\text{Tol}(S_1)$ is a five-element lattice depicted in Fig. 3. and obviously $\text{Tol}(S_1) \in \mathcal{P}_2 \setminus \mathcal{P}_1$.

\[ \begin{array}{c}
 \nabla \\
 \downarrow \\
 T(a_1, 0) \\
 \downarrow \\
 \Delta \\
 \uparrow \\
 T(a_2, 0) \\
 \end{array} \]

Fig. 3.

For $n \geq 2$ we denote $S_n$ the $\land$-semilattice from Fig. 4.

\[ \begin{array}{c}
 b \\
 \downarrow \\
 a_1 \quad a_2 \\
 \downarrow \\
 a_3 \quad a_4 \\
 \downarrow \\
 \ldots \\
 \downarrow \\
 a_{n+1} \\
 \downarrow \\
 0 \\
 \end{array} \]

Fig. 4

Let $T_j$ be a tolerance generated by the set $\{(a_i, 0) : i \neq j\}$, i.e. $T_j = \{(a_i, 0), (0, a_i) : i \neq j\} \cup \Delta, j = 1, 2, \ldots, n$. Hence $T_j^* \supseteq T(a_j, 0), j = 1, 2, \ldots, n$ and
T_1 \wedge T_2 \wedge \ldots \wedge T_n = T(a_{n+1}, 0),
T_1^* \wedge T_2 \wedge \ldots \wedge T_n = T(a_1, 0),
\ldots
T_1 \wedge T_2 \wedge \ldots \wedge T_n^* = T(a_n, 0).

It yields that \((b, c) \notin T^*(a_i, 0)\) for \(i = 1, 2, \ldots, n + 1\) and since \(b, c\) are both maximal elements, \((b, c) \notin T^*(a_1, 0) \lor T^*(a_2, 0) \lor \ldots \lor T^*(a_{n+1}, 0)\).

Therefore
\((T_1 \wedge T_2 \wedge \ldots \wedge T_n)^* \lor (T_1^* \wedge T_2 \wedge \ldots \wedge T_n)^* \lor \ldots \lor (T_1 \wedge T_2 \wedge \ldots \wedge T_n^*)^* \neq \nabla\)
and \(\text{Tol}(S_n) \notin \mathcal{P}_n\).

Now we wish to prove that \(\text{Tol}(S_n) \in \mathcal{P}_{n+1}\). Let \(T_1, T_2 \ldots T_{n+1}\) be arbitrary tolerances on \(S_n\) and \(U_1 = T_1 \wedge T_2 \wedge \ldots \wedge T_{n+1}, U_2 = T_1^* \wedge T_2 \wedge \ldots \wedge T_{n+1}, \ldots, U_{n+2} = T_1 \wedge T_2 \wedge \ldots \wedge T_{n+1}\). Since \(U_1, U_2 \ldots U_{n+2}\) are \(n + 2\) pairwise disjoint tolerances there exists \(j \in \{1, 2 \ldots n + 2\}\) such that \((a_i, 0) \notin U_j\) for \(i = 1, 2 \ldots n + 1\). Two possibilities can occur:

(i) If \(n > 2\) then \(U_j = \Delta\) and trivially \(U_1^* \lor U_2^* \lor \ldots \lor U_{n+2}^* = \nabla\).

(ii) If \(n = 2\) then \(U_j = \Delta\) or \(U_j = T(a_3, c)\).

In the second case \(U_j^* = (S \times S) \setminus \{(a_3, c), (c, a_3)\}\). Since for any tolerance \(U\) such that \((a_3, c) \notin U\) is \((a_3, c) \in U^*\) we obtain \(U_1^* \lor U_2^* \lor \ldots \lor U_{n+2}^* \supseteq U_1^* \cup U_2^* \cup U_{n+2}^* = S \times S = \nabla\). ∎

References


