RECTANGULAR MODULUS AND GEOMETRIC PROPERTIES OF NORMED SPACES

Ioan Șerb

Department of Mathematics, "Babeș-Bolyai" University, 3400 Cluj-Napoca, România

Received: August 1997

MSC 1991: 46 B 20; 46 C 15

Keywords: Moduli for normed spaces, orthogonality, geometry of normed spaces.

Abstract: Recently, [18] we have introduced the rectangular (*-rectangular) modulus of a normed space $X$. It is a convex function strongly related to some known constants of $X$. The aim of this paper is to characterize some geometric properties of normed spaces in terms of the rectangular modulus. We prove that a normed space of dimension $\geq 3$ is an inner product space if and only if the right derivative in 0 of the rectangular modulus is zero. The case of two-dimensional spaces is also treated. A characterization of the uniform convexity of $X$ is given in terms of the *-rectangular modulus.

1. Introduction and notation

The geometry of a real linear normed space $X$ with dim $X \geq 2$ may be described, among others, using some moduli attached to $X$ and their properties. For instance, the moduli of convexity [5], and of smoothness [11] are well known and often used in various applications.

Let us denote by $B(x, r)$ the closed ball of $X$, (dim $X \geq 2$) with center $x$ and radius $r > 0$ and by $B = B(0, 1)$ the closed unit ball of $X$. Let $S(x, r)$, respectively $S = S(0, 1)$ be the corresponding spheres of $X$. The symbol $\perp$ will be used for Birkhoff orthogonality in the normed space $(X, \| \cdot \|)$, namely $x \perp y$ iff $\|x\| \leq \|x + \mu y\|$ holds for all $\mu \in \mathbb{R}$. 
For \(x, y \in X, x \neq y\) denote \(L(x, y)\) the straight line passing through \(x\) and \(y\). Similarly, \([x; y]\) will be the suitable closed segment. Recall that the \textit{modulus of convexity} of \(X\) is the function \(\delta_X : [0, 2] \rightarrow \mathbb{R}\) defined by:

\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{\|x - y\|} : x, y \in S, \|x - y\| = \varepsilon \right\}, \quad \varepsilon \in [0, 2],
\]

while the \textit{modulus of smoothness} of \(X\) is the function \(\rho_X : [0, \infty) \rightarrow \mathbb{R}\) defined by:

\[
\rho_X(\tau) = \sup \left\{ \frac{1}{2}(\|x + \tau y\| + \|x - \tau y\| - 2) : x, y \in S \right\}, \quad \tau \geq 0.
\]

The following modulus of smoothness, modified with a condition of orthogonality, was defined in [9] as being the function \(\overline{\rho}_X : [0, \infty) \rightarrow \mathbb{R}\)

\[
\overline{\rho}_X(\tau) = \sup \left\{ \frac{1}{2}(\|x + \tau y\| + \|x - \tau y\| - 2) : x, y \in S, x \perp y \right\}, \quad \tau \geq 0.
\]

T. Figiel [9] has proved that \(\rho_X\) and \(\overline{\rho}_X\) are equivalent, more precisely we have:

(1) \[\frac{1}{8} \rho_X(\tau) \leq \overline{\rho}_X(\tau) \leq \rho_X(\tau), \forall \tau \geq 0.\]

Now, a normed space is said to be \textit{uniformly convex} if \(\delta_X(\varepsilon) > 0, \forall \varepsilon \in (0, 2]\) and \textit{uniformly smooth} if \(\lim_{\tau \searrow 0} \rho_X(\tau)/\tau = 0\), (or equivalently if \(\lim_{\tau \searrow 0} \overline{\rho}_X(\tau)/\tau = 0\).

The normed space \(X\) is said to be \textit{smooth} at \(x_0 \in S\) whenever there exists a unique \(f \in X^*, \|f\| = 1\) such that \(f(x_0) = 1\). If \(X\) is smooth at each point of \(S\) then we say that \(X\) is \textit{smooth}, [8, p.21]. A normed space \(X\) is said to be \textit{strictly convex} whenever \(S\) contains no non-trivial line segments, [8, p.23]. A uniformly smooth space is said to have \textit{modulus of smoothness of power type} \(p\), with \(p > 1\) if there exists a number \(C > 0\) such that \(\rho_X(\tau) \leq C\tau^p, \forall \tau \geq 0\), [12, p.63].

K. Przeslawski and D. Yost [13], [14] have introduced the modulus of squareness. It appears, in a natural way, in some estimates for the Lipschitz constants of multivalued mappings in Banach spaces. They considered a pair \((x, y)\) of points in \(X\) with \(\|y\| < 1 < \|x\|\). Then there is a unique \(z = z(x, y)\) in the line segment \([x; y]\) with \(\|z\| = 1\). As in [14] we put

\[
\omega(x, y) = \frac{\|x - z(x, y)\|}{\|x\| - 1}
\]

and define the \textit{modulus of squareness} \(\xi_X : [0, 1) \rightarrow \mathbb{R}\) by

\[
\xi_X(\beta) = \sup \{ \omega(x, y) : \|y\| \leq \beta < 1 < \|x\| \}, \quad \beta \in [0, 1).
\]

In [15] we have obtained the following alternative formula for \(\xi_X\):
\[ (2) \quad \xi_X(\beta) = \sup \{\|x - y\| : x \in S, y \in X, x \perp y, \min_{\lambda \geq 0} \|(1 - \lambda)x + \lambda y\| = \beta \}, \]

\( \beta \in [0, 1) \). Surprisingly, from the behaviour of \( \xi_X \) in the neighbourhood of 1 and of 0 respectively, it is possible to characterize uniformly convex and uniformly smooth normed spaces. The relation

\[ (3) \quad \lim_{\beta \searrow 1}(1 - \beta)\xi_X(\beta) = 0, \]

characterizes the uniform convexity of \( X \), [3, 13], while the relation

\[ (4) \quad \lim_{\beta \searrow 0}\frac{\xi_X(\beta) - 1}{\beta} = 0, \]

characterizes the uniform smoothness of \( X \), [4, 16]. On the other hand \( \xi_X \) is an increasing function, convex in the neighbourhood of 1, it verifies a Day-Nordlander type inequality and characterizes inner product spaces (i.p.s for short) [4, 17]. Recently, we have introduced the rectangular modulus of \( X \) [18], as the function \( \mu_X : (0, \infty) \to \mathbb{R} \)

\[ \mu_X(\lambda) = \sup \{\max\{\varphi_{\lambda,x,y}(t), \lambda\varphi_{\lambda,x,y}(t)\} : t > 0, x, y \in S, x \perp y\}, \lambda > 0, \]

where

\[ \varphi_{\lambda,x,y}(t) = \frac{\lambda + t}{\|x + ty\|}, \lambda, t > 0, x, y \in S, x \perp y. \]

The function \( \varphi_{\lambda,x,y} \) is a useful ingredient in some characterizations of i.p.s in terms of Birkhoff orthogonality. In the same paper it was also proved that

a) \( \mu_X \) is a convex function; if \( H \) is an i.p.s then \( \mu_H(\lambda) = \sqrt{1 + \lambda^2} \);

b) \( \mu_X \) verifies a Day-Nordlander inequality i.e.: \( \mu_X(\lambda) \geq \mu_H(\lambda) = \sqrt{1 + \lambda^2} \), \( \forall \lambda > 0 \);

c) \( \mu_X(\lambda) = \sqrt{1 + \lambda^2} \) for a fixed \( \lambda > 0 \), then \( X \) is an i.p.s.

The rectangular modulus [18] defined by the simpler formula

\[ \mu_X^*(\lambda) = \sup \{\varphi_{\lambda,x,y}(t) : t > 0, x, y \in S, x \perp y\}, \lambda > 0, \]

verifies also the properties a), b) and c). Moreover \( \mu_X^*(\lambda) \leq \lambda + 2, \forall \lambda > 0 \).

On the other hand \( \mu_X(1) = \mu_X^*(1) = \mu(X) \), where \( \mu(X) \) is the rectangular constant of \( X \) defined by J.L. Joly [10]. Let \( \mu_X(0+) \) be given by \( \mu_X(0+) = \lim_{\lambda \searrow 0}\mu_X(\lambda) \). Then \( \mu_X(0+) = \mu_X^*(0+) \in [1, 2] \) and \( \mu_X(0+) \) is the known radial constant of \( X \), denoted by \( k(X) \), [20], which in turn is equal to other four constants of \( X \), denoted by \( MPB(X), MPB'(X), \overline{MPB}(X), \beta(X) \), respectively. For more information on this subject see [2], [3], [6], [7], [19], [20].
2. Main results

In this paper we obtain some relations between the properties of \( \mu_X, (\mu_X^*) \) and the geometry of the normed space \( X \). A characterization of i.p.s of dimension \( \geq 3 \) is deduced from the knowledge of the right derivative of \( \mu_X^* \) in the origin. The two-dimensional case is partially treated. A characterization of uniformly convex spaces is obtained from the behaviour of \( \mu_X^* \) at infinity.

For \( x, y \in X \) let \( \tau(x, y) \) be defined by:

\[
\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} = \inf_{t > 0} \frac{\|x + ty\| - \|x\|}{t}.
\]

It is clear that \( X \) is smooth if and only if \( \tau(x, y) = -\tau(x, -y) \), for any pair \( (x, y) \in X \times X \) with \( x \neq 0 \).

**Lemma A** [3]. A normed space \( X \) is smooth if and only if the following condition holds:

\[
\{(x, y) \in S \times S : x \perp y \} = \{(x, y) \in S \times S : \tau(x, y) = 0 \}.
\]

A uniformly smooth variant of Lemma A is given by

**Lemma 2.1.** A normed space \( X \) is uniformly smooth if and only if the following condition holds:

\[
\alpha) \quad x, y \in S, x \perp y \Rightarrow \|x + ty\| = 1 + o(x, y, t),
\]

where \( \lim_{t \to 0} o(x, y, t)/t = 0 \), uniformly with respect to \( x, y \in S, x \perp y \).

**Proof.** i) If \( X \) is uniformly smooth and \( x, y \in S, x \perp y \) then by Lemma A

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t \to 0} \frac{o(x, y, t)}{t} = \tau(x, y) = 0.
\]

By uniform smoothness this limit is uniform with respect to \( x, y \in S, x \perp y \), and \( \alpha \) follows.

ii) Suppose that \( \alpha \) holds and that \( X \) is not uniformly smooth. Then \( \lim_{t \to 0} \rho_X(t)/t = \inf_{t > 0} \rho_X(t)/t = a > 0 \). Using (1) it follows that \( \lim_{t \to 0} \bar{\rho}_X(t)/t = \inf_{t > 0} \bar{\rho}_X(t)/t \geq a/8 \). There exists then a sufficiently small \( \varepsilon > 0 \) such that \( \bar{\rho}_X(t)/t > a/16 \), for all \( t \in (0, \varepsilon) \). For any \( t \in (0, \varepsilon) \) choose a pair \( (x_t, y_t) \in S \times S, x_t \perp y_t \) such that

\[
\frac{1}{2t}(\|x_t + ty_t\| + \|x_t - ty_t\| - 2) > a/32.
\]

Let \( \bar{y}_t \in \{y_t, -y_t\} \) be such that \( \|x_t + t\bar{y}_t\| = \max \{\|x_t + ty_t\|, \|x_t - ty_t\|\} \).

One obtains \( (\|x_t + t\bar{y}_t\| - 1)/t > a/32 \), for all \( t \in (0, \varepsilon) \). It follows that

\[
\frac{o(x_t, \bar{y}_t, t)}{t} \geq a/32, \quad \forall t \in (0, \varepsilon),
\]

contradicting \( \alpha \). ◊
The relation (4) characterizes the uniform smoothness in terms of the squareness modulus. In the sequel we will see that similar formula for \(\ast\)-rectangular modulus of \(X\) has a different interpretation.

**Lemma 2.2.** If the \(\ast\)-rectangular modulus of \(X\) verifies the relation

\[
\lim_{\lambda \searrow 0} \frac{\mu_X^\ast(\lambda) - \mu_X^\ast(0+)}{\lambda} = 0,
\]

then the Birkhoff orthogonality in \(X\) is symmetric.

**Proof.** Let \(\lambda \in (0, 1)\) and \(x, y \in S, x \perp y\) be given. It follows that

\[
\varphi_{\lambda, x, y}(t) = \frac{\lambda + t}{\|x + ty\|} = \lambda \cdot \left(\frac{\lambda x + t\lambda y}{\lambda + t}\right)^{-1}, \quad \forall t > 0,
\]

and

\[
\psi(x, y, \lambda) \overset{\text{def}}{=} \sup_{t > 0} \varphi_{\lambda, x, y}(t) = \lambda \cdot \left(\min_{\mu \in [0, 1]} \|\mu x + (1 - \mu)\lambda y\|\right)^{-1} = \\
\lambda \cdot \|\mu_0 x + (1 - \mu_0)\lambda y\|^{-1},
\]

where \(\mu_0 = \mu_0(x, y, \lambda) \in [0, 1]\) and \(\mu_0\) is not necessarily unique. If any \(\mu_0(x, y, \lambda)\) is \(\neq 0\), then the straight line \(L(x, \lambda y)\) is a support line for the sphere \(S(0, \|\mu_0 x + (1 - \mu_0)\lambda y\|)\) and \(\psi(x, y, \lambda) > 1\). In the opposite case \(\psi(x, y, \lambda) = 1\). Supposing that \(\psi(x, y, \lambda) = 1\), for all \(x, y \in S, x \perp y\) one obtains that \(\mu_X^\ast(\lambda) = 1 < \sqrt{1 + \lambda^2}\), in contradiction with the property b) of \(\mu_X^\ast\). This means that in order to obtain \(\sup_{x, y \in S, x \perp y} \psi(x, y, \lambda) = \mu_X^\ast(\lambda)\), we can consider only the pairs \(x, y \in S, x \perp y\) with any \(\mu_0(x, y, \lambda) \in (0, 1)\).

A parallel to the straight line \(L(x, \lambda y)\) from the origin intersects the parallel to the straight line \(L(0, \mu_0 x + (1 - \mu_0)\lambda y)\) from \(y\) in \(y_0 = y_0(x, y, \lambda)\). The triangle with vertices 0, \(\mu_0 x + (1 - \mu_0)\lambda y, \lambda y\) is similar to the triangle with vertices \(y, y_0, 0\). From this we obtain:

\[
\lambda \cdot \|\mu_0 x + (1 - \mu_0)\lambda y\|^{-1} = \|y\| \cdot \|y - y_0\|^{-1},
\]

and

\[
\mu_X^\ast(\lambda) = \sup_{x, y \in S, x \perp y} \psi(x, y, \lambda y) = \left(\inf\{|y - y_0| : x, y \in S, x \perp y}\right)^{-1}.
\]

On the other hand we have

\[
\mu_X^\ast(0+) = \sup\{|tx + y|^{-1} : t > 0, x, y \in S, x \perp y\} = \\
= \left(\inf\{|y - x_0| : x, y \in S, x \perp y}\right)^{-1},
\]

where \(x_0 = x_0(x, y) \in L(0, x), y - x_0 \perp x\). By changing \(x\) in \(-x\) we can suppose that \(0 \in [x_0, x]\) (see Fig. 1).

In general, \(x_0(x, y)\) is not uniquely determined. Let \(z_0 = z_0(x, y, \lambda)\) be defined by \(\{z_0\} = L(y, x_0) \cap L(0, y_0)\). Since \(y - y_0 \perp y_0\) we have \(\|y - y_0\| \leq \|y - z_0\|\). One obtains
\( \mu_x^*(\lambda) \geq \frac{1}{\inf\{|y - z_0(x, y, \lambda)| : x, y \in S, x \perp y\}}. \)

Let \( z_1 = z_1(x, y, \lambda) \) be given by \( \{z_1\} = L(y, x_0) \cap L(x, \lambda y) \). Since for two nonzero collinear vectors \( u \) and \( v, \|u\|/\|v\| \) is independent of the norm, we can apply the Menelaus Theorem in a two-dimensional normed space. Consider the triangle with vertices \( 0, x_0, y \) and the transversal \( L(x, \lambda y) \). The relation

\[
\frac{\lambda}{1 - \lambda} \cdot \frac{\|y - z_1\|}{\|z_1 - x_0\|} \cdot \frac{1 + \|x_0\|}{1} = 1,
\]

implies

\[
\|y - z_1\| = \frac{1 - \lambda}{1 + \lambda \|x_0\|} \cdot \|y - x_0\|,
\]

and

\[
\|y - z_0\| = \frac{1}{1 + \lambda \|x_0\|} \cdot \|y - x_0\|.
\]

Finally, suppose that (5) holds and that Birkhoff orthogonality in \( X \) is not symmetric. Then from [17] \( \mu_x^*(0+) > 1 \). Let again \( \lambda \in (0, 1) \) be fixed. By (6) there exists a pair \( x', y' \in S, x' \perp y' \) such that

\[
\mu_x^*(0+) \geq \frac{1}{\|y' - x_0(x', y')\|} > \mu_x^*(0+) - \lambda^2,
\]

and such that \( 0 \in [x_0(x', y'), x'] \). It follows that

\[
\frac{1}{\|y' - z_0(x', y', \lambda)\|} = \frac{1 + \lambda \|x_0(x', y')\|}{\|y' - x_0(x', y')\|}.
\]

From (7), (8) and (9) we obtain:

\[
\frac{\mu_x^*(\lambda) - \mu_x^*(0+)}{\lambda} > \frac{1}{\|y' - z_0(x', y', \lambda)\|} - \frac{1}{\|y' - x_0(x', y')\|} - \lambda^2
\]
\[
\frac{\lambda \| x_0(x', y') \|}{\| y' - x_0(x', y') \|} - \lambda^2 \geq \frac{\| x_0(x', y') \|}{\| y' - x_0(x', y') \|} - \lambda \geq \frac{\| y' \| - \| y' - x_0(x', y') \|}{\| y' - x_0(x', y') \|} - \lambda \geq \mu_X^*(0+) - \lambda^2 - \lambda - 1.
\]

We have
\[
\lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} \geq \mu_X^*(0+) - 1 > 0,
\]
in contradiction with (5). \(\diamondsuit\)

**Theorem 2.3.** Let \( X \) be a real normed space, \( \dim X \geq 3 \). The following are equivalent:

1) \( \lim_{\lambda \searrow 0} \frac{\mu_X(\lambda) - \mu_X(0+)}{\lambda} = 0. \)

2) \( \lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = 0. \)

3) \( X \) is an inner product space.

**Proof.** 1) \( \Rightarrow \) 2). Suppose that 1) is valid. This implies that
\[
0 \leq \lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} \leq \lim_{\lambda \searrow 0} \frac{\mu_X(\lambda) - \mu_X(0+)}{\lambda} = 0,
\]
and 2) follows.

2) \( \Rightarrow \) 3). By Lemma 2 and 2) we have that the Birkhoff orthogonality is symmetric. Since \( \dim X \geq 3 \), it follows (see [1, p. 143]) that \( X \) is an i.p.s.

3) \( \Rightarrow \) 1). \( X \) being an i.p.s we have \( \mu_X(\lambda) = \sqrt{1 + \lambda^2}, \forall \lambda > 0 \), and 1) is obvious. \( \diamondsuit \)

**Theorem 2.4.** Let \( X \) be a two-dimensional real Banach space. If
\[
\lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = 0,
\]
then \( X \) is strictly convex.

**Proof.** Suppose that \( X \) is not strictly convex. Then, using the notation from Lemma 2, there exists a pair \( x'', y'' \in S, x'' \perp y'' \) such that \( \| x_0(x'', y'') \| > 0 \), and \( 0 \in [x_0(x'', y''), x''] \). The symmetry of orthogonality implies that \( \| y'' - x_0(x'', y'') \| = 1 \). As in Lemma 2 we have:
\[
\frac{\mu_X^*(\lambda) - \mu_X(0+)}{\lambda} \geq \frac{\| x_0(x'', y'') \|}{\| y'' - x_0(x'', y'') \|} - \lambda = \| x_0(x'', y'') \| - \lambda.
\]
One obtains that
\[
\lim_{\lambda \to 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} \geq \|x_0(x'', y'')\| > 0,
\]
a contradiction. Now, it is well-known that a two-dimensional space with symmetric orthogonality is strictly convex, iff it is smooth (see [1, p.78]). So, \(X\) is uniformly smooth, uniformly convex and the Birkhoff orthogonality in \(X\) is symmetric.\(\diamondsuit\)

**Theorem 2.5.** Let \(X\) be a two-dimensional real Banach space. We suppose that the Birkhoff orthogonality in \(X\) is symmetric and that \(X\) is smooth with the modulus of smoothness of power type \(p, p > (\sqrt{5} + 1)/2,\) then

\[
\lim_{\lambda \to 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = 0.
\]

**Proof.** Let \(x, y \in S, x \perp y\) and \(\lambda > 0\) be fixed. We have

\[
\frac{\|x - \lambda y\| - 1}{2} \leq \frac{\|x + \lambda y\| - 1 + \|x - \lambda y\| - 1}{2} \leq \bar{\rho}_X(\lambda) \leq \rho_X(\lambda) \leq C\lambda^p.
\]

Then \(\|x - \lambda y\| \leq 1 + C_1 \lambda^p, \forall \lambda > 0, \forall x, y \in S, x \perp y,\) and by Lemma 1, \(o(x, -y, \lambda) \leq C_1 \lambda^p, \forall x, y \in S, x \perp y, \forall \lambda > 0.\) Moreover the function \(o(x, -y, \cdot)\) is increasing in \((0, \infty).\) Using again the notation in Lemma 2, we observe that \(h = h(x, y, \lambda) = \mu_0 x + (1 - \mu_0)\lambda y \perp x - \lambda y, (h\) is unique) and by the symmetry of orthogonality \(h - x \perp h.\) We have \(\|h - x\| \leq \|x\| = 1\) and \(\|h - \lambda y\| \leq \lambda.\) Let \(h_1\) be the unique vector in the line segment \([x; \lambda y]\) verifying \(\|h_1 - x\| = 1.\) A parallel from \(h_1\) to the straight line \(L(0, h)\) intersects \(L(0, x)\) in \(h_2.\) We have

\[
\frac{\|x - h\|}{1} = \frac{\|h\|}{\|h_2 - h_1\|},
\]

and \(h_2 - h_1 \perp x - \lambda y.\) Now, by Lemma 1

\[
\|h_2 - h_1\| = \frac{\|h\|}{1 + o(x, -y, \lambda) - \|h - \lambda y\|} \leq \frac{\|h\|}{1 + o(x, -y, \lambda) - \lambda}.
\]

By orthogonality and Lemma 1 it follows that:

\[
\|h - h_1\| \leq \|x - h_2\| - \|x\| = \|x - h_2\| - \|x - h_1\| = 1 + o \left( \frac{x - \lambda h}{\|x - \lambda h\|}, h \right) = 1 \leq C_1 \|h_2 - h_1\|^p \leq C_1 (1 + o(x, -y, \lambda) - \lambda)^p.
\]

But from \(y - y_0 \perp x - \lambda y\) we obtain
\[ 1 = \|y\| = \|y - y_0 + y_0\| = \|y - y_0\| \cdot \left( \frac{y - y_0}{\|y - y_0\|} + \frac{y_0}{\|y - y_0\|} \right) = \|y - y_0\| \cdot \left( 1 + o \left( \frac{y - y_0}{\|y - y_0\|}, \frac{x - \lambda y}{\|x - \lambda y\|}, \frac{y_0}{\|y - y_0\|} \right) \right) \leq \|y - y_0\| \cdot \left( 1 + C_1 \frac{\|y_0\|^p}{\|y - y_0\|^p} \right). \]

The triangle with vertices 0, h, \lambda y is similar to the triangle with vertices y, y_0, 0 and this means that
\[ \frac{\|y_0\|}{\|y - y_0\|} = \frac{\|h - \lambda y\|}{\|h\|} = \frac{\|h - h_1\| + \|h_1 - \lambda y\|}{\|h\|} \leq \frac{\|h - h_1\|}{\|h\|} + \frac{\|x - \lambda y\| - 1}{\|h\|} \leq C_1 \frac{\|h\|^{p-1} \lambda^p}{(1 + o(x, -y, \lambda) - \lambda)^p} + \frac{\|x - \lambda y\| - 1}{\|h\|}. \]

Since \( \lambda/\|h\| \leq \mu_X^*(\lambda) \leq \lambda + 2 \), for \( \lambda > 0 \) small enough
\[ \frac{\|y_0\|}{\|y - y_0\|} \leq 2C_1 \|h\|^{p-1} + C_1 \frac{\lambda^p}{\|h\|} \leq 2C_1 \lambda^{p-1} + 3C_1 \lambda^{p-1} = 5C_1 \lambda^{p-1}, \]
which implies that
\[ 1 - \|y - y_0\| \leq \|y - y_0\| \frac{\|y_0\|^p}{\|y - y_0\|^p} \leq \|y - y_0\| \cdot C_1 \cdot 5^p \cdot C_1^p \lambda^{p-1} = \|y - y_0\| \cdot 5^p \cdot C_1^{p+1} \lambda^{p-1}. \]

The symmetry of orthogonality yields \( \mu_X^*(0+) = 1 \) and:
\[ 0 \leq \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = \sup \left\{ \frac{1}{\|y - y_0(x, y, \lambda)\|} : x, y \in S, x \perp y \right\} - 1 \leq \frac{5^p C_1^{p+1} \lambda^{p-1}}{\lambda} = 5^p \cdot C_1^{p+1} \cdot \lambda^{p-1}, \]
with \( \lambda \) close to 0. If \( p > (\sqrt{5} + 1)/2 \) then \( \lim_{\lambda \downarrow 0} (\mu_X^*(\lambda) - \mu_X^*(0+))/\lambda = 0 \).

**Remark.** Denoting by \( H \) an inner product space it is well-known [11] that
\[ \rho_X(\tau) \geq \rho_H(\tau) = \sqrt{1 + \tau^2} - 1 = \frac{\tau^2}{2} + o(\tau^2). \]

This implies that \( \rho_X \) is of power type at most 2.

**Example.** Let \( p \in ((\sqrt{5} + 1)/2, 2) \) be a given number and let \( q \) be its conjugate \( 1/p + 1/q = 1 \). In \( \mathbb{R}^2 \) define the norm:
\[
\|(\alpha, \beta)\| = \begin{cases} 
(|\alpha|^p + |\beta|^p)^{1/p} = \|(\alpha, \beta)\|_p, & \text{for } \alpha \beta \geq 0 \\
(|\alpha|^q + |\beta|^q)^{1/q} = \|(\alpha, \beta)\|_q, & \text{for } \alpha \beta < 0.
\end{cases}
\]

Then \((\mathbb{R}^2, \| \cdot \|)\) is a Banach space and the Birkhoff orthogonality is symmetric, (see [1, p.77]). Let \(x_1 = (\alpha_1, \beta_1), x_2 = (\alpha_2, \beta_2)\) be two unit vectors with \(x_1 \perp x_2\). We have

\[
\|x_1 + \lambda x_2\| - 1 \leq \max\{\|x_1 + \lambda x_2\|_p - 1, \|x_1 + \lambda x_2\|_q - 1\} \leq \max\{C_1 \lambda^p, C_2 \lambda^2\} \leq (C_1 + C_2) \lambda^p, \forall \lambda \in [0, 1).
\]

The space \((\mathbb{R}^2, \| \cdot \|)\) is two-dimensional, uniformly convex and uniformly smooth with modulus of smoothness of power type > \((\sqrt{5} + 1)/2\). From Th. 2.5 it follows that \(\lim_{\lambda \to 0}(\mu_X^*(\lambda) - \mu_X^*(0+))/\lambda = 0\). However \((\mathbb{R}^2, \| \cdot \|)\) is not a Hilbert space.

**Theorem 2.6.** The real normed space \(X\) is uniformly convex if and only if

\[
\lim_{\lambda \to \infty}(\mu_X^*(\lambda) - \lambda) = 0.
\]

**Proof.** Let \(x, y \in S, x \perp y\) and \(\lambda > 1\) be fixed. Denote by \(h_0(\lambda) = \inf\{\|h(x, y, \lambda)\| : x, y \in S, x \perp y\}\) where \(h(x, y, \lambda)\) is as in Th. 2.5. In the two-dimensional subspace \(X\) of \(X\), generated by \(x\) and \(y\) we consider the ball \(B(0, h_0(\lambda))\) and a support line \(l_x\) to \(B(0, h_0(\lambda))\) passing through \(x\). Suppose that \(\{\lambda_1 y\} = L(0, y) \cap l_x\) is chosen such that \(\lambda_1 > 0\). Then \(0 < \lambda_1 \leq \lambda\) and from \(x \perp y\) it follows:

\[
\|x - \lambda_1 y\| \leq \|x - \lambda y\| \leq 1 + \lambda.
\]

Using formula (2) for the definition of the squareness modulus we obtain:

\[
\xi_X(h_0(\lambda)) \leq 1 + \lambda, \forall \lambda > 0.
\]

From \(\mu_X^*(\lambda) = \lambda/h_0(\lambda) \geq \sqrt{1 + \lambda^2} = \mu_H^*(\lambda), \lambda > 0\), we have that \(h_0(\lambda) + 1/(4\lambda^2) < 1\), for all \(\lambda > 1\). Pick now \(x, y \in S, x \perp y\) such that \(\|h(x, y, \lambda)\| \leq h_0(\lambda) + 1/(4\lambda^2)\). For large \(\lambda\) one obtains

\[
\xi_X\left(h_0(\lambda) + \frac{1}{4\lambda^2}\right) \geq \xi_X(\|h(x, y, \lambda)\|) \geq \|x - \lambda y\| \geq \lambda - 1,
\]

implying \(h_0(\lambda) \geq \xi_X^{-1}(\lambda - 1) - 1/(4\lambda^2)\). On the other hand \(h_0(\lambda) \leq \xi_X^{-1}(\lambda + 1)\), and

\[
\lambda(1 - \xi_X^{-1}(\lambda + 1)) \leq \lambda(1 - h_0(\lambda)) \leq \lambda(1 - \xi_X^{-1}(\lambda - 1)) + \frac{1}{4\lambda}.
\]

Letting \(\beta(\lambda) = \xi_X^{-1}(\lambda + 1), \gamma(\lambda) = \xi_X^{-1}(\lambda - 1)\), it follows \(\beta(\lambda), \gamma(\lambda) \to 1\) for \(\lambda \to \infty\) and
(1 - \beta(\lambda))\xi_X(\beta(\lambda)) - 1 + \beta(\lambda) \leq \frac{\lambda}{\mu_X^*(\lambda)}(\mu_X^*(\lambda) - \lambda) \leq (1 - \gamma(\lambda))\xi_X(\gamma(\lambda)) + 1 - \gamma(\lambda) + \frac{1}{4\lambda}.

Suppose that X is uniformly convex. Using formula (4) we get

$$\lim_{\lambda \to \infty} \frac{\lambda}{\mu_X^*(\lambda)} \cdot (\mu_X^*(\lambda) - \lambda) = 0.$$ 

Now, from [18] we have

$$\frac{\lambda}{\lambda + 2} \leq \frac{\lambda}{\mu_X^*(\lambda)} \leq \frac{\lambda}{\sqrt{1 + \lambda^2}}; \lim_{\lambda \to \infty} \frac{\lambda}{\mu_X^*(\lambda)} = 1.$$ 

and \(\lim_{\lambda \to \infty} (\mu_X^*(\lambda) - \lambda) = 0\). Finally, if (10) holds then

$$0 \leq \lim_{\lambda \to \infty} [(1 - \beta(\lambda))(\xi_X(\beta(\lambda)) - 1)] \leq \lim_{\lambda \to \infty} \frac{\lambda}{\mu_X^*(\lambda)}(\mu_X^*(\lambda) - \lambda) = 0,$$

implying \(\lim_{\lambda \to \infty} (1 - \beta)\xi_X(\beta) = 0\), i.e. X is uniformly convex. \(\diamond\)

**Corollary 2.7.** The real normed space X is uniformly smooth if and only if

$$\lim_{\lambda \to \infty} (\mu_X^*(\lambda) - \lambda) = 0.$$

**References**


