NEAR-RINGS IN WHICH EACH PRIME FACTOR IS SIMPLE

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Abstract: In this paper we investigate connections between the condition that every prime ideal is maximal and various generalizations of von Neumann regularity. As a corollary of our results we show that if \( N \) is a reduced zero-symmetric right near-ring, then every prime ideal is maximal if and only if \( N \) is left weakly regular (i.e., \( x \in \langle x \rangle x \), for all \( x \in N \), where \( \langle x \rangle \) denotes the ideal generated by \( x \)).

Throughout this paper all near-rings are zero symmetric right near-rings, and \( N \) denotes such a near-ring. It can be shown that if \( N^2 = N \), then every maximal ideal is a prime ideal. Since, in general, prime ideals are not maximal even in near-rings with unity, it is natural to ask: when is every prime ideal of \( N \) also a maximal ideal of \( N \)? Near-rings satisfying this condition (equivalently, every prime factor is simple) are said to satisfy the pm condition. Surprisingly, the pm condition (a condition on ideals) has been shown to be equivalent to various generalizations of von Neumann regularity (a condition on elements) for large classes of rings \([2], [4], [8], [9], [11], [12], [13], [14], [19], [20], [28], [29], [31] and [32]. The survey paper [5] gives an overview of the research in this area.
For the case of commutative rings, the first clearly established equivalence between the pm condition and a generalization of von Neumann regularity seems to have been made by Storrer [28] in the following result. If \( R \) is a commutative ring with unity, then the following statements are equivalent: (1) \( R \) is \( \pi \)-regular; (2) \( R/\mathcal{P}(R) \) is von Neumann regular (\( \mathcal{P}(R) \) is the prime radical of \( R \)); (3) \( R \) satisfies the pm condition. For a near-ring \( N \) with unity it is well known [24, p.349] that if \( N \) is strongly regular then \( N \) satisfies the pm condition. In the paper we will investigate the connections between the pm condition and various generalizations of von Neumann regularity in the class of near-rings. In particular, we will extend the main results of [8] to near-rings. We provide examples which illustrate the contrast between the ring and the near-ring cases for our results.

Let \( \mathcal{P}_0(N) \) denote the prime radical and \( \mathcal{N}(N) \) the set of nilpotent elements of the near-ring \( N \). From [6], an ideal \( I \) of \( N \) is a 2-primal ideal of \( N \) if \( \mathcal{P}_0(N/I) = \mathcal{N}(N/I) \). If \( I \) is the zero ideal of \( N \), then \( N \) is a 2-primal near-ring. (This is equivalent to \( \mathcal{P}_0(N) = \mathcal{N}(N) \). A near ring is said to be reduced if \( \mathcal{N}(R) = 0 \). Recall from [24] that an ideal \( P \) is called a minimal prime ideal of an ideal \( I \) if \( P \) is minimal in the set of all prime ideals containing \( I \). If \( I \) is the zero ideal, then \( P \) is called a minimal prime ideal of \( N \). By \( \mathcal{B}(I) \) we denote the intersection of all prime ideals of \( N \) containing \( I \). From [16], \( \mathcal{B}(I) \) is the intersection of all minimal prime ideals in \( N \) containing \( I \). An ideal \( I \) of \( N \), denoted by \( I \triangleleft N \), is a completely prime ideal (completely semi-prime ideal) if for \( a, b \in N \), \( ab \in I \) implies \( a \in I \) or \( b \in I \) (\( a^2 \in I \) implies \( a \in I \)). The completely prime radical \( \mathcal{P}_c(N) \) of the near-ring \( N \) is the intersection of all the completely prime ideals of \( N \). If follows from [17] that \( \mathcal{P}_c(N) \) is completely semi-prime ideal of \( N \). Moreover from [7], \( N \) is 2-primal if and only if \( \mathcal{P}_0(N) = \mathcal{P}_c(N) \).

We use \( \mathcal{N}_r(N) \), \( \mathcal{J}_2(N) \) and \( \mathcal{G}(N) \) to represent the nilradical of \( N \), \( \mathcal{J}_2 \)-radical of \( N \) and the Brown-McCoy radical of \( N \), respectively. \( N \) is said to fulfill the insertion-of-factors property (IFP) provided that for all \( a, b, x \in N \), then \( ab = 0 \) implies \( axb = 0 \). Also for \( X \subseteq N \), \( (0 : X) \) and \( \langle X \rangle \) denote the left annihilator of \( X \) and the ideal of \( N \) generated by \( X \), respectively. For other notation and/or terminology see [24].

1. Preliminaries

In this section we discuss the various generalizations of von Neumann regularity which will be used in our main results in section 3.
In [25], Ramamurthi defined weakly regular rings, and in [18] Gupta defined weakly \( \pi \)-regular rings. Jat and Choudhary [21] extended weak regularity to near-rings, and Goyal and Choudhary [15] did likewise for \( \pi \)-regularity. Furthermore Ramakotaiah [26] gave a nonring example of a \( \pi \)-regular subnear-ring of \( M_0(Z_4) \). The following definitions will be used in this paper (note that we are introducing the concept of a left pseudo \( \pi \)-regular near-ring).

**Definition 1.1.** (i) \( N \) is said to be left (right) weakly regular if \( x \in \langle x \rangle x \) (\( x \in x \langle x \rangle \)) for all \( x \in N \).

(ii) \( N \) is said to be \( \pi \)-regular if for every \( x \in N \) there exists a natural number \( n = n(x) \) such that \( x^n \in x^n N x^n \).

(iii) \( N \) is said to be left (right) weakly \( \pi \)-regular if for every \( x \in N \) there exists a natural number \( n = n(x) \) such that \( x^n \in \langle x^n \rangle x^n \) (\( x^n \in x^n \langle x^n \rangle \)).

(iv) We say \( N \) is left (right) pseudo \( \pi \)-regular if for every \( x \in N \) there exists a natural number \( n = n(x) \) such that \( x^n \in \langle x \rangle x^n \) (\( x^n \in x^n \langle x \rangle \)).

In the above definitions if \( N \) satisfies both the left and right version, then the adjective "left" or "right" is omitted. Observe that we have the following implications from Definition 1.1: (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv). Moreover the classes of near-rings of Definition 1.1 as well as the class of \( pm \) near-rings are closed under homomorphic images. Observe that if \( N \) is left (right) weakly regular and \( I \) is an ideal of \( N \), then \( I = I^2 \). From [23], \( N \) is left strongly regular if for all \( x \in N \), there exists \( a \in N \) with \( x = ax^2 \). Hence if \( N \) is left strongly regular, then \( N \) is left weakly regular. Furthermore, from [21], \( N \) is bipotent if \( Na = Na^2 \) for \( a \in N \). So if \( N \) is bipotent and \( a \in N \), then \( \langle a \rangle a^2 \subseteq Na^2 = (Na)a = (Na^2)a = (Na)a^2 \subseteq \langle a \rangle a^2 \). Thus \( Na = \langle a \rangle a^2 \), so \( a^2 \in \langle a \rangle a^2 \) for every \( a \in N \). Hence every bipotent near-ring is left pseudo \( \pi \)-regular.

**Lemma 1.2.**

(i) Let \( N \) be reduced. Then \( N \) is left weakly regular if and only if \( N \) is left pseudo \( \pi \)-regular.

(ii) Let \( N \) be commutative. Then \( N \) is weakly \( \pi \)-regular if and only if \( N \) is pseudo \( \pi \)-regular.

**Proof.** (i) Clearly if \( N \) is left weakly regular then \( N \) is left pseudo \( \pi \)-regular. So assume \( N \) is left pseudo \( \pi \)-regular. Let \( a \in N \). Then there exists \( s \in \langle a \rangle \) and a natural number \( n \) such that \( a^n = sa^n \). If \( n = 1 \), we are finished. For \( n > 1 \), then \( (a - sa)a^{n-1} = 0 \). Since 0 is a completely semiprime ideal, [17, Lemma 2.1] yields \( (a - sa)a = 0 = a(a - sa) \). Hence \( (a - sa)^2 = 0 \). So \( a = sa \in \langle a \rangle \). Therefore \( N \) is left weakly regular.
(ii) Assume \( N \) is left pseudo \( \pi \)-regular. Let \( a \in N \). There exists \( s \in \langle a \rangle \) and a natural number \( n \) such that \( a^n = sa^n = s(a^n) = s^2a^n = \cdots = s^n a^n \in \langle a^n \rangle a^n \). Hence \( N \) is left weakly \( \pi \)-regular. The converse is clear. \( \diamond \)

The following result generalizes [15, Th. 1.14].

**Proposition 1.3.** Let \( a \in N \).

(i) \( a^k \in Na^{k+1} \) for some positive integer \( k \) if and only if the descending chain \( Na \supseteq Na^2 \cdots \) stabilizes after a finite number of steps.

(ii) If \( N \) is finite, then \( N \) is left and right weakly \( \pi \)-regular.

**Proof.**

(i) Assume \( a^k \in Na^{k+1} \). Then there exists \( x \in N \) such that \( a^k = xa^{k+1} \). Then \( Na^k = Na^{k+1} \subseteq Na^{k+1} \subseteq Na^k \). Hence the chain stabilizes. Conversely assume \( Na^m = Na^{m+1} \). Then there exists \( y \in N \) such that \( a^{m+1} = ya^m = ya^{m+1} \). There exists \( y_1 \in N \) such that \( ya^m = y_1a^{m+1} \). So \( a^{m+1} = (ya^m)a = (y_1a^{m+1})a \in Na^{m+2} \). Take \( k = m + 1 \).

(ii) By part (i), there exists a positive integer \( k \) and \( x \in N \) such that \( a^k = xa^{k+1} = xa^k a = x(a^{k+1})a = x^2 a^k a^2 = \cdots = x^k a^k a^k \in \langle a^k \rangle a^k \).

Hence \( N \) is left weakly \( \pi \)-regular. Since part (i) is left-right symmetric, \( N \) is also right weakly \( \pi \)-regular. \( \diamond \)

Observe that every finite ring satisfies the \( pm \) condition. However there are finite \( d.g. \) prime near-rings with unity which are not simple [22]. Thus determining when a finite near-ring satisfies the \( pm \) condition is a nottrivial problem. Also from [3] there exist uncountable \( d.g. \) near-rings with unity, but with only finitely many \( N \)-subgroups. By Prop. 1.3(ii) such near-rings are left weakly \( \pi \)-regular. Also note that integral near-rings which are finite or simple illustrate Lemma 1.2(i), and near-rings of nilpotent index two illustrate Lemma 1.2(ii).

**Proposition 1.4.** Let \( N \) be a near-ring with left unity \( e \), and \( k \) and \( n \) are natural numbers.

(i) If \( N = (0 : a^n) + \langle a^k \rangle \), then \( a^n \in \langle a^k \rangle a^n \).

(ii) If \( (0 : a^n) \triangleleft N \) and \( a^n \in \langle a^k \rangle a^n \), then \( N = (0 : a^n) + \langle a^k \rangle \).

**Proof.**

(i) There exists \( v \in (0 : a^n) \) and \( s \in \langle a^k \rangle \) such that \( e = v + s \).

Then \( a^n = va^n + sa^n \). So \( (e - s) \in (0 : a^n) \).

Then for any \( t \in N \), we have \( t = (e - s + s)t = (e - s)t + st \). Since \( (0 : a^n) \triangleleft N \), \( (e - s) \in (0 : a^n) \). Therefore \( N = (0 : a^n) + \langle a^k \rangle \). \( \diamond \)

**Corollary 1.5.** Let \( N \) be an IFP near-ring with a left unity. Then \( N \) is left pseudo (weakly) \( \pi \)-regular if and only if for every \( a \in N \) there exists a natural number \( n = n(a) \) such that

\[
N = (0 : a^n) + \langle a \rangle, \quad (N = (0 : a^n) + \langle a^n \rangle).
\]
Proof. Since $N$ is IFP, then $(0 : x) \triangleleft N$ for every $x \in N$. Now the result is an immediate consequence of Prop. 1.4.$\diamond$

**Proposition 1.6.** Let $I$ be any proper ideal of left pseudo $\pi$-regular near-ring $N$. Every nonzero element of $I$ is a divisor of zero.

**Proof.** Let $0 \neq a$ be any element of the ideal $I$. Assume $a$ is not a divisor of zero. Since $N$ is left $\pi$-weakly regular there exists $n(a)$, a positive integer, such that $a^n \in \langle a \rangle$. Hence $a^n = xa^n$ for some $x \in \langle a \rangle$. For every $z \in N$ we have $za^n = zxa^n$. Hence $(z - zx)a^n = 0$. Since $a$ is not a divisor of zero, we have $z = zx \in I$. Hence $N = I$ which is a contradiction.$\diamond$

**Corollary 1.7.** If $N$ is weakly $\pi$-regular with nonzero divisors of zero, then $N$ is simple.

2. Completely prime ideals and the $pm$ condition

**Proposition 2.1.** $\rho(N)$ is completely semiprime if and only if every prime ideal which is minimal among the prime ideals containing $\rho(N)$ is completely prime (where $\rho(N) = \mathcal{P}_0(N), \mathcal{N}_r(N), \mathcal{J}_2(N)$ or $\mathcal{G}(N)$).

**Proof.** Let $P$ be a prime ideal which is minimal amongst the prime ideals containing $\rho(N)$. Now clearly, $P/\rho(N)$ is a minimal prime ideal of $\bar{N} = N/\rho(N)$. Since $\rho(N)$ is completely semiprime, $\bar{N}$ is reduced. Since $\bar{N}$ is reduced, it is also 2-primal and from [7, Cor. 1.3] we have that $P/\rho(N)$ is a completely prime ideal of $\bar{N}$. Hence $N/P \cong (N/\rho(N))/(P/\rho(N))$ is a completely prime near-ring and consequently $P$ is a completely prime ideal of $N$.

Now let $B$ be the intersection of all the prime ideals of $N$ which are minimal among prime ideals of $N$ containing $\rho(N)$. Let $D$ be the intersection of all the prime ideals of $N$ containing $\rho(N)$. For $\rho(N) = \mathcal{P}_0(N)$ we have $\mathcal{P}_0(N) = B$ and from our assumption $\mathcal{P}_0(N)$ is the intersection of completely prime ideals and hence $\mathcal{P}_0(N)$ is completely semiprime.

Case $\rho(N) = \mathcal{N}_r(N)$. Recall from [30] that $\mathcal{N}_r(N)$ is the intersection of all $s$-prime ideals and that each $s$-prime ideal is also a prime ideal of $N$ which contains $\mathcal{N}_r(N)$. Hence $\mathcal{N}_r(N) \subseteq B = D \subseteq \subseteq \cap \{s$-prime ideals of $N\} = \mathcal{N}_r(N)$.

Case $\rho(N) = \mathcal{J}_2(R)$. Recall that a 2-primitive ideal of $N$ is a prime ideal of $N$ which contains $\mathcal{J}_2(N)$. Now $\mathcal{J}_2(N) \subseteq B = D \subseteq \subseteq \cap \{s$-primitive ideals of $N\} = \mathcal{J}_2(N)$.

Case $\rho(N) = \mathcal{G}(N)$. From [1] we know that $\mathcal{G}(N) = \cap \{M < N : N/M$ is a simple near-ring with identity$\}$. Each of the ideal in the inter-
section is clearly a prime ideal and contains $\mathcal{G}(N)$. Hence $\mathcal{G}(N) \subseteq B = D \subseteq \cap \{M : N/M \text{ is simple with identity}\} = \mathcal{G}(N)$.

Thus in all four cases $B = \rho(N)$ and since $B$ is completely semi-prime, then so is $\rho(N)$.$\blacklozenge$

The following well known result is an immediate corollary of Prop. 2.1.

Corollary 2.2. Let $N$ be a reduced near-ring. Every minimal prime ideal of $N$ is completely prime.

Proof. Since $N$ is reduced, $\mathcal{P}_0(N) = 0$. The corollary now follows from Prop. 2.1.$\blacklozenge$

Proposition 2.3. If $\rho(N)$ is a completely semi-prime ideal of $N$ and $N/\rho(N)$ is left pseudo $\pi$-regular, then $N/P$ is a simple integral near-ring with a right unity for every prime ideal $P$ of $N$ with $\rho(N) \subseteq P$ (where $\rho(N) = \mathcal{P}_0(N), \mathcal{N}_r(N), \mathcal{J}_2(N)$ or $\mathcal{G}(N)$).

Proof. Let $P$ be any prime ideal of $N$ such that $\rho(N) \subseteq P$. Now there exists a prime ideal $X$ of $N$ which is minimal among prime ideals containing $\rho(N)$ and $X \subseteq P$. From Prop. 2.1, $X$ is completely prime. Let $\tilde{N} = N/X$.

Since $X$ is a completely prime ideal, $\tilde{N}$ is an integral near-ring. We show that $N$ is simple with a right unity. Let $0 \neq I \not\triangleleft \tilde{N}$ and $0 \neq v \in I$. Since $\tilde{N}$ is weakly $\pi$-regular, there exists $y \in \langle v \rangle$ such that $y^k = yv^k$. Now we have $yu^k = y^2v^k$. Hence $(y - y^2)v^k = 0$. Since $\tilde{N}$ is an integral near-ring and $v \neq 0$, we have $y = y^2$. Hence for any $t \in \tilde{N}$ we have $ty = ty^2$. Now $(t - ty)y = 0$ and therefore $t = ty$. Hence $y$ is a right unity of $\tilde{N}$. Now also $t = ty \in t\langle v^k \rangle \subseteq tI \subseteq I$. Hence $\tilde{N} = I$ and, therefore, $\tilde{N}$ is simple with right unity $y$. Hence $X$ is maximal and $X \subseteq P$, therefore $X = P$.$\blacklozenge$

Corollary 2.4. If the near-ring $N$ is 2-primal and $N/\mathcal{P}_0(N)$ is left pseudo $\pi$-regular, then every prime ideal of $N$ is maximal.

Proposition 2.5. If $I \not\triangleleft N$, then $B(I)$ is completely semi-prime if and only if every minimal prime ideal of $I$ is completely prime. In particular, if $N$ also satisfies the pm condition, then every prime ideal containing $I$ is completely prime.

Proof. The result is a consequence of [6, Lemma 2.2(v)] and [7, Th. 1.2].$\blacklozenge$

Corollary 2.6. If $N$ is 2-primal (e.g., if $N$ is reduced) and satisfies the pm condition, then every prime ideal is completely prime.

3. Left weakly regular near-rings

In this section we characterise reduced left weakly regular near-rings.
Lemma 3.1. If $I$ is a completely semi-prime ideal of the near-ring $N$ and $x_1, x_2, \ldots, x_n \in I$ then $x_{\sigma(1)}, \ldots, x_{\sigma(n)} \in I$, where $\sigma$ is any permutation of $\{1, 2, \ldots, n\}$.

**Proof.** This follows by applying Lemma 2.1 of [17].

**Proposition 3.2.** Let $N$ be an IFP right near-ring with left unity $e$ such that every completely prime ideal is maximal. Let $a \in N$ such that $(0 : a)$ is a 2-primal ideal of $N$, there exists $s \in \langle a \rangle$ such that:

(i) $a^3 = sa^3 + x$ where $s \in \langle a \rangle$ and $x \in \mathcal{N}(N)$.

(ii) If $a^3(e - s)a^3 = (e - s)a^k$ for some $k$, then there exists $m$ such that $a^m \in \langle a \rangle a^m$.

**Proof.** (i) Let $0 \neq a \in N$. Since $N$ has IFP, it follows from [24, p. 289] that $(0 : a) \triangleleft N$. Let $\bar{N} = N/(0 : a)$. It is easy to see that every completely prime ideal of $\bar{N}$ is also maximal. Let $M$ be the multiplicative semigroup generated by all elements of the form $\bar{a} = \bar{x}a$ where $x \in \langle a \rangle$.

We claim $\mathcal{P}_0(\bar{N}) \cap M \neq \emptyset$. To see this, assume $\mathcal{P}_0(\bar{N}) \cap M = \emptyset$. Since $\mathcal{P}_0(\bar{N}) = \mathcal{N}(\bar{N})$ (because $(0 : a)$ is a 2-primal ideal - see [7]), $\mathcal{P}_0(N)$ is a completely semiprime ideal of $N$. If follows from [17, Lemma 3.1] that there exists a completely prime ideal $\bar{P}$ and $\bar{N}$ such that $\bar{P} \cap M = \emptyset$. Now $\langle \bar{a} \rangle \subseteq \bar{P}$ or there exists $\bar{a} \in \langle \bar{a} \rangle$ such that $\bar{a} \notin \bar{P}$. If $\langle \bar{a} \rangle \subseteq \bar{P}$, then $\bar{a} - \bar{a} \bar{a} = \bar{P} \cap M \neq \emptyset$, a contradiction. So assume there exists $\bar{a} \in \langle \bar{a} \rangle$ such that $\bar{a} \notin \bar{P}$. Since $\bar{P}$ is maximal, we have $\bar{P} + \langle \bar{a} \rangle = \bar{N}$. So $\bar{e} = \bar{p} + \bar{i}$, where $\bar{p} \in \bar{P}$ and $\bar{i} \in \langle \bar{a} \rangle \subseteq \langle \bar{a} \rangle$. From this we have $\bar{a} - \bar{t} \bar{a} = (\bar{e} - \bar{i})\bar{a} = \bar{p}\bar{a} \in \bar{P} \cap M$, a contradiction. Hence $\mathcal{P}_0(\bar{N}) \cap M \neq \emptyset$.

So $\bar{e} - \bar{s} \bar{a} = \bar{k} \bar{a} \in \mathcal{P}_0(\bar{N})$. From [17, Lemma 2.1(iii)],

$$\bar{e} - \bar{s} \bar{a} = \bar{k} \bar{a} \in \mathcal{P}_0(\bar{N}).$$

So $(\bar{e} - \bar{a})\bar{a}^2 = \bar{k} \bar{a} \in \mathcal{P}(\bar{N})$. Thus $(\bar{a} - \bar{s} \bar{a} - \bar{k})\bar{a} = \bar{0}$. Hence $(a - sa - k)a = 0$, where $ka = 0$ for some positive integer $j$. Then $a^3 = sa^3 + ka^2$. Since $N$ has IFP, $(ka)^j = 0$ and $(ka^2)^j = 0$. Therefore, $ka^2 \in \mathcal{N}(N)$.

(ii) By part (i) there exists $n$ such that $0 = ((e - s)a^3)^n = (e - s)a^3(e - s)a^3 \ldots (e - s)a^3 = (e - s)a^m$ for some $m$ and some $s \in \langle a \rangle$. The last equation involves a reduction technique which we will illustrate with $n = 3$.

$$0 = (e - s)a^3(e - s)a^3(e - s)a^3 = (e - s)^2a^k(e - s)a^3.$$

Case (i) Assume $k \leq 3$. Since $N$ has the IFP, then
0 = (e - s)^2 a^k a^{k-3} (e - s) a^3 = (e - s)^3 a^k.

Observe there exists \( \bar{s} \in \langle a \rangle \) such that \((e - s)^3 = e - \bar{s}\). Hence \(0 = (e - \bar{s}) a^k\), so \(a^k = \bar{s} a^k \in \langle a \rangle a^k\).

Case (ii) Assume \(k > 3\). Let \(p\) be the least positive integer such that \(k \leq 3p\). Since \(N\) has the IFP, then

\[
0 = (e - s)^2 a^{3p} (e - s) a^3 = (e - s)^2 a^{3(p-1)} a^3 (e - s) a^3 =
\]

\[
= (e - s)^2 a^{3(p-1)} (e - s) a^k = (e - s)^2 a^{3(p-2)} a^3 (e - s) a^3 a^{3(p-1)} =
\]

\[
= (e - s)^2 a^{3(p-2)} (e - s) a^{k+3(p-1)} = \ldots = (e - s)^3 a^m = (e - \bar{s}) a^m.
\]

Hence \(a^m \in \langle a \rangle a^m\).\)

From [6, Th. 4.4], we have that if every prime ideal of \(N\) is completely prime then every ideal is 2-primal (hence if \((0 : a) \triangleleft N\), then \((0 : a)\) is 2-primal). Examples of such near-rings with unity are provided in [6].

From [10], \(N\) satisfies the CZ1 condition if for any \(x, y \in N\) and positive integer \(k\) such that \((xy)^k = 0\), then there exists a positive integer \(m\) such that \(x^m y^m = 0\). Observe if \(\mathcal{N}(N)\) is contained in the multiplicative center of \(N\), then \(N\) satisfies the CZ1 condition. However [10, Example 2.5] provides an example of a ring \(R\) with unity such that \(R\) satisfies CZ1, but \(\mathcal{N}(R)\) is not contained in the center of \(R\).

**Theorem 3.3.** Let \(N\) be an IFP near-ring with left unity \(e\) which satisfies the CZ1 condition, and \((0 : a)\) is a 2-primal ideal for all \(a \in N\). Then the following conditions are equivalent:

(i) \(N\) is left pseudo \(\pi\)-regular.

(ii) Every prime ideal is maximal.

(iii) Every completely prime ideal is maximal.

(iv) For every \(a \in N\) there exists \(n\), possibly depending on \(a\), such that \(N = (0 : a^n) + \langle a \rangle\).

**Proof.** (i) \(\Rightarrow\) (ii). This implication follows from Cor. 2.4.

(ii) \(\Rightarrow\) (iii). This implication is obvious.

(iii) \(\Rightarrow\) (i). Let \(a \in N\). By Prop. 3.2(i), there exists a positive integer \(k\) and \(s \in \langle a \rangle\) such that \(0 = ((e - s) a^3)^k\). Since \(N\) satisfies the CZ1 condition, there exists a positive integer \(m\) such that \(0 = (e - s)^m a^{3m}\). There exists \(\bar{s} \in \langle a \rangle\) such that \((e - s)^m = e - \bar{s}\). Hence \(a^{3m} = \bar{s} a^{3m}\). Therefore \(N\) is left pseudo \(\pi\)-regular.

(i) \(\Leftrightarrow\) (iv). This equivalence follows from Cor. 1.5.\)

**Lemma 3.4.** If \(N\) is a reduced near-ring, and \(0 \neq a \in N\), then \(N/(0 : a)\) is reduced and \(\bar{a} \in N/(0 : a)\) is not a divisor of zero.

**Proof.** Let \(0 \neq a \in N\). From [24, Prop. 9.3], it follows that \((0 : a) \triangleleft N\). Let \(x^m \in (0 : a)\). Hence \(x^m a = 0\). Since \((0)\) is a completely semiprime ideal, it follows from [17, Lemma 2.1(ii)] that \(xa = 0\). Hence
$x \in (0 : a)$ and it follows that $(0 : a)$ is a completely semi-prime ideal and consequently $N/(0 : a)$ is reduced. The element $\bar{a} \in N/(0 : a)$ is nonzero since $N$ is reduced. Now suppose $\bar{b} \bar{a} = \bar{0}$. From [17, Lemma 2.1], it follows that $\bar{a} \bar{b} = \bar{0}$. Hence $aba = 0$ and, therefore, $(ba)^2 = 0$. Since $N$ is reduced, we have $\bar{b} = \bar{0}$.

**Lemma 3.5.** If $N$ is a reduced near-ring, then $N$ has the IFP and $(0 : a)$ is a 2-primal ideal of $N$ for all $a \in N$.

**Proof.** From [17, Lemma 2.1(ii)] $N$ is IFP. By Lemma 3.4 $N/(0 : a)$ is reduced. Thus $(0 : a)$ is a 2-primal ideal of $N$.

**Lemma 3.6.** A near-ring $N$ with left unity $e$ is reduced and left weakly regular if and only if $N = (0 : a) \oplus \langle a \rangle$ for every $a \in N$.

**Proof.** Assume $N$ is reduced and left weakly regular. Since $N$ is reduced, 0 is a completely semiprime ideal. By [17, Lemma 2.1], $(0 : a^n) = (0 : a)$. From Cor. 1.6, we have $N = (0 : a) + \langle a \rangle$. We show this sum is direct. Let $x \in (0 : a) \cap \langle a \rangle$. Now $xa = 0$ and $x \in \langle a \rangle$. Since $N$ is reduced, we can show that $x \langle a \rangle = 0$. Hence $x^2 = 0$ and since $N$ is reduced, we have $x = 0$.

For the converse, suppose $N = (0 : a) \oplus \langle a \rangle$ for all $a \in N$. We first show $N$ is reduced. Let $a \in N$ such that $a^2 = 0$. Now $a \in (0 : a) \cap \langle a \rangle = 0$. Next we show $N$ is weakly regular. Let $a \in N$. Since $e \in N = (0 : a) \oplus \langle a \rangle$, we have $a = e \cdot a = (t_1 + t_2)a$ with $t_1 \in (0 : a)$ and $t_2 \in \langle a \rangle$. Hence $a = t_1 a + t_2 a = t_2 a \in \langle a \rangle a$.

As in [27], we define $O_p$ to be $\{a \in N \mid ba = 0, \text{ for some } b \notin P\}$, where $P$ is a prime ideal of $N$.

Observe if $N$ is a reduced near-ring and $e$ is a left unity, then $e$ is a unity. To see this observe $(xe - x)^2 = xe(xe - x) - x(xe - x) = 0$. Hence $xe = x$.

The following corollary generalizes [23, Lemma 4 and Th. 2].

**Corollary 3.7.** Let $N$ be a reduced near-ring with unity. The following conditions are equivalent:

(i) $N$ is left weakly regular.

(ii) $N$ is left pseudo $\pi$-regular.

(iii) Every prime ideal of $N$ is maximal.

(iv) Every completely prime ideal of $N$ is maximal.

(v) For every $a \in N$ we have $N = (0 : a) \oplus \langle a \rangle$.

(vi) For each prime ideal $P$ of $N$, $P = O_p$.

**Proof.** The equivalence of parts (i) through (v) follows from Th. 3.3, Lemma 3.5, and Lemma 3.6.

(i) $\Rightarrow$ (vi). Let $P$ be any prime ideal. Since (i) $\Leftrightarrow$ (iii) we have that $N$ has the $pm$ condition and from Cor. 2.7 we have $P$ is completely
prime. Let \( x \in O_p \). Then there exists \( b \notin P \) such that \( bx = 0 \in P \). Now the fact that \( P \) is completely prime forces \( x \in P \) and with \( (1-a) \in N \setminus P \). Hence \( x \in O_p \) and we have \( O_p = P \).

\[(vi) \Rightarrow (iii). \] Suppose \( P = O_p \) for each \( P \). Let \( M \) be a maximal ideal such that \( P \subseteq M \). From [24, Cor. 2.72], \( M \) is also a prime ideal. Now from our assumption we have \( M = O_M \subseteq O_p = P \).

**Corollary 3.8.** Let \( N \) be a reduced near-ring with unity. \( N \) is left weakly regular if and only if every prime factor near-ring of \( N \) is a simple integral near-rings.

Observe from [24, Th. 9.36] and Corrolary 3.8, we have: if \( N \) is a reduced left weakly regular near-ring with unity, then \( N \) is a subdirect product of simple integral near-rings.

**Corollary 3.9.** Let \( N \) be a reduced near-ring with unity and DCC on \( N \)-subgroups. Then every prime factor near-ring of \( N \) is a near-field.

**Proof.** The proof follows from Cor. 3.7, Prop. 1.3, and [24, Remarks 9.48d].

This corollary is in contrast to the statement after Prop. 1.3. An alternative proof of Cor. 3.9 can be given using [24, Prop. 9.41]. Observe there are finite simple reduced near-rings which are not near-fields [24, Remark 9.40].

**Corollary 3.10.** Let \( N \) be a 2-primal near-ring with a left unity. The following conditions are equivalent:

\( (i) \) \( N/P_0(N) \) is left weakly regular.

\( (ii) \) \( N/P_0(N) \) is left pseudo \( \pi \)-regular.

\( (iii) \) Every prime ideal of \( N \) is maximal.

\( (iv) \) Every completely prime ideal of \( N \) is maximal.

**Proof.** Since \( N \) is 2-primal, \( N/P_0(N) \) is a reduced near-ring. The remainder of the proof follows routinely from Cor. 3.7.

Observe that in [8] an example was given to show that the condition in Cor. 3.10 "\( N \) is 2-primal" is not superfluous. Also there are nonreduced nearrings satisfying the hypothesis of Th. 3.4. For example, let \( B \) be a reduced near-ring with unity and \( C \) is nonreduced commutative ring with unity. Then \( N = B \oplus C \) satisfies the hypothesis of Th. 3.4.

**Question.** Is a reduced left weakly regular near-ring (with unity) also right weakly regular?

If we let \( N_P = \{ a \in N : ba \in P_0(N) \text{ for some } b \in P \setminus N \} \) we can now characterize minimal prime ideals in reduced near-rings similar to that for rings in [27].

**Theorem 3.11.** Let \( N \) be a reduced near-ring. If \( P \) is any prime ideal of \( N \), then
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\[ O_p = \cap \{ Q < N \mid Q \text{ prime and } O_p \subseteq Q \} \]
\[ = \cap \{ Q < N \mid Q \text{ prime and } Q \subseteq P \}. \]

**Proof.** Let \( P \) be any prime ideal and suppose \( Q \) is a prime ideal such that \( Q \subseteq P \). We first show that \( O_Q \subseteq Q \). Let \( z \in O_Q \). Now \( bz = 0 \) for some \( b \notin Q \). Since \( bz = 0 \) and \( N \) reduced, it follows from [6, Lemma 2.5] that \( \langle b \rangle \langle z \rangle = 0 \subseteq Q \). Since \( Q \) is a prime ideal and \( b \notin Q \) we must have \( z \in Q \) hence \( O_Q \subseteq Q \) for any prime ideal \( Q \). For \( Q \subseteq P \) we now have \( O_p \subseteq Q \) and consequently
\[ O_p \subseteq \cap \{ Q \mid O_p \subseteq Q \} \subseteq \cap \{ Q \mid Q \subseteq P \}. \]

Suppose now \( a \notin O_p \). We shall find a prime ideal \( Q \) such that \( a \notin Q \) and \( Q \subseteq P \). Let \( S = \{ a, a^2, a^3, \ldots \} \). \( S \) is a multiplicative system that does not contain 0. Let \( L = N \setminus P \), i.e. \( L \) is an \( m \)-system. Let \( T \) be the set of all nonzero elements of \( N \) of the form \( a^{t_0}x_1a^{t_1}x_2 \ldots a^{t_n-1}x_na^{t_n} \) where \( x_i \in L \) and the \( t_i \)'s are positive integers with \( t_0 \) and \( t_n \) allowed to be zero. Clearly \( L \subseteq T \). Let \( M = T \cup S \). We show that \( M \) is an \( m \)-system. Let \( x, y \in M \). If \( x, y \in S \) then \( xy \in S \subseteq M \) and we are done. Let \( x \in S \) and \( y \in T \), say \( x = a^s \) and \( y = a^{t_0}y_1a^{t_1}y_2 \ldots y_ma^{t_m} \). If \( \langle x \rangle \langle y \rangle \neq 0 \), then \( xy \neq 0 \). This follows form the fact that \( N \) is reduced and from the contrapositive of Lemma 2.5 of [6]. Since \( xy \neq 0 \) we have \( xy \in T \), hence \( \langle x \rangle \langle y \rangle \cap M \neq \emptyset \). We show \( xy = 0 \) is impossible. Suppose \( xy = 0 \), then we have \( xy = a^s \cdot a^{t_0}y_1a^{t_1}y_2 \ldots a^{t_m-1}y_ma^{t_m} = 0 \). Since 0 is a completely semi-prime ideal, Lemma 3.1 yields \( xy = a^l \), where \( l = s + t_0 + \ldots + t_m \). From [6, Lemma 2.5] it follows that \( \langle a^l \rangle \langle y_1 \rangle \langle y_2 \rangle \ldots \langle y_m \rangle = 0 \). Let \( 0 \neq w \in \langle y_1 \rangle \langle y_2 \rangle \ldots \langle y_m \rangle \cap L \). This is possible since \( L \) is an \( m \)-system. Now we have \( a^lw = 0 \). Since \( N \) is reduced, we have \( aw = 0 \). Hence \( wa = 0 \) with \( w \in N \setminus P \). So \( a \in O_p \), a contradiction. So a similar argument can be used for \( x \in T \) and \( y \in S \) and for \( x, y \in T \). This shows that \( M \) is an \( m \)-system disjoint from 0. From [17, Lemma 3.1] there exists a completely prime ideal \( Q \) disjoint from \( M \). Hence \( a \notin Q \) and \( Q \subseteq P \), completing the proof.∴

We have the following corollary:

**Corollary 3.12.** If \( N \) is a 2-primal near-ring and \( P \) is any prime ideal, then:

(i) \( N_p = \cap \{ Q \mid Q \text{ is a prime ideal and } Q \subseteq P \} \);

(ii) \( P \) is a minimal prime ideal if and only if \( P = N_p \).

**Proof.** If \( N \) is 2-primal then \( P_0(N) = P_c(N) \) and, therefore, \( N/P_0(N) \) is reduced. Hence part (i) follows from the previous theorem, and part (ii) is a consequence of part (i).∴
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References

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