FINITARY LINEAR IMAGES
OF FINITARY LINEAR GROUPS

B.A.F. Wehrfritz

School of Mathematical Sciences, Queen Mary & Westfield College,
Mile End Road, London E1 4NS, England

Received: March 1998

MSC 1991: 20 H 20

Keywords: Finitary linear group, images.

Abstract: A well-known and very useful result, I believe originally due to
C. Chevalley, is that if \( G \) is a linear group of finite degree and if \( H \) is Zariski-
closed normal subgroup of \( G \), then \( G/H \) is isomorphic to some linear group of
finite degree over the same field. We show in this paper firstly that this theorem
does not extend to (infinite-dimensional) finitary linear groups and secondly in
substantial special cases it does so extend.

Throughout this paper \( F \) denotes a (commutative) field and \( V \) a
vector space over \( F \). The full finitary linear group on \( V \) we denote by
\( FGL(V) \). Let \( H \) be a normal subgroup of the subgroup \( G \) of \( FGL(V) \).
We consider here circumstances which force the group \( G/H \) to be iso-
morphic to some finitary linear group over \( F \).

A well-known and very useful result, I believe originally due to
C. Chevalley is that if \( \dim_F V \) is finite and if \( H \) is Zariski-closed, then
\( G/H \) is isomorphic to a linear group over \( F \) of finite degree (see [1], p.
299 or [7] 6.4). Moreover in special cases, for example if \( H = C_G(X) \)
for some subset \( X \) of \( \text{End}_F V \) or if \( H = N_G(U) \) for some subspace \( U \)
of \( V \), the degree of the representation of \( G/H \) can be bounded in terms
of \( \dim_F V \) only. Now the notion of Zariski topology does extend in a
natural way to \( FGL(V) \), for example see §25 below. Unfortunately the
above theorem of Chavalley does not.
Example. a) For every characteristic $q \geq 0$ there is a completely reducible, periodic nilpotent, finitary linear group $G$ of characteristic $q$ with a Zariski-closed normal subgroup $H$ such that $G/H$ is not isomorphic to any finitary linear group.

b) For every positive characteristic $p$ there is a unipotent nilpotent finitary linear group $G$ of characteristic $p$ with a Zariski-closed normal subgroup $H$ such that $G/H$ is not isomorphic to any finitary linear group.

In a) above $H$ is certainly periodic with trivial unipotent radical. In [6] and [10] criteria other than Chevalley's are given for $G/H$ to be isomorphic to a linear group of finite degree, whenever $G$ is given as a linear group of finite degree. They all involve $H$ being periodic with a very restricted unipotent radical and the Ex. a) above prevents any direct generalization of any of these results of [6] and [10] to finitary groups.

We are left to consider special cases of Chevalley's Theorem. Now in a linear group of finite degree, the Zariski-closure of a soluble (resp. nilpotent) subgroup is again soluble (resp. nilpotent) of the same derived length (resp. class) and this also applies to subgroups of finitary linear groups. The main result of this paper is the following.

Theorem. Let $H$ be a completely reducible normal subgroup of the subgroup $G$ of $FGL(V)$. Then there is a normal subgroup $K \supset H$ of $G$ with $G/K$ isomorphic to a finitary linear group over $F$ and such that $H, K$ and $F$ satisfy either

a) $H$ and $K$ are abelian and completely reducible, or
b) $F$ is perfect and $H$ and $K$ are nilpotent of class $c$, or
c) $char F = 0$ and $H$ and $K$ are soluble of derived length $d$.

The following is immediate from the Theorem.

Corollary 1. Let $H$ be a completely reducible normal subgroup of the subgroup $G$ of $FGL(V)$. Under any one of the following three conditions the group $G/H$ is isomorphic to a finitary linear group over $F$.

a) $H$ is a maximal abelian normal subgroup of $G$.

b) $F$ is perfect and $H$ is a maximal nilpotent-of-class-$c$ normal subgroup of $G$.

c) $char F = 0$ and $H$ is a maximal soluble-of-derived-length-$d$ normal subgroup of $G$.

To prove the Theorem we need to consider some less specialized conditions more akin to the special cases of Chevalley's Theorem mentioned above, where the degree can be bounded. The following summarizes some of the more simply stated of these conditions.

Proposition. Let $H$ be a completely reducible normal subgroup of the subgroup $G$ of $FGL(V)$ and let $U$ be a subspace of $V$. Under any one of
the following four conditions, the group $G/H$ is isomorphic to a finitary linear group over $F$.

a) $H = C_G(X)$ for some subset $X$ of $\text{End}_F V$.

b) $H = N_G(U)$.

c) $H = \{g \in N_G(U) : g_U \text{is scalar}\}$.

d) $H = C_G(U)$.

Here a) is the basic result and b) and c) we derive from a). Part d) is very easy; we included it here for completeness. We also record some variants of a) to d); for these, as well as a proof of the Proposition, see Points 1. to 8. below. Parts a) and b) of the Proposition cannot be proved by following the usual proofs of Chevalley's Theorem, for example as in [1] or [7]. If one tries to do just that, at some point in the argument one arrives at an $FG$-module upon which $G$ may not act finitarily. At that point the approach breaks down and one needs to attack the Proposition somewhat differently.

In the situation of both the Theorem and Proposition, I have little information if $H$ is not completely reducible. However from Cor. 1 and the Proposition one can at least derive the following.

**Corollary 2.** Let $H$ be a normal subgroup of the subgroup $G$ of $\text{FGL}(V)$ and suppose the unipotent radical $u(g)$ of $G$ is trivial. If either $H = C_G(X)$ for some subset $X$ of $G$, or if one of the conditions a), b) or c) of Cor. 1 hold, then $G/H$ is isomorphic to a finitary linear group over $F$.

To derive Cor. 2, note that $u(G)$ is the kernel of the action of $G$ on the direct sum of the factors in any composition series of $V$ as $FG$-module. Thus, replacing $V$ by this direct sum, we may assume that $G \leq \text{FGL}(V)$ is completely reducible. Then $H$ too is completely reducible by Clifford's Theorem (see [3], 5.1 or [8] Prop. 5.9). Cor. 2 now follows from Part a) of the Proposition and Cor. 1. ◊

We now embark upon the proof of the Proposition and related results. For the remainder of this paper $G$ denotes a subgroup of $\text{FGL}(V)$ and $H$ a completely reducible normal subgroup of $G$. Let $V = V_0 \oplus (\oplus_{i \in I} V_i)$, where $V_0 = C_V(H)$ and the $V_i$ for $i$ in $I$ are the (remaining) non-zero homogeneous components of $V$ as $FH$-module. Let $E = \text{End}_F V$ and $W_i = \text{End}_{FH} V_i$ for each $i$ (including $W_0 = \text{End}_F V_0$). Let $W' = \oplus_{i \in I} W_i \leq C_E(H) = \text{End}_{FH} V$.

1. **Let $G$ act on $W'$ by conjugation. Then $W'$ is a finitary $FG$-module.**

**Proof.** We may assume that $V_0 = \{0\}$. Since $H$ is normal in $G$, so $G$ permutes the $V_i$. If $g \in G$ and $i, j \in I$ with $V_i g = V_j$, then $g^{-1} W_i g = W_j$. Certainly, therefore, $W'$ is an $FG$-module. For any $g$ in $G$, the space
\[ [V, g] = V(g - 1) \text{ is finite dimensional, so } [V, g] \leq \oplus_{j \in J} V_j \text{ for some finite subset } J \text{ of } I. \text{ If } i \in I \setminus J, \text{ then } g \text{ centralizes } V_i \text{ and hence } g \text{ centralizes } W_i. \text{ Therefore } g \text{ centralizes the subspace } \oplus_{j \in J} W_i \text{ of } V. \]

If \( \dim_F V_i \) is finite, then so is \( \dim_F W_i \). Suppose \( \dim_F V_i \) is infinite, and suppose \( U \) is an irreducible \( FH \)-submodule of \( V_i \). Since, by hypothesis, \( U \) is not \( H \)-trivial and \( V_i \) is \( FH \)-finitary, so \( V_i \) is a direct sum of a finite number, say \( r_i \), of copies of \( U \). There exists \( h \in H \) with \( 0 < \dim_F [U, h] < \infty \). If \( D = \text{End}_{FH} U \), then \( D \) is a division \( F \)-algebra and \( [U, h] \) is a \( D \)-submodule. Therefore \( \dim_F D \leq \dim_F [U, h] < \infty \) and \( \dim_F W_i = r_i^2 (\dim_F D) < \infty \). Consequently \( \dim_F W_i \) is finite for every \( i \) in \( I \) and hence \( \dim_F (\oplus_{j \in J} W_j) \) too is finite. Thus \( g \) acts finitarily on \( W' \) as claimed.\( \Diamond \)

**Remarks.** In the above proof \( Ug \) is also an infinite-dimensional irreducible \( FH \)-submodule of \( V \), so finitariness yields that \( Ug \cap U \neq \{0\} \) and \( Ug = U \). Thus \( U \) is an \( FG \)-submodule of \( V \) and consequently so too is \( V_i \). Therefore \( W_i \) is normalized by \( G \) for all \( i \) in \( I \) with \( \dim_F V_i \) infinite.

If \( \dim_F V_0 \) is infinite, then \( W_0 \) is not a finitary \( FG \)-module and \( V_0 \) has to be handled separately.

2. Suppose \( H = C_G(X) \) for some subset \( X \) of \( E \) and assume \( V_0 = \{0\} \). Then \( H = C_G(W') \) and the group \( G/H \) embeds into \( FGL(W') \).

**Proof.** Here \( X \subseteq C_E(H) = \Pi_I W_i \geq W'. \) If \( g \in C_G(W') \), then \( g \) normalizes each \( W_i \) and so \( g \) normalizes each \( V_i \). Hence \( g \in \Pi_I \text{End}_F V_i \) and, since \( g \in C_G(W') \), we have

\[ g \in C_G(\Pi_I W_i) \leq C_G(X) = H \quad \text{and} \quad C_G(W') \leq H. \]

Trivially \( H \leq C_G(W') \). The remainder of 2. follows.\( \Diamond \)

3. Suppose \( H = C_G(X) \) for some subset \( X \) of \( E \). Then \( G/H \) is isomorphic to some finitary linear group over \( F \).

**Proof.** Set \( V' = \oplus_I V_i \), so \( V = V_0 \oplus V' \). Set \( W = V_0 \oplus W' \). Clearly

1. implies that \( W \) is an \( H \)-trivial, finitary \( FG \)-module. Let \( X' \) be the projection of \( X \) into \( \text{End}_{FH} V' \) along \( \text{End}_F V_0 \) (for \( X \) is a subset of \( \text{End}_{FH} V = (\text{End}_{FH} V_0) \oplus (\text{End}_{FH} V') \)). As in the proof of 2. we obtain \( C_G(W') \leq C_G(X') \). Hence

\[ C_G(W) \geq H = C_G(V_0) \cap C_G(X) = \]

\[ = C_G(V_0) \cap C_G(\text{End}_F V_0) \cap C_G(X') \geq C_G(V_0) \cap C_G(W') = C_G(W). \]

Therefore \( H = C_G(W) \) and consequently \( G/H \) embeds into \( FGL(V) \).

4. **Corollary.** Let \( K \) be a normal subgroup of the subgroup \( G \) of \( FGL(V) \). Suppose \( K = N_G(Fu) \) for some \( u \in V \). Then \( G/K \) is isomorphic to a finitary linear group over \( F \).
Proof. Let \( U = FuG \). Then \( U \) is spanned by the set \( uG \) of common eigenvectors of \( K \). In particular \( U \) is completely reducible as \( F K \)-module. Clearly

\[
C_G(U) \leq C_G(u) \leq N_G(Fu) = K.
\]

Consider the action of \( G/C_G(U) \) on \( U \). Now \( U \) has a basis \( B \subseteq uG \) containing \( u \), with respect to which \( K \) is diagonal. The full diagonal algebra \( X \) in \( \text{End}_F U \) with respect to \( B \) is self-centralizing, so \( K \leq C_G(X) \leq \leq N_G(Fu) = K \), using that \( u \in B \). The result now follows from 3. with \( U, G/C_G(U) \) and \( K/C_G(U) \) playing the roles of \( V, G \) and \( H \).

5. Suppose \( H = N_G(U) \) for some subspace \( U \) of \( V \). Then \( G/H \) is isomorphic to some finitary linear group over \( F \).

Proof. Since \( H \) is completely reducible, so \( V = U \oplus U' \) for some \( FH \)-submodule \( U' \) of \( V \). Let \( x = 1_U + 0_{U'} \in E \). Then \( C_{GL(V)}(x) = GL(U) \times GL(U') \). Hence

\[
C_G(x) = N_G(U) \cap N_G(U') = N_H(U') = H.
\]

The claim now follows from 3.

6. Let \( U \) be a subspace of \( V \) and suppose \( H = \{ g \in N_G(U) : g|_U \text{ is scalar} \} \). Then \( G/H \) is isomorphic to some finitary linear group over \( F \).

Proof. Again since \( H \) is completely reducible, we have \( V = U \oplus U' \) as \( FH \)-module. Set \( X = (\text{End}_F U) + 0_{U'} \leq E \). Then \( C_E(X) = (F1_U) \oplus (\text{End}_F U') \). Hence \( H \leq C_G(X) \leq H \cap N_G(U') = H \). Consequently \( H = C_G(X) \) and 3. yields the result.

7. Let \( K \) be a normal subgroup of \( G \leq FGL(V) \) and \( U \) a subspace of \( V \) such that \( K = C_G(U) \). Then \( G/H \) is isomorphic to some finitary linear group over \( F \).

Proof. Since \( K \) is normal in \( G \), we have \( UG \leq C_V(K) \). Thus \( K = C_G(UG) \) and \( G/K \) embeds into \( FGL(UG) \).

The Proposition, Parts a), b), c) and d) follows, respectively, from 3., 5., 6. and 7.

We can combine 5., 6. and 7. as follows.

8. Let \( \{ U_j : j \in J \} \) be a family of subspaces of \( V \) and for each \( j \) in \( J \) suppose that \( H_j \) is a normal subgroup of \( G \) such that \( H_j \) is either \( N_G(U_j) \) or \( \{ g \in N_G(U_j) : g \text{ is scalar on } U_j \} \) or \( C_G(U_j) \). If \( H = \cap_{j \in J} H_j \), then \( G/H \) is isomorphic to some finitary linear group over \( F \).

Proof. Replace \( V \) by \( V \oplus Fu_\infty \), where \( u_\infty \) is non-zero and fixed by \( G \). If \( H_j = C_G(U_j) \), replace \( U_j \) by \( U_j \oplus Fu_\infty \). Then \( H_j = \{ g \in N_G(U_j) : g \text{ is scalar on } U_j \} \). Since \( H \) is completely reducible for each \( j \) in \( J \)
there is an $FH$-submodule $U_i \subset V$ with $V = U_j \oplus U_i$. If $H_j = N_G(U_j)$ set $X_j = \{1u_j + 0u_i\}$. Otherwise set $X_j = (\text{End}_FU_j) + 0u_i$. Then $H = C_G(\cup_{j \in J} X_j)$. Now apply 3.$\diamond$

9. Let $N$ be a subgroup of $GL(n, F)$, where $n$ is an integer, $F$ is perfect and $\text{char } F = p \geq 0$. If $N$ is locally nilpotent Statements a), b) and c) below are equivalent. If $N$ is soluble Statements a), b) and d) are equivalent.

   a) Every cyclic subgroup of $N$ is completely reducible.
   b) Every subgroup of $N$ is completely reducible.
   c) $N$ is completely reducible.
   d) $N$ is completely reducible and if $p > 0$, then the index $(N : N^0)$ is prime to $p$. (Here $N^0$ denotes, as usual, the connected component of the identity of $N$ in the Zariski topology.)


10 Corollary. Let $N$ be a subgroup of $FGL(V)$ where $F$ is perfect and $N$ is a hypercentral group. Of the following conditions, a) $\Rightarrow$ b) $\Leftrightarrow$ c) c) $\Rightarrow$ d).

   a) $V$ is completely reducible as $FN$-module.
   b) $V$ is completely $F<x>$-reducible for every $x$ in $N$.
   c) $V$ is completely $FX$-reducible for every finitely generated subgroup $X$ of $N$.
   d) $[V, N]$ is completely $FN$-reducible.

Proof. a) $\Rightarrow$ b). Every subgroup of $N$ is ascendant in $N$. Thus here we just apply Clifford's Theorem, see [3] 5.1 or [8] Prop. 9.

   b) $\Rightarrow$ c). Let $X$ be a finitely generated subgroup of $N$. If $U$ is a finite-dimensional subspace of $V$ containing $[V, X]$, then $U$ is completely reducible as $FX$-module by 9. above. Hence $U = [V, X] + C_U(X)$ for all such $U$ and so $V = [V, X] + C_V(X)$. Therefore $V$ is completely $FX$-reducible.

   c) $\Rightarrow$ b). This is trivial.

   c) $\Rightarrow$ d). By [4] or [9] the $FN$-module $[V, N]$ is completely reducible.$\diamond$

   We need to restrict to $[V, N]$ in 10d); that is b) does not imply a) in 10.

11 Example. For any field $F$ there is an abelian finitary linear group over $F$ that is not completely reducible, while all its finitely generated subgroups are completely reducible.
Proof. We modify the example in [9]. The example there is non-abelian, but with just two composition factors in the standard module. To make the group abelian we abandon the latter property.

Let $F$ be a field with non-zero, non-identity element $\alpha$. Set $W = \oplus_{1 \leq i < \infty} F v_i$, where each $v_i \neq 0$. Let $a_i \in GL(W)$ act on $W$ by $v_i a_i = \alpha v_i$ and $v_j a_i = v_j$ for $j \neq i$. Set $G = \langle a_i : i \geq 1 \rangle$. Then $G$ is abelian, being a direct product of copies of $\langle \alpha \rangle$. Let $\delta : G \to W$ be defined by $\delta : a \mapsto \sum_i (\alpha_i - 1) v_i$ for all $a \in G$, where $v_i a = \alpha_i v_i$ for each $i$. Almost all the $\alpha_i$ are 1, so $\delta : G \to W$ is well defined. Further $\delta$ is a derivation.

Set $V = F v_0 \oplus W$, where $v_0 \neq 0$. Let $a \in G$ act on $V$ via the matrix $\begin{pmatrix} 1 & a \delta \\ 0 & a \end{pmatrix}$; that is $a$ maps $v_0 + w$ for $\beta \in F$ and $w \in W$, to $\beta v_0 + (+\beta(a\delta) + wa$. Then $V$ becomes a finitary $FG$-module and, since $W \leq V$, trivially $V$ is $G$-faithful. $W$ is completely $FG$-ducible. We claim that $V$ is not completely $FG$-ducible. Since $V/W$ is $G$-trivial, while no irreducible $FG$-submodule of $W$ is $G$-trivial, the following are equivalent: $V$ is completely $FG$-ducible; $V$ splits as $FG$-module over $W$; $C_V(G) \not\leq W$; $C_V(G) \neq \{0\}$.

Suppose $v_0 + w \in C_V(G)$ for some $w \in W$. Then for any $a \in G$ we have $(v_0 + w)a = v_0 + w$, so $a \delta + wa = w$ and $a \delta = w(a - 1)$. Now $w \in \oplus_{1 \leq i \leq n} F v_i$ for some finite $n$. Then $a_n \delta = w(a_n - 1) = 0$. But by definition $a_n \delta = (\alpha - 1)v_n \neq 0$. This contradiction proves that no such element $v_0 + w$ exists. Therefore $V$ is not completely reducible as $FG$-module.

Let $H$ be any finitely generated subgroup of $G$. Suppose $\alpha$ is a root of unity. Then $H$ has finite order dividing some power of $|\alpha|$ and the latter is prime to char $F$ if char $F \neq 0$. Thus the group algebra $FH$ is semisimple and so $V$ is completely reducible as $FH$-module.

There remains the case where $F$ contains no non-trivial root of unity (so char $F = 2$ etc.). Suppose $|\alpha|$ above is infinite. Now $v_0 + v_1 + \cdots + v_n$ is a fixed point for $a_1, a_2, \ldots, a_n$. Then $V = F(\sum_{0 \leq i \leq n} v_i) \oplus W$ as $F(a_1, a_2, \ldots, a_n)$-module. It follows that every finitely generated subgroup of $G$ is completely reducible. There remains the case where $|F| = 2$. Let $\alpha$ be a primitive cube root of unity over $F$. Repeat the above construction with $F$ replaced by $F(\alpha)$ and then regard $V$ as an $FG$-module. If $H$ is a finitely generated subgroup of $G$ then $|H|$ has order a power of 3 and hence $H$ is completely reducible. If $V$ is completely reducible as $FG$-module, then $C_V(G) \neq \{0\}$ and $V$ is completely reducible as $F(\alpha)G$-module, which is not. The construction is complete. \end{proof}
12. Let \( x \in FGL(V) \) act diagonalizably on \( V \). Then \( x \) acts diagonalizably via conjugation on \( E \). (The element \( x \) acts diagonalizably on \( V \) if \( V \) spanned by eigenvectors of \( x \).)

**Proof.** Now \( V = U \oplus C \) as \( F < x > \)-module, for some finite-dimensional subspace \( U \) and some subspace \( C \leq C_V(x) \). Also \( U = \oplus_{1 \leq i \leq n} F u_i \) for some basis \( u_1, \ldots, u_n \) of \( U \) of eigenvectors of \( x \). Let \( e_{ij} \) be the element of \( E \) defined by \( u_i e_{ij} = u_j \); \( u_k e_{ij} = 0 \) for \( k \neq i \) and \( C e_{ij} = \{0\} \). Then \( E \) as \( F \)-space is equal to

\[
(\oplus_{ij} F e_{ij}) \oplus \text{End}_F C(\oplus_i \text{Hom}_F(F u_i, C)) \oplus (\oplus_j \text{Hom}_F(C, F u_j)).
\]

Suppose \( u_i x = \xi_i u_i \) for each \( i \), where \( \xi_i \in F \). Then \( x^{-1} e_{ij} x = \xi_i^{-1} \xi_j e_{ij} \). Clearly \( x \) centralizes \( C \). Also \( x \) acts on \( \text{Hom}_F(F u_i, C) \) by multiplying by \( \xi_i^{-1} \) and acts on \( \text{Hom}_F(C, F u_j) \) by multiplying by \( \xi_j \). Thus \( E \) is spanned by eigenvectors of \( x \) and the claim follows. \( \Diamond \)

13. Let \( x \in FGL(V) \) be such that \( V \) is completely reducible as \( F < x > \)-module and suppose \( E \) is perfect. Then \( E \) is completely reducible as \( F < x > \)-module.

**Proof.** Let \( F^- \) be the algebraic closure of \( F \). Now \( V = \oplus_{\lambda \in \Lambda} I_\lambda \) for some irreducible \( F < x > \)-modules \( I_\lambda \). Also each \( \dim_F I_\lambda \) is finite. Then [1] p. 64 or [7] 1.24 yields that \( F^- \oplus_F V \cong \oplus_{\lambda} F^- \oplus F I_\lambda \) and is completely reducible as \( F^- < x > \)-module. By 12. it follows that \( x \) acts diagonalizably on \( \text{End}_F(F^- \oplus_F V) \). But there is an \( F^- < x > \) embedding of \( F^- \oplus_F E \) into \( \text{End}_F(F^- \oplus_F V) \). Therefore \( x \) acts diagonalizably on \( F^- \oplus_F E \). Consequently \( E \) is completely reducible over \( F < x > \) by [1] p. 64 or [7] 1.24 again. \( \Diamond \)

14. Consider the proof of 3. Suppose that \( F \) is perfect and that every element of \( G \) acts completely reducibly on \( V \). Then every element of \( G \) acts completely reducibly on \( W \).

**Proof.** Now \( V = V_0 \oplus V' \) and \( W = V_0 \oplus W' \), see the proof of 3. If \( x \in G \), than \( V \) is completely reducible as \( F < x > \)-module and hence \( V_0 \) and \( V' \) are too. Further \( W' \leq \text{End}_F V' \) is also completely reducible as \( F < x > \)-module by 13. The claim follows. \( \Diamond \)

15. Consider the situation of 1. Assume \( G \) is soluble and completely reducible. Then \( \dim_F(F w^G) \) is finite for every \( w \) in \( W' \). (Here \( F w^G = \sum_{g \in G} F w g = \sum_{g \in G} F g^{-1} w g \).)

**Proof.** Let \( w \in W' \). Then \( w \in \bigoplus_{j \in J} W_j \) for some finite subset \( J \) of \( I \). Since \( G \) is soluble, every irreducible \( FG \)-submodule and every irreducible \( FH \)-submodule of \( V \) is finite-dimensional (see [5] Prop. 3). It follows that \( \dim_F(\bigoplus_{J} V_j) \) is finite. But then \( \bigoplus_{J} V_j \) lies in a direct sum of a finite
number of irreducible $FG$-submodules of $V$. Consequently $JG$ is a finite subset of $I$. But $w^G \subseteq \oplus_{j \in JG} W_j$ and the latter is finite dimensional. Therefore $\dim_F (Fw^G)$ is finite.\ }

16. In the situation of 3., suppose $F$ is perfect and $G$ is nilpotent and completely reducible. Then $W$ is completely reducible as $FG$-module.

**Proof.** By definition $V = V_0 \oplus V'$ and $W = V_0 \oplus W'$, see the proof of 3. By 9. and [5] Prop. 3 every cyclic subgroup of $G$ is completely reducible, so by 14. every element of $G$ acts completely reducibly on $W'$. Consequently $G$ does too by 9. and 15. Clearly $G$ acts completely reducibly on $V_0$. Therefore $W$ is a completely reducible $FG$-module.\ }

17. In the situation of 3., suppose $\text{char } F = 0$ and $G$ is soluble and completely reducible. Then $W$ is a completely reducible $FG$-module.

**Proof.** Repeat the proof of 16.\ }

18. Let $K$ and $L$ be soluble, completely reducible, normal subgroups of the subgroup $M$ of $GL(n, F)$. Under any one of the following three conditions, $KL$ is completely reducible.

a) $\text{char } F = 0$.

b) $\text{char } F = p > 0$, $F$ is perfect and $(K : K^0)$ and $(L : L^0)$ are prime to $p$.

c) $\text{char } F = p > 0$, $F$ is perfect and $K$ and $L$ are locally nilpotent. ($K^0$ denotes the connected component of $K$ containing the identity in the Zariski topology.)

**Proof.** In all three cases $F$ is perfect, so ([1] p. 64 or [7] 1.24) we may assume that $F$ is algebraically closed.

a) If $K^0 L^0$ is completely reducible, then so is $KL$ by Maschke’s Theorem ([7] 1.5). Hence assume $K$ and $L$ are connected. Then $K$ and $L$ are diagonalizable and $KL$ is triangularizable by the Lie–Kolchin Theorems ([7] 5.8). Hence $[K, L] \leq K \cap L = \langle 1 \rangle$ and $KL$ is abelian. Therefore $KL$ is completely reducible by [7] 7.1.

b) Here $(K : K^0)$ and $(L : L^0)$ are prime to $p$ and $(KL : K^0 L^0)$ divides the product of $(K : K^0)$ and $(L : L^0)$. Thus Maschke’s Theorem still applies and we can repeat the proof of Part a).

c) By [7] 7.6 and 7.7 the indices $(K : K^0)$ and $(L : L^0)$ are coprime to $p$. Now apply Part b).\ }

19. Let $K$ and $L$ be soluble, completely reducible, normal subgroups of the subgroup $M$ of $FGL(V)$. Suppose that either $\text{char } F = 0$, or $F$ is perfect and $K$ and $L$ are locally nilpotent. Then $KL$ is completely reducible.

**Proof.** Let $X$ and $Y$ be finite subsets of $K$ and $L$ respectively and set $M_1 = \langle X, Y \rangle$, $K_1 = K \cap M_1$ and $L_1 = L \cap M_1$. Then $K_1$ and $L_1$ are
normal in $M_1$ and $M_1 = K_1L_1$. By 9. and [5] Prop. 3 both $K_1$ and $L_1$ are completely reducible. Further, since $M_1$ is finitely generated and finitary, so $V = U \oplus C$ as $FM_1$-module, where $\dim_F U$ is finite and $C \leq C_V(M_1)$. By 18. applied to the action of $M_1$ on $U$ it follows that $U$ and $V$ are completely reducible as $FM_1$-modules.


$$V = [V, KL] + C_V(K) = T \oplus C_V(K)$$

for some $FKL$-submodule $T$ of $[V, KL]$; note that $C_V(K)$ is a $KL$-submodule of $V$. Since $T \leq [V, KL]$, so $T$ is completely reducible as $FKL$-module, and since $L$ is completely reducible, $C_V(K)$ is completely reducible as $FL$-module and thus as $FKL$-module. Therefore $V$ is completely reducible as $FKL$-module.\)

20. In 1. it would be useful for induction purposes, if, when given that $G$ is also completely reducible, one could deduce that $W'$ is completely reducible as $FG$-module. Unfortunately this is false, which is the reason for the apparent detour through the previous series of lemmas. The simplest counterexample is probably the following. Suppose char $F = 2$ and consider $G = GL(2, 2) \leq GL(2, F)$. Then $G$ is solube and absolutely irreducible. In fact $G$ is dihedral of order 6, so let $H$ be the normal subgroup of $G$ of order 3. Then $G/H$ acts faithfully and linearly on the centralizer space $C(\equiv W'$ here) of $H$ in the matrix algebra $F^{2 \times 2}$. Since char $F = 2 = |G/H|$, so $G/H$ acts unipotently, but not completely reducibly, on $C$.

21. Suppose char $F = 0$ and $H$ is soluble of derived length $d$. Then there is a soluble normal subgroup $K \supseteq H$ of $G$ of derived length $d$ such that $G/K$ is isomorphic to a finitary linear group over $F$.

Proof. We induct on $d$; if $d = 0$, the claim is trivial. Let $H_1$ by the last non-trivial term (the $(d - 1)$st term) of the derived series of $H$ and set $K_1 = C_G(\text{End}_{FH_1}V)$. Then $H_1 \leq K_1 \leq G$. Also $H_1$ is abelian, so $H_1 \subseteq \text{End}_{FH_1}V$ and $K_1$ lies in the centre of $\text{End}_{FH_1}V$. In particular $K_1$ is an abelian normal subgroup of $G$.

By Clifford's Theorem (applied to $H$) the group $H_1$ is completely reducible. Set $V = U_0 \oplus (\oplus_{j \in J} U_j)$, where $U_0 = C_V(H_1)$ and the $U_j$ are non-zero homogeneous components of $V$ as $FH_1$-module. Since $H_1$ is abelian and finitary, $\dim_F U_j$ is finite for each $j \in J$. Also $\text{End}_{FH_1}V = (\text{End}_F U_0) \oplus \prod_{j} \text{End}_{FH_1}(U_j)$, where $\text{End}_{FH_1}(U_j)$ is a matrix ring of finite degree over $D_j = \text{End}_{FH_1}(I_j)$ for $I_j$ any irreducible $FH_1$-submodule
of $U_j$ and the centralizer of $\text{End}_{FH_1} V$ in $E$ is $F1_0 \oplus \prod_j \zeta_1(D_j) 1_j$, where $1_0$ is the identity on $U_0$, the element $1_j$ is the identity on $U_j$ and $\zeta_1(D_j)$ denotes the centre of $D_j$. Consequently

$$C_{FGL(V)}(\text{End}_{FH_1} V) = \begin{cases} 1_0 + (\oplus_j \zeta_1(D_j^*) 1_j) & \text{if dim}_F U_0 \text{ is infinite} \\ F^* 1_0 + (\oplus_j \zeta_1(D_j^*) 1_j) & \text{if dim}_F U_0 \text{ is finite.} \end{cases}$$

Either way this subgroup of $FGL(V)$ is abelian and completely reducible and consequently, by Clifford’s Theorem, $K_1$ is completely reducible. By 3. we can represent $G/K_1$ finitarily and faithfully on some $F$-space $W$ and we take $W$ to be the space constructed in the proof above of 3. with $V, G$ and $K_1$ in the place of $V, G$ and $H$.

Taking $H = H_1$; that is, assuming $H$ is abelian, we have proved the following.

**22. Suppose $H$ is abelian. Then there is an abelian, completely reducible, normal subgroup $K$ of $G$ containing $H$ such that $G/K$ is isomorphic to a finitary linear group over $F$.**

We now return to the proof of 21. By 19. the subgroup $HK_1$ of $G$ is completely reducible. By 17. the $F(HK_1/K_1)$-module $W$ is completely reducible. By induction on $d$ there is a soluble normal subgroup $K/K_1$ of $G/K_1$ containing $HK_1/K_1$ of derived length $d - 1$ such that $G/K$ is isomorphic to a finitary linear group over $F$. Clearly $K$ is soluble of derived length at most $1 + (d - 1)$ and $H \leq K$, so the derived length of $K$ is exactly $d$. This completes the proof of 21.$\diamondsuit$

**23. Question.** Does there exist $K$ as in 21. such that $K$ is completely reducible as a subgroup of $G \leq FGL(V)$? (Point 22. above says 'yes' to this if $H$ is abelian.)

**24. Suppose $F$ is perfect and $H$ is nilpotent of class $c$. Then there is a nilpotent normal subgroup $K \geq H$ of $G$ of class $c$ such that $G/K$ is isomorphic to a finitary linear group over $F$.**

**Proof.** By 22. we may assume that $c > 1$. We induct on $c$ and copy the proof of 21. with $H_1$ the centre of $H$. Thus set $K_1 = C_G(\text{End}_{FH_1} V)$. Here $H \leq \text{End}_{FH_1} V$ and $K_1$ is an abelian normal subgroup of $G$ centralizing $H$. As in the proof of 21. we have that $HK_1$ is completely reducible. By induction, cf. the proof of 21., there is a normal subgroup $K_2 \geq HK_1$ of $G$ such that $G/K_2$ is isomorphic to a finitary linear group over $F$ and $K_2/K_1$ is nilpotent of class $c - 1$. 
As in the proof of 21., the group $K_1$ is completely reducible. Let 
\[ V = U_0 \oplus (\oplus_{j \in J} U_j), \]
where here $U_0 = C_V(K_1)$ and where the $U_j$ are non-zero homogeneous components of $V$ as $FK_1$-module. Now $U_j$ is finite $F$-dimensional and a direct sum of a finite number, $r_j$ say, of copies of an irreducible $FK_1$-module $I_j$. Set $D_j = \text{End}_{FK_1} I_j$. Then 
\[ C = FGL(V) \cap \text{End}_{FK_1} V = FGL(U_0) \times (\times_j S_j), \]
where $S_j$ is isomorphic to $GL(\tau_j, D_j)$, acting in the obvious way on $U_j$. Thus $U_0$ and each $U_j$ are irreducible as $FC$-modules and hence $C$ is completely reducible. By 3. the group $N_{FGL(V)}(C)/C$ is isomorphic to some finitary linear group over $F$. Then so too is $G/C_G(K_1) (\cong GC/C \leq N_{FGL(V)}(C)/C)$.

Set $K = K_2 \cap C_G(K_1)$. Since $K_2/K_1$ is nilpotent of class $c - 1$, so $K$ is nilpotent of class at most $c$. But $H$ centralizes $K_1$, so $H \leq K$ and $K$ is nilpotent of class exactly $c$. Finally if $X$ and $Y$ are $F$-spaces such that $G/K_2$ is embeddable in $FGL(X)$ and $G/C_G(K_1)$ is embeddable in $FGL(Y)$ then $G/K$ is embeddable into $FGL(X \oplus Y)$. The proof is complete.\)

Note that the Theorem follows from 21., 22. and 24.

We have yet to exhibit any example of a Zariski-closed normal subgroup $H \leq G \leq FGL(V)$ such that $G/H$ is not isomorphic to any finitary linear group. First we summarize the basic properties of the Zariski topology.

25. The Zariski Topology. With $V$, as above, a vector space over the field $F$, set, for the moment, $G = FGL(V)$. Suppose $V = U \oplus W$ as $F$-space, where $\dim_F U$ is finite. Set $G(U, W) = GL(U) + 1_W \leq G$. Now $GL(U)$ carries its usual Zariski topology. Transfer this to $G(U, W)$ via the obvious isomorphism of $GL(U)$ to $G(U, W)$. Call a subset $X$ of $G$ (Zariski) closed in $G$ if and only if $X \cap G(U, W)$ is closed in $G(U, W)$ for all decompositions $V = U \oplus W$ with $\dim_F U$ finite. This defines a topology on $G$ that coincides with the usual Zariski topology on $G$ whenever $\dim_F V$ is finite. Of course we call this topology the Zariski topology on $G$.

This topology does not, in general, make $G$ into a CZ-group (see[7] Chap. 5), but we do at least have the following.

a) Each finite subset of $G$ is closed in $G$.

b) If $a \in G$, then the following four maps of $G$ into itself are continuous: $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$, $x \mapsto x^{-1} ax$. 

c) If $X$ is a subset of $G$, then $C_G(X)$ is a closed subgroup of $G$.

d) If $X$ is a closed subset of $G$, then $N_G(X)$ is a closed subgroup of $G$.

e) Let $X$ be a subspace of $V$. Then $N_G(X)$ and $C_G(X)$ are closed in $G$.

f) If $X$ is a subgroup of $G$ with closure $X^\sim$ in $G$, then $X^\sim$ is a subgroup of $G$ and if $X$ is normal in $G$, respectively soluble of derived length $d$, respectively nilpotent of class $c$, then so is $X^\sim$.

26. **Some examples.** Returning to the notation $H \lhd G \leq FGL(V)$ of the introduction, if it was true, that $G/H$ is isomorphic to a finitary linear group over $F$ whenever $H$ is Zariski-closed in $G$ (or at least when $H$ is also completely reducible), then the Theorem and the Proposition would follow almost at once. Unfortunately, the example, which we now construct, shows this is not the case.

For $i = 1, 2, \ldots$ let

$$G_i = \langle x_i, y_i \mid x_i^p = y_i^p = 1 \text{ and } G \text{ nilpotent of class } 2 \rangle.$$  

Here $p$ is any prime. Let $q$ be a characteristic and $F$ any field of characteristic $q$. Clearly $G_i$ acts faithfully on a vector space $V_i$ over $F$ of degree $p^6$. Let $V = \bigoplus_{i \geq 1} V_i$ and $G = \times_{i \geq 1} G_i$, regarded as a subgroup of $FGL(V)$ in the obvious way. Let $z_i = [x_i, y_i]$ for each $i$; so $< z_i >$ is the centre of $G_i$ and has order $p^2$. (To see this consider the unitriangular group $Tr_1(3, \mathbb{Z}/p^2)$.) Set

$$H = \langle z_i^{-1} z_j : i, j \geq 1 \rangle = \times_{i \geq 1} z_i^{-1} z_i.$$  

Let $V = U \oplus W$ as $F$-space, where $\dim_F U$ is finite. Then $U \leq \bigoplus_{j \in J} V_j$ for some finite subset $J$ of $I$. Let $G_0 = G \cap G(U, W)$. Since $[V, G_0] \leq U$, so $G_0$ centralizes $\bigoplus_{j \in J} V_j$. Thus $G_0 \leq C_G(\bigoplus_{j \in J} V_j) = \times_{j \in J} G_i$, which is finite. Therefore $H \cap G(U, W)$ is finite and hence closed in $G(U, W)$. Consequently $H$ is Zariski-closed in $FGL(V)$.

If $q = p$, then $G$ is unipotent. If $q \neq p$, then $V_i$ is completely reducible as $FG_i$-module, so $G$ and hence $H$ (by Clifford's Theorem) are completely reducible subgroups of $FGL(V)$. Also $G$ and $G/H$ are nilpotent of class 2 and $p$-groups of finite exponent ($p^2$ if $p > 2$ and 8 if $p = 2$).

By 3.6 of [2] the group $G/H$ is not isomorphic to any finitary linear group of characteristic $p$. Suppose $X$ is a finitary linear group of characteristic not $p$ that is isomorphic to $G/H$. Then $X$ has trivial unipotent radical, so we may choose $X$ completely reducible. Since $X$ is nilpotent and finitary, each irreducible constituent of $X$ is finite dimensional ([5]
Prop. 3). Now a nilpotent irreducible linear group of finite degree and finite exponent is finite. Thus $X$ is residually finite. The centre $Z$ of $X$ has order $p^2$. Thus there is a normal subgroup $Y$ of $X$ of finite index with $Y \cap Z = \langle 1 \rangle$. It follows from the nilpotence of $X$ that $Y = \langle 1 \rangle$. But $X/Z$ is infinite, being a direct product of infinitely many cyclic groups of order $p^2$. Therefore $G/H$ is not isomorphic to any finitary linear group. Choosing $q \neq p$ gives Ex. a) and choosing $q = p$ gives Ex. b).

References