A NEW CLASS OF QUASI-UNIFORM SPACES

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Abstract: We introduce and study the notion of a fitting quasi-uniform space. Every quiet quasi-uniform space is fitting. We show that any bicompletion of a fitting quasi-uniform space is fitting and deduce that every fitting totally bounded quasi-uniformity is a uniformity. We also characterize those $T_1$ quasi-uniform spaces whose bicompletion is $T_1$. Finally, we discuss other kinds of completeness on fitting quasi-uniform spaces.

1. Introduction and preliminaries

In [9] Doitchinov introduced the notion of a quiet quasi-uniform space and obtained a consistent theory of completion for these spaces (see also [10]). By using Doitchinov's completion, Fletcher and Hunsaker proved in [11] the interesting result that every quiet totally bounded...
bounded quasi-uniform space is a uniform space (see [15] for an alternative proof). In [7] Deák generalized quietness in several directions and extended in this way parts of the theory of quiet quasi-uniform spaces, although at cost of losing the good property, cited above, that every quiet totally bounded quasi-uniformity is a uniformity (see [17] Section 6).

In this paper we introduce and study the notion of a fitting quasi-uniform space. Every quiet quasi-uniform space is fitting and each nonregular quasi-metric space \((X, d)\) such that the supremum metric \(d \vee d^{-1}\) is the discrete metric on \(X\) provides an example of a fitting quasi-uniform space which is not quiet. We show that fittingness is preserved by bicompletion of each quasi-uniform space and deduce that every fitting totally bounded quasi-uniform space is a uniform space. Finally, other completeness properties on fitting quasi-uniform spaces are discussed. In many cases, these properties are obtained in the more general context of uniformly weakly regular quasi-uniform spaces, concept introduced in [1], which permits us to generalize and extend several known results.

The letter \(\mathbb{N}\) will denote the set of positive integers and \(\mathbb{R}\) the set of real numbers.

Our basic references for quasi-uniform and quasi-metric spaces are [13] and [16].

We recall some pertinent concepts.

If \(\mathcal{U}\) is a quasi-uniformity on a set \(X\), then \(\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}\) is also a quasi-uniformity on \(X\) called the conjugate of \(\mathcal{U}\), and

\[ T(U) = \{A \subseteq X : \text{ for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\} \]

is the topology generated by \(\mathcal{U}\), where as usual \(U(x) = \{y \in X : (x, y) \in U\}\) for all \(U \in \mathcal{U}\). The coarsest uniformity on \(X\) finer than the quasi-uniformity \(\mathcal{U}\) will be denoted by \(\mathcal{U}^s\), i.e. \(\mathcal{U}^s = \mathcal{U} \vee U^{-1}\). If \(U \in \mathcal{U}\), the element \(U \cap U^{-1}\) of \(\mathcal{U}^s\) will be denoted by \(U^s\).

A quasi-uniform space \((X, \mathcal{U})\) is called bicomplete if \((X, \mathcal{U}^s)\) is a complete uniform space. In this case we say that \(\mathcal{U}\) is a bicomplete quasi-uniformity.

A bicompletion of a quasi-uniform space \((X, \mathcal{U})\) is a bicomplete quasi-uniform space \((Y, \mathcal{V})\) such that \((X, \mathcal{U})\) is quasi-unimorphic to a \(T(V^s)\)-dense subset of \(Y\). It was proved in [3] and in [24] (see also [13]) that every quasi-uniform space admits a bicompletion. Moreover \((Y, \mathcal{V}^{-1})\) is a bicompletion of \((X, \mathcal{U}^{-1})\). In addition, if \((X, \mathcal{U})\)
is a $T_0$ quasi-uniform space, then $(X, \mathcal{U})$ has a unique (up to quasi-isomorphism) $T_0$ bicompletion $(\tilde{X}, \tilde{\mathcal{U}})$. In this case $(\tilde{X}, \tilde{\mathcal{U}})$ is called the bicompletion of (the $T_0$ quasi-uniform space) $(X, \mathcal{U})$.

A quasi-pseudometric on a set $X$ is a nonnegative real-valued function $d$ on $X \times X$ such that for all $x, y, z \in X : d(x, x) = 0$ and $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric on $X$ is a quasi-pseudometric $d$ on $X$ such that $d(x, y) = 0$ if and only if $x = y$.

Each quasi-pseudometric $d$ on $X$ generates a topology $\mathcal{T}(d)$ on $X$, where the basic open sets of $\mathcal{T}(d)$ are the $d$-balls $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$. Note that if $d$ is a quasi-metric, then $\mathcal{T}(d)$ is a $T_1$ topology.

Each quasi-(pseudo)metric $d$ on $X$ induces a (pseudo)metric $d^s$ on $X$ defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ for all $x, y \in X$, where $d^{-1}$ is the conjugate quasi-(pseudo)metric of $d : d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$.

A quasi-pseudometric $d$ on $X$ induces a quasi-uniformity $\mathcal{U}_d$ on $X$ with basic entourages of the form $\{(x, y) : d(x, y) < 2^{-n}\}, n \in \mathbb{N}$.

2. Bicompletion of fitting quasi-uniform spaces

In order to obtain a consistent theory of quasi-uniform completion based on a notion of completeness which provided $\mathcal{T}(U)$-convergence of Cauchy filters, Doitchinov introduced the concept of a quiet quasi-uniform space ([9], [10]).

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is said to be $D$-Cauchy ([9], [12]) if there is a so-called co-filter $\mathcal{G}$ of $\mathcal{F}$ such that $(\mathcal{G}, \mathcal{F}) \rightarrow 0$, where $(\mathcal{G}, \mathcal{F}) \rightarrow 0$ if for each $U \in \mathcal{U}$ there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $G \times F \subseteq U$.

A quasi-uniform space $(X, \mathcal{U})$ is called quiet ([9], [12]) if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that whenever $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$ satisfying $(\mathcal{G}, \mathcal{F}) \rightarrow 0$ and $x$ and $y$ are points of $X$ satisfying $V(x) \in \mathcal{F}$ and $V^{-1}(y) \in \mathcal{G}$, then $(x, y) \in U$.

In this case, we say that $V$ is quiet for $U$ and $\mathcal{U}$ is called a quiet quasi-uniformity.

The Sorgenfrey line and the Kofner plane are interesting examples of nonmetrizable spaces which admit compatible quasi-metrics such that the induced quasi-uniformities are quiet.
Clearly, each uniform space is quiet. It is also well known that each quiet quasi-uniform space $(X, \mathcal{U})$ is regular, i.e. $\mathcal{T}(U)$ is a regular topology, and that a quasi-uniform space $(X, \mathcal{U})$ is quiet if and only if $(X, \mathcal{U}^{-1})$ is quiet. Thus, in a certain sense, quietness is a symmetric property.

Motivated by the existence of many interesting examples of non-regular quasi-uniform spaces having useful symmetry properties (see Ex. 1 below), we introduce the following generalization of the notion of a quiet quasi-uniform space.

**Definition 1.** A quasi-uniform space $(X, \mathcal{U})$ is called **fitting** if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that whenever $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$ satisfying $(\mathcal{G}, \mathcal{F}) \to 0$ and $x$ and $y$ are points of $X$ satisfying $V^\circ(x) \in \mathcal{F}$ and $V^\circ(y) \in \mathcal{G}$, then $(x, y) \in U$.

In this case, we say that $V$ is **fitting** for $U$ and $\mathcal{U}$ is called a fitting quasi-uniformity.

**Remark 1.** (a) It is easily seen that every fitting quasi-uniform space is $R_0$, and, hence, every fitting $T_0$ quasi-uniform space is $T_1$. Furthermore, a quasi-uniform space $(X, \mathcal{U})$ is fitting if and only if $(X, \mathcal{U}^{-1})$ is fitting.

(b) Similarly to the quiet case, fittingness is a hereditary and productive property.

**Example 1.** Let $(X, d)$ be a quasi-metric space such that $d^\circ$ is the discrete metric on $X$. We show that then $\mathcal{U}_d$ is a fitting quasi-uniformity on $X$.

Indeed, given $k \in \mathbb{N}$ put $U_k = \{(x, y) \in X \times X : d(x, y) < 2^{-k}\}$.
We shall prove that $U_k$ is fitting for $U_k$ Let $(\mathcal{G}, \mathcal{F}) \to 0$ and $x, y \in X$ such that $U_k^\circ(x) \in \mathcal{F}$ and $U_k^\circ(y) \in \mathcal{G}$. Since $d^\circ$ is the discrete metric on $X$, $U_k^\circ(x) = \{x\}$ and $U_k^\circ(y) = \{y\}$. Therefore the filters $\mathcal{F}$ and $\mathcal{G}$ are generated by $\{x\}$ and $\{y\}$, respectively. Thus $d(y, x) = 0$, so $x = y$.

Next we give an example of a fitting quasi-uniform space $(X, \mathcal{U})$ such that both $\mathcal{T}(U)$ and $\mathcal{T}(U^{-1})$ are the discrete topology on $X$ but $\mathcal{U}$ is not a quiet quasi-uniformity.

**Example 2.** Let $d$ be the function defined on $\mathbb{N} \times \mathbb{N}$ by

$$
\begin{align*}
    d(n, m) &= \frac{1}{|n - m|} & \text{if } n \text{ is odd and } m \text{ is even;} \\
    d(n, m) &= \frac{1}{|n - m|} & \text{if } n \text{ and } m \text{ are odd with } n > m; \\
    d(n, m) &= \frac{1}{|n - m|} & \text{if } n \text{ and } m \text{ are even with } n < m; \\
    d(n, n) &= 0 & \text{for all } n \in \mathbb{N}; \\
\end{align*}
$$

and

$$
    d(n, m) = 1 \text{ otherwise.}
$$
It is easily seen that $d$ is a quasi-metric on $\mathbb{N}$ such that $T(d)$ and $T(d^{-1})$ are the discrete topology on $\mathbb{N}$. Moreover $d^3$ is the discrete metric on $\mathbb{N}$, so $\mathcal{U}_d$ is a fitting quasi-uniformity (see Ex. 1).

Put $U = \{(n, m) \in \mathbb{N} \times \mathbb{N} : d(n, m) < 1\}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the filters on $\mathbb{N}$ generated by $\{\{2k : k \geq n\} : n \in \mathbb{N}\}$ and $\{\{2k - 1 : k \geq n\} : n \in \mathbb{N}\}$, respectively. Clearly $(\mathcal{G}, \mathcal{F}) \to 0$ with respect to $\mathcal{U}_d$. Furthermore, for each $\varepsilon > 0$ there are $j, k \in \mathbb{N}$ such that $B_d(2j, \varepsilon) \in \mathcal{F}$ and $B_d^{-1}(2k - 1, \varepsilon) \in \mathcal{G}$. But $d(2j, 2k - 1) = 1$, so $(2j, 2k - 1) \notin U$. We have shown that $\mathcal{U}_d$ is not a quiet quasi-uniformity on $\mathbb{N}$.

It was proved in [18] that any bicompletion of a quiet quasi-uniform space is quiet. We next extend this result to fitting quasi-uniform spaces.

**Proposition 1.** Let $(Y, \mathcal{U})$ be a quasi-uniform space and let $X$ be a $T(U^3)$-dense subset of $Y$. Then $(Y, \mathcal{U})$ is fitting if and only if $(X, \mathcal{U} | X \times X)$ is fitting.

**Proof.** Since the necessity is obvious (see Remark 1 (b)), we only prove the sufficiency. Let $U_0 \in \mathcal{U}$. Choose $U_1 \in \mathcal{U}$ such that $U_1^3 \subseteq U_0$. Then, there is $V \in \mathcal{U}$ such that $V \subseteq U_1$ and $V \cap (X \times X)$ is fitting for $U_1 \cap \cap (X \times X)$. Let $W \in \mathcal{U}$ such that $W^3 \subseteq V$. We shall show that $W$ is fitting for $U_0$.

Indeed, let $\mathcal{F}$ and $\mathcal{G}$ be filters on $Y$ such that $(\mathcal{G}, \mathcal{F}) \to 0$ and let $x, y \in Y$ be such that $W^s(x) \in \mathcal{F}$ and $W^s(y) \in \mathcal{G}$. Since $X$ is a $T(U^3)$-dense subset of $Y$, $\mathcal{F}_1$ and $\mathcal{G}_1$ are filter bases on $X$, where

$$\mathcal{F}_1 = \{U^s(F) \cap X : F \in \mathcal{F}, U \in \mathcal{U}\} \quad \text{and} \quad \mathcal{G}_1 = \{U^s(G) \cap X : G \in \mathcal{G}, U \in \mathcal{U}\}.$$ 

Furthermore $(\mathcal{G}_1, \mathcal{F}_1) \to 0$ because $(\mathcal{G}, \mathcal{F}) \to 0$. Now let $a \in W^s(x) \cap \cap X$ and $b \in W^s(y) \cap X$. Then $W^s(W^s(x)) \cap X \subseteq V^s(a) \cap X$. Since $W^s(x) \in \mathcal{F}$ we deduce that $V^s(a) \cap X \in \mathcal{F}_1$. Similarly, we obtain that $V^s(b) \cap X \in \mathcal{G}_1$. So, by our assumption, $(a, b) \in U_1$. Thus $(x, y) \in \in U_1^3 \subseteq U_0$. We conclude that $(Y, \mathcal{U})$ is fitting. \hfill \Box

From Prop. 1 we immediately deduce the following result.

**Theorem 1.** Let $(Y, \mathcal{V})$ be a bicompletion of a fitting quasi-uniform space $(X, \mathcal{U})$. Then $(Y, \mathcal{V})$ is a fitting quasi-uniform space.

**Corollary 1.** The bicompletion of any fitting $T_0$ quasi-uniform space is a fitting $T_0$ quasi-uniform space.
3. Fitting totally bounded quasi-uniformities

In this section we shall extend Fletcher and Lindgren's theorem cited in Section 1 to fitting quasi-uniform spaces.

Let us recall that a quasi-uniform space \((X, \mathcal{U})\) is totally bounded provided that the uniform space \((X, \mathcal{U}^s)\) is totally bounded (see for instance [13]).

**Theorem 2.** Let \((X, \mathcal{U})\) be a fitting totally bounded quasi-uniform space. Then \((X, \mathcal{U})\) is a uniform space.

**Proof.** We first show that \(\mathcal{U}^{-1} \subseteq \mathcal{U}\). Assume the contrary. Then, there exists \(U \in \mathcal{U}\) such that \(V \setminus U^{-1} \neq \emptyset\) for all \(V \in \mathcal{U}\). Let \((x_V, y_V) \in V \setminus U^{-1}\) whenever \(V \in \mathcal{U}\). Since \((X, \mathcal{U})\) is totally bounded we can construct, without loss of generality, two subnets \((a_\alpha)_{\alpha \in \Lambda}\) and \((b_\alpha)_{\alpha \in \Lambda}\) of the nets \((x_V)_{V \in \mathcal{U}}\) and \((y_V)_{V \in \mathcal{U}}\) respectively, such that both \((a_\alpha)_{\alpha \in \Lambda}\) and \((b_\alpha)_{\alpha \in \Lambda}\) are Cauchy nets in the uniform space \((X, \mathcal{U}^s)\). Moreover, for each \(\alpha \in \Lambda\) we may put \(a_\alpha = x_{V_\alpha}\) and \(b_\alpha = y_{V_\alpha}\), where the corresponding map from \(\Lambda\) to \(\mathcal{U}\) witness that \((a_\alpha)_{\alpha \in \Lambda}\) is a subnet of \((x_V)_{V \in \mathcal{U}}\) and \((b_\alpha)_{\alpha \in \Lambda}\) is a subnet of \((y_V)_{V \in \mathcal{U}}\). Since \((X, \mathcal{U})\) is totally bounded there is a bicompletion \((Y, \mathcal{V})\) of \((X, \mathcal{U})\) such that \((Y, \mathcal{V}^s)\) is a compact uniform space ([24] p. 80). Thus \((a_\alpha)_{\alpha \in \Lambda}\) converges to a point \(a \in Y\) with respect to \(\mathcal{T}(\mathcal{V}^s)\) and \((b_\alpha)_{\alpha \in \Lambda}\) converges to a point \(b \in Y\) with respect to \(\mathcal{T}(\mathcal{V}^s)\). (Recall that both \((a_\alpha)_{\alpha \in \Lambda}\) and \((b_\alpha)_{\alpha \in \Lambda}\) are Cauchy nets in \((X, \mathcal{U}^s)\) and hence in \((Y, \mathcal{V}^s)\).) It immediately follows that \((a, b) \in \cap \{W : W \in \mathcal{V}\}\). On the other hand \((Y, \mathcal{V})\) is fitting by Th. 1, so \(\mathcal{T}(\mathcal{V})\) is an \(R_0\) topology, and, thus, \((b, a) \in \cap \{W : W \in \mathcal{V}\}\) ([13] Prop. 1.9). Therefore the net \(((y_{V_\alpha}, x_{V_\alpha}))_{\alpha \in \Lambda}\) is eventually in \(U\), which contradicts our assumption that \((x_V, y_V) \notin U^{-1}\) for all \(V \in \mathcal{U}\). We conclude that \(\mathcal{U}^{-1} \subseteq \mathcal{U}\). Similarly, we prove that \(\mathcal{U} \subseteq U^{-1}\). Hence \(\mathcal{U}\) is a uniformity on \(X\) and the proof is complete.\(\diamond\)

**Corollary 2 ([11], [15]).** Every quiet totally bounded quasi-uniformity is a uniformity.

Since a topological space is \(T_1\) if and only if it is \(T_0\) and \(R_0\), and each fitting quasi-uniform space is \(R_0\), it seems interesting to characterize those \(T_1\) quasi-uniform spaces \((X, \mathcal{U})\) whose bicompletion \((\tilde{X}, \tilde{\mathcal{U}})\) is \(T_1\). Our next result provides such a characterization. In order to help the reader we recall the construction of the bicompletion of a \(T_0\) quasi-uniform space \((X, \mathcal{U})\), as is given in [13] Th. 3.33.

Let \(\tilde{X}\) be the set of all minimal \(\mathcal{U}^s\)-Cauchy filters on \(X\). (Recall that a \(\mathcal{U}^s\)-Cauchy filter is minimal provided that it contains no \(\mathcal{U}^s\)-
Cauchy filters other than itself.) For each \( U \in \mathcal{U} \) let
\[
\tilde{U} = \{(F, G) \in \tilde{X} \times \tilde{X} : \text{there exist } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ with } F \times G \subseteq U\}.
\]

Then \( \{\tilde{U} : U \in \mathcal{U}\} \) is a base for a quasi-uniformity \( \tilde{U} \) on \( \tilde{X} \) such that \((\tilde{X}, \tilde{U})\) is a bicomplete \( T_0 \) quasi-uniform space and \((X, \mathcal{U})\) is quasi-uniformly embedded as a \( T(\mathcal{U}) \)-dense subspace of \((\tilde{X}, \tilde{U})\) by the map \( i : X \to \tilde{X} \) such that for each \( x \in X \), \( i(x) \) is the \( T(U^s) \)-neighborhood filter of \( x \). Furthermore, any \( T_0 \) bicompletion of \((X, \mathcal{U})\) is quasi-unimorphic to \((\tilde{X}, \tilde{U})\) (see [13] Th. 3.34) and we may identify \( X \) with \( i(X) \).

**Proposition 2.** A \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) has a \( T_1 \) quasi-uniform bicompletion if and only if whenever \( F \) and \( G \) are \( U^s \)-Cauchy filters such that \((F, G) \to 0\), then \((G, F) \to 0\).

**Proof.** We first suppose that the bicompletion \((\tilde{X}, \tilde{U})\) of the \( T_1 \) quasi-uniform space \((X, \mathcal{U})\) is \( T_1 \). Let \( F \) and \( G \) be \( U^s \) -Cauchy filters on \( X \) such that \((F, G) \to 0\). Then \( i(F) \) and \( i(G) \) are \( U^s \)-Cauchy filter bases on \( \tilde{X} \), so there exist \( a \) and \( b \) in \( \tilde{X} \) such that \( i(F) \) and \( i(G) \) are \( T(\tilde{U}^s) \)-convergent to \( a \) and \( b \), respectively. Since \((i(F), i(G)) \to 0\) and \((\tilde{X}, T(\tilde{U}))\) is \( T_1 \), it follows that \( a = b \). Hence \((i(G), i(F)) \to 0\), and, thus, \((G, F) \to 0\).

Conversely, suppose that \( F \) and \( G \) are minimal \( U^s \)-Cauchy filters on \( X \) such that \((F, G) \in \tilde{U}\) for all \( U \in \mathcal{U} \). Then, for each \( U \in \mathcal{U} \) there exist \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \) such that \( F \times G \subseteq U \). Thus \((F, G) \to 0\) and, by assumption, \((G, F) \to 0\). This implies that \((G, F) \) \( \in \tilde{U}\) for all \( U \in \mathcal{U} \). Since \((\tilde{X}, T(\tilde{U}))\) is \( T_0 \), we obtain \( F = G \). Consequently \((\tilde{X}, T(\tilde{U}))\) is a \( T_1 \) topological space. \( \Diamond \)

4. **Other completeness properties on fitting quasi-uniform spaces**

Some authors have investigated several kinds of completeness on quiet quasi-uniform spaces ([5], [12], [6], [19], [7], [22], etc.). In this direction, many interesting results were obtained in the more general context of uniformly regular quasi-uniform spaces, which were introduced by Császár in [4]. (Let us recall that a quasi-uniform space \((X, \mathcal{U})\) is uniformly regular provided that for each \( U \in \mathcal{U} \) there is \( V \in \mathcal{U} \) such that \( \text{cl}_{T(U)} V(x) \subseteq U(x) \) for all \( x \in X \).)

In this section we study several kinds of completeness for fitting
and uniformly $R_0$ quasi-uniform spaces.

A quasi-uniform space $(X, \mathcal{U})$ is uniformly $R_0$ (uniformly weakly regular in [1]) provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\text{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$ for all $x \in X$.

Obviously, every uniformly $R_0$ quasi-uniform space is $R_0$ and every uniformly regular quasi-uniform space is uniformly $R_0$.

**Proposition 3.** Let $(X, \mathcal{U})$ be a fitting quasi-uniform space. Then both $(X, \mathcal{U})$ and $(X, \mathcal{U}^{-1})$ are uniformly $R_0$.

**Proof.** Let $U \in \mathcal{U}$. By assumption, there is $V \in \mathcal{U}$ which is fitting for $U$. Fix $x \in X$. We shall show that $\text{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$. Indeed, given $z \in \text{cl}_{\mathcal{T}(U)} V^s(x)$, denote by $\mathcal{F}$ the filter on $X$ for which $\{V^s(x) \cap W(z) : W \in \mathcal{U}\}$ is a base and denote by $\mathcal{G}$ the filter on $X$ generated by $\{z\}$. Clearly $\mathcal{G} \subseteq \mathcal{F}$ and $V^s(x) \in \mathcal{F}$, it follows that $(x, z) \in U$. We conclude that $(X, \mathcal{U})$ is uniformly $R_0$. By symmetry we show that $(X, \mathcal{U}^{-1})$ is uniformly $R_0$.

Next we recall some pertinent concepts.

Let $(X, \mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ be a filter on $X$. Then $\mathcal{F}$ is said to be left $K$-Cauchy ([22]) if for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $U(x) \in F$ for all $x \in F$. $\mathcal{F}$ is called right $K$-Cauchy ([22]) if it is left $K$-Cauchy on $(X, \mathcal{U}^{-1})$.

A quasi-uniform space $(X, \mathcal{U})$ is said to be left (right) $K$-complete ([22]) if each left (right) $K$-Cauchy filter on $(X, \mathcal{U})$ is $\mathcal{T}(U)$-convergent. $(X, \mathcal{U})$ is Smyth completable if and only if each left $K$-Cauchy filter on $(X, \mathcal{U})$ is a $\mathcal{U}^s$-Cauchy filter ([16]), and it is Smyth complete if and only if each left $K$-Cauchy filter on $(X, \mathcal{U})$ is $\mathcal{T}(\mathcal{U}^s)$-convergent ([16]). According to [5], $(X, \mathcal{U})$ is said to be half-complete if each $\mathcal{U}^s$-Cauchy filter is $\mathcal{T}(\mathcal{U})$-convergent.

A Smyth complete quasi-uniform space is both bicomplete and left $K$-complete, and a bicomplete or left $K$-complete quasi-uniform space is half-complete. However, it is well known that the converse implications do not hold in general. Furthermore, right $K$-completeness implies half-completeness, while left $K$-completeness and right $K$-completeness are independent concepts ([22]).

It is interesting to recall that Smyth completeness and left $K$-completeness constitute useful tools to explain properties of some interesting examples of quasi-uniform and quasi-metric spaces which arise in several fields of Theoretical Computer Science (see [26], [27], [23], [25], etc.), while bicompleteness, right $K$-completeness and half-completeness are appropriate notions of completeness in the study of function
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and multifunction spaces from a quasi-uniform point of view (see [20], [21], [2], etc.).

**Lemma 1.** Let \( (X, \mathcal{U}) \) be a uniformly \( R_0 \) quasi-uniform space. If a \( \mathcal{U}^s \)-Cauchy filter is \( \mathcal{T}(U) \)-convergent to a point \( x_0 \in X \), then it is \( \mathcal{T}(U^s) \)-convergent to \( x_0 \).

**Proof.** Let \( \mathcal{F} \) be a \( \mathcal{U}^s \)-Cauchy filter on \( X \) such that \( \mathcal{F} \) is \( \mathcal{T}(U) \)-convergent to \( x_0 \in X \). By assumption, given \( U \in \mathcal{U} \) there is \( V \in \mathcal{U} \) such that \( \text{cl}_{\mathcal{T}(U)}V^s(x) \subseteq U(x) \) for all \( x \in X \). Furthermore, there is \( F \in \mathcal{F} \) such that \( V^s(x) \in \mathcal{F} \) for all \( x \in F \). So \( W(x_0) \cap V^s(x) \neq \emptyset \) for all \( W \in \mathcal{U} \) and all \( x \in F \). Hence \( x_0 \in \text{cl}_{\mathcal{T}(U)}V^s(x) \subseteq U(x) \) for all \( x \in F \). Therefore \( F \subseteq U^{-1}(x_0) \) and, thus, \( U^{-1}(x_0) \in \mathcal{F} \) for all \( U \in \mathcal{U} \). Consequently \( \mathcal{F} \) is \( \mathcal{T}(U^{-1}) \)-convergent to \( x_0 \). We conclude that \( \mathcal{F} \) is \( \mathcal{T}(U^s) \)-convergent to \( x_0 \).

**Proposition 4 ([1]).** Every uniformly \( R_0 \) half-complete quasi-uniform space is bicomplete.

**Proof.** Let \( \mathcal{F} \) be a \( \mathcal{U}^s \)-Cauchy filter on a uniformly \( R_0 \) half-complete quasi-uniform space \( (X, \mathcal{U}) \). Since \( (X, \mathcal{U}) \) is half-complete \( \mathcal{F} \) is \( \mathcal{T}(U) \)-convergent to a point \( x_0 \in X \). By Lemma 1, \( \mathcal{F} \) is \( \mathcal{T}(U^s) \)-convergent to \( x_0 \). We conclude that \( (X, \mathcal{U}) \) is bicomplete.

**Proposition 5.** For a uniformly \( R_0 \) Smyth completable quasi-uniform space \( (X, \mathcal{U}) \) the following are equivalent:

1. \( (X, \mathcal{U}) \) is Smyth complete;
2. \( (X, \mathcal{U}) \) is bicomplete;
3. \( (X, \mathcal{U}) \) is half-complete;
4. \( (X, \mathcal{U}) \) is left \( K \)-complete.

**Proof.** (1) \( \Rightarrow \) (2), (1) \( \Rightarrow \) (4), (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (3) are obvious. We show that (3) \( \Rightarrow \) (1): By Prop. 4, \( (X, \mathcal{U}) \) is bicomplete. Since by assumption \( (X, \mathcal{U}) \) is Smyth completable, every left \( K \)-Cauchy filter on \( (X, \mathcal{U}) \) is a \( \mathcal{U}^s \)-Cauchy filter. Hence, every left \( K \)-Cauchy filter on \( (X, \mathcal{U}) \) is \( \mathcal{T}(U^s) \)-convergent. So \( (X, \mathcal{U}) \) is Smyth complete.

**Corollary 3.** Let \( (X, \mathcal{U}) \) be a uniformly \( R_0 \) Smyth completable left \( K \)-complete quasi-uniform space. Then \( (X, \mathcal{U}^{-1}) \) is right \( K \)-complete.

**Proof.** By Prop. 5, \( (X, \mathcal{U}) \) is Smyth complete. Let \( \mathcal{F} \) be a right \( K \)-Cauchy filter on \( (X, \mathcal{U}^{-1}) \). Then \( \mathcal{F} \) is left \( K \)-Cauchy on \( (X, \mathcal{U}) \). So, it is \( \mathcal{T}(U^s) \)-convergent and, in particular, \( \mathcal{T}(U^{-1}) \)-convergent. We conclude that \( (X, \mathcal{U}^{-1}) \) is right \( K \)-complete.

A quasi-uniform space \( (X, \mathcal{U}) \) is said to be \( D \)-complete ([9], [12]) if every \( D \)-Cauchy filter on \( (X, \mathcal{U}) \) is \( \mathcal{T}(U) \)-convergent and it is said
to be strongly $D$-complete ([14]) if whenever $G$ and $F$ are filters on $X$ such that $(G,F) \rightarrow 0$, then $G$ has a $\mathcal{T}(U)$-cluster point. It is known that each strongly $D$-complete quasi-uniform space is $D$-complete but the converse implication does not hold in general ([14]).

It was proved in [8] that each co-stable quiet half-complete quasi-uniform space is strongly $D$-complete. The space of Ex. 2 shows that this result does not hold when “quiet” is replaced by “fitting”. (Let us recall ([8]) that a quasi-uniform space $(X,\mathcal{U})$ is co-stable provided that for each pair $G, F$ of filters on $X$ such that $(G,F) \rightarrow 0$, then $G$ is stable, where $G$ is said to be stable ([4]) if for each $U \in \mathcal{U}$, $\cap_{G \in G} U(G) \in G$.)

In fact, it is easy to see that the quasi-uniform space $(\mathbb{N},\mathcal{U}_d)$ of Ex. 2 is co-stable and half-complete. Clearly, it is not $D$-complete. Note also that both $\mathcal{U}_d$ and $(\mathcal{U}_d)^{-1}$ are uniformly regular because $\mathcal{T}(d)$ and $\mathcal{T}(d^{-1})$ are the discrete topology on $\mathbb{N}$.

The interesting question of obtaining conditions under which a uniformly regular quasi-uniform space is quiet has been investigated by several authors ([6], [12], [19], etc.). In particular, it was shown in [12] that every uniformly regular strongly $D$-complete quasi-uniform space is quiet. Here we show that strong $D$-completeness is also an appropriate property for a uniformly $R_0$ quasi-uniform space to be fitting.

**Proposition 6.** Every uniformly $R_0$ strongly $D$-complete quasi-uniform space is fitting.

**Proof.** Let $(X,\mathcal{U})$ be a uniformly $R_0$ strongly $D$-complete quasi-uniform space. Let $U \in \mathcal{U}$ and $W \in \mathcal{U}$ such that $W^2 \subseteq U$. There is $V \in \mathcal{U}$ such that $V^2 \subseteq W$ and $\text{cl}_{\mathcal{T}(U)} V^s(x) \subseteq W(x)$ for all $x \in X$. We shall prove that $V$ is fitting for $U$. Indeed, suppose $(G,F) \rightarrow 0$ and let $x, y \in X$ such that $V^s(x) \in F$ and $V^s(y) \in G$. Since $(X,\mathcal{U})$ is strongly $D$-complete, the filter $G$ has a $\mathcal{T}(U)$-cluster point $x_0 \in X$. Thus $F$ is $\mathcal{T}(U)$-convergent to $x_0$. Hence $x_0 \in (\text{cl}_{\mathcal{T}(U)} V^s(y)) \cap (\text{cl}_{\mathcal{T}(U)} V^s(x))$. Consequently $y \in V^2(x_0)$ and $x_0 \in W(x)$. Therefore $y \in W^2(x) \subseteq U(x)$. We conclude that $V$ is fitting for $U$ and, thus, $(X,\mathcal{U})$ is a fitting quasi-uniform space. ♦

**References**


