THE NEUBERG CUBIC IN LOCUS PROBLEMS

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Dedicated to Professor Gino Tironi on his 60th birthday

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Abstract: In this paper we continue the exploration of various locus problems whose solutions involve the Neuberg cubic of the scalene triangle in the plane. We use analytical geometry and the complex numbers to show that the Neuberg equation describes the essential part of the locus in many geometric constructions. In this way we discover new characteristics of the Neuberg cubic that has been extensively studied recently.

1. Introduction

Let $ABC$ be a scalene triangle in the plane. The author has considered in [5] numerous locus problems whose solutions involve the circular cubic $N$ which Neuberg [18] calls the 21-point cubic and which is known today as the Neuberg cubic of the triangle $ABC$. It is evident from the extensive list of references on this curve given below that the Neuberg cubic has attracted a lot of attention lately. The present paper is yet another such contribution. It adds more than two dozens of new instances when the Neuberg cubic appears in various geometric constructions. Most results utilise the notion of the homology for triangles but there are also those that use the concurrency of lines and the concept of the power of a point with respect to a circle.
Our proofs use the analytical geometry of complex numbers. This choice leads to the simplest expressions and appears to be the most natural for our search for the Neuberg cubic. It is suitable for implementation on computers. In fact, our results are all discovered with the help of a computer (PC Pentium 200 MHz, 64 MB RAM) and the software Maple V (version 4). We leave out many details because Maple V (or any other package with symbolic algebra computation capability) performs all factorisations and simplifications easily.

The paper is organised as follows. After the introduction we describe our notation and give basics on the use of complex numbers in geometry. In the remaining sections we present and prove some new results of our search for the Neuberg cubic that all give new characterisations of this remarkable curve by various geometric constructions or locus problems. The section titles are chosen to suggest the method of recognition.

Of course, since our results are characterisations of the same curve, in some cases one can show easily that one method follows from the other(s). Observations of this kind and other comments on possible extensions and special cases are included in remarks.

2. Complex numbers in geometry

In this paper we shall use complex numbers in proofs because they provide simple expressions and arguments. There are several books, for example [16], [9], [23], [12], [26], and [20], that give introductions into the method which we utilise.

A point $P$ in the Gauss plane is represented by a complex number. This number is called the affix of $P$. It is customary to denote the affix of a point $P$ with the corresponding small Latin letter $p$ and to identify a point and its affix. The complex conjugate of $p$ is denoted $\bar{p}$. This rule has an important exception in that the vertices $A$, $B$, and $C$ of the reference triangle are represented by numbers $u$, $v$, and $w$ on the unit circle. The letters $a$, $b$, and $c$ are reserved for the lengths of sides of $ABC$. Hence, the circumcentre $O$ of $ABC$ is the origin. The affix of $O$ is number 0 (zero) and complex conjugates of $u$, $v$, and $w$ are $u^{-1}$, $v^{-1}$, and $w^{-1}$.

Let $\varphi$ and $\psi$ denote the first and the second cyclic permutation on triples of letters. For example, $\varphi(a) = b$, $\psi(a) = c$, $\varphi(u x) = v y$, and $\psi(u x) = w z$. Finally, if $f$ is an expression, $\mathcal{S}f$ and $\mathcal{P}f$ replace
\( f + \varphi(f) + \psi(f) \) and \( f \varphi(f) \psi(f) \). The expressions \( \varphi(f) \) and \( \psi(f) \) are called relatives of \( f \).

Most interesting points, curves, ... associated with the triangle \( ABC \) are expressions that involve symmetric functions of \( u \), \( v \), and \( w \) that we denote as follows: \( \mu = uvw \), \( \sigma = u + v + w \), \( \tau = vw + wu + uv \), \( \sigma_a = -u + v + w \), \( \sigma_b = \varphi(\sigma_a) \), \( \sigma_c = \psi(\sigma_a) \), \( \mu_a = vw \), \( \mu_b = wu \), \( \mu_c = uv \), \( \tau_a = -v w + wu + uv \), \( \tau_b = \varphi(\tau_a) \), \( \tau_c = \psi(\tau_a) \), \( \delta_a = v - w \), \( \delta_b = \varphi(\delta_a) \), \( \delta_c = \psi(\delta_a) \), \( \zeta_a = v + w \), \( \zeta_b = \varphi(\zeta_a) \), \( \zeta_c = \psi(\zeta_a) \). For each \( k \geq 2 \), \( \sigma_k \), \( \sigma_{ka} \), \( \sigma_{kb} \), and \( \sigma_{kc} \) are derived from \( \sigma \), \( \sigma_a \), \( \sigma_b \), and \( \sigma_c \) with the substitution \( u = u^k \), \( v = v^k \), \( w = w^k \). In a similar fashion we can define analogous expressions using letters \( \tau \), \( \mu \), \( \delta \), and \( \zeta \).

Let us close this section on preliminaries with a few words on analytic geometry that we shall use and on triangle notation. Any of the books mentioned above contains more than enough information on basic constructions (line through two points, perpendicular and parallel to a line through a point, condition for concurrency of three lines, condition for collinearity of three points) that are needed to follow our arguments. As a convenience for the reader we repeat them here.

In geometry lines are important so that we have the special notation \([m, n]\) for the set of all points \( P \) that satisfy the equation \( mp - m\bar{p} + n = 0 \), where \( n \) is purely imaginary.

Let \( X \), \( Y \), and \( Z \) be three points with affixes \( x \), \( y \), and \( z \) and let \( \ell \) be a line \([f, h]\) in the plane. Then the line \( XY \) is \([x - y, x\bar{y} - y\bar{x}]\), the parallel to \( \ell \) through \( X \) is \([f, f\bar{x} - \bar{f}x]\) and the perpendicular to \( \ell \) through \( X \) is \([-f, -f\bar{x} + \bar{f}x]\).

The conditions for points \( X \), \( Y \), and \( Z \) to be collinear and for lines \([m, n]\), \([p, q]\), and \([s, t]\) to be concurrent are \( S\bar{x}(y - z) = 0 \) and \( S\bar{m}(pt - sq) = 0 \). If \( X \), \( Y \), and \( Z \) are not collinear, they determine the unique circle \( k(X, Y, Z) \) which goes through them.

The centroid, the circumcentre, and the centre of the nine-point circle of the triangle \( XYZ \) are \((S \bar{x})/3, (S \bar{x}(y - z))/(S \bar{x}(y - z)), \) and \((S \bar{x}(y^2 - z^2))/(2S \bar{x}(y - z))\).

Let \( P \) and \( Q \) be points and let \( \ell \) be a line. Then \( pa(P, \ell), pe(P, \ell), pr(P, \ell), re(P, \ell), \) and \( re(P, Q) \) denote the parallel to \( \ell \) through \( P \), the perpendicular to \( \ell \) through \( P \), the projection of \( P \) onto \( \ell \), the reflection of \( P \) in \( \ell \), and the reflection of \( P \) at \( Q \), respectively.

For a point \( P \) not on the circumcircle of a triangle \( XYZ \), let \( ig(P, XYZ) \) be the isogonal conjugate of \( P \) with respect to \( XYZ \). This point is the intersection of lines which make equal angles with internal angle bisectors as do the lines \( XP,YP,\) and \( ZP \).
Let $G$, $O$, $H$, $F$, $K$, $I_v$, and $I_u$ be the centroid, the circumcentre, the orthocentre, the centre of the nine-point circle, the symmedian or Grebe-Lemoine point, the first isogonic point, and the second isogonic point of the base triangle $ABC$.

We shall need triangles $A_xB_xC_x$, where the index $x$ is either $e$, $r$, $t$, $u$, and $v$. In order to describe $A_xB_xC_x$ it suffices to give description of the vertex $A_x$ because $B_x$ and $C_x$ are its relatives. The point $A_x$ is the excentre on $AI$, the reflection $re(A, BC)$, the intersection of tangents to the circumcircle at $B$ and $C$, and the apexes of equilateral triangles constructed inwards and outwards on $BC$, respectively. $A_xB_xC_x$ is the excentral, $A_tB_tC_t$ the tangential, and $A_rB_tC_r$ the three images triangle of $ABC$.

Triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are homologous if lines $X_1X_2$, $Y_1Y_2$, and $Z_1Z_2$ are concurrent.

3. Homology of triangles — circumcentres

Among the oldest known methods of recognising the Neuberg cubic are the following two theorems which use the condition that two families of triangles are families of homologous triangles. In this and the next five sections we consider some other uses of this method for recognition of the Neuberg cubic $N$. Division into sections reflects different ways of defining families of variable triangles.

Let $W_0$ be the complement of the union of sidelines of the base triangle $ABC$ in the plane. For a point $P$ in the plane, let $O_x$, $O_y$, and $O_z$ denote the circumcentres of the triangles $BCP$, $CAP$, and $ABP$. Neuberg [18] has first proved the following result. As a convenience to the reader we shall give easy proofs of this and the next theorem using complex numbers.

**Theorem 3.1.** The locus of all points $P$ in $W_0$ such that $ABC$ is homologous to $O_xO_yO_z$ is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

**Proof.** The $O_x$ is $\mu_x M/n_x$, while $O_y$ and $O_z$ are its relatives, where $n_x$ and $M$ are equations $p + \mu_x \bar{p} - \zeta_x$ and $p \bar{p} - 1$ of the sideline $BC$ and of the circumcircle. The line $AO_x$ has the form $[f/n_x, g/(u n_x)]$, where $f = u p + \mu_x \bar{p} - u p \bar{p} - \tau_x$ and $g = M (u^2 - \mu_x)$, while lines $BO_y$ and $CO_z$ are its relatives. The triangles $ABC$ and $O_xO_yO_z$ are homologous if and only if $M N P \delta_x n_x^{-1} = 0$, where $N = \tau p^2 \bar{p} - \mu \sigma \bar{p}^2 p + \mu \tau \bar{p}^2 - \sigma p^2 + +\sigma_2 p - \tau_2 \bar{p}$ is the equation of the Neuberg cubic [16]. ◊
The following result is proved on page 199 of [16]. It was well known to readers of Mathesis (see [13]) and was mentioned again recently in [21].

**Theorem 3.2.** The Neuberg cubic of $ABC$ is the locus of all points $P$ such that $ABC$ is homologous to the triangle on the reflections of $P$ in the sidelines of $ABC$.

**Proof.** The reflection $R_\alpha$ of the point $P$ in the side $BC$ is $\zeta_\alpha - \mu_\alpha \bar{p}$ and the line $AR_\alpha$ is $[\mu_\alpha \bar{p} - \sigma_\alpha, u^{-1} \mu_\alpha \bar{p} - \mu_\alpha^{-1} \mu p + \mu^{-1} \zeta_\alpha (u^2 - \mu_\alpha)]$. Hence, triangles $ABC$ and $R_\alpha R_\beta R_\gamma$ are homologous if and only if $\mu_2^{-1} N \mathbb{P} \delta_\alpha = 0$.

Our first theorem is similar to the Th. 3.1. We just replace a point $P$ with its isogonal conjugate $Q$ with respect to $ABC$. Let $W_1$ denote the complement of the circumcircle of $ABC$ in the plane.

**Theorem 3.3.** The locus of all points $P$ in $W_1$ such that $ABC$ is homologous to the triangle on the circumcentres of $BCQ$, $CAQ$, and $ABQ$ is the intersection with $W_1$ of the Neuberg cubic of $ABC$.

**Proof.** The affix of the isogonal conjugate $Q$ is $(p + \tau \bar{p} - \mu \bar{p}^2 - \sigma)/M$ and the circumcentre $S_\alpha$ of $BCQ$ is $n_\alpha/M$. The circumcentres $S_b$ and $S_c$ of $CAQ$ and $ABQ$ have analogous affixes. The triangles $ABC$ and $S_a S_b S_c$ are homologous if and only if $M^{-2} N \mathbb{P} \delta_\alpha/u^2 = 0$. \(\diamondsuit\)

In the next theorems, we shall replace circumcentres of triangles $BCQ$, $CAQ$, and $ABQ$ with the circumcentres of $B_r C_r Q$, $C_r A_r Q$, and $A_r B_r Q$, where $A_r$, $B_r$, and $C_r$ are vertices of the three images triangle of $ABC$.

The locus of all points $P$ whose isogonal conjugates lie on the sideline $B_r C_r$ of $A_r B_r C_r$ is a conic $\Delta_a$. Let $W_3$ denote the complement in the plane of the union of the circumcircle of $ABC$, of the conic $\Delta_a$, and of two other related conics $\Delta_b$ and $\Delta_c$.

**Theorem 3.4.** The locus of all points $P$ in $W_3$ such that $ABC$ is homologous to the triangle on the circumcentres of the triangles $B_r C_r Q$, $C_r A_r Q$, and $A_r B_r Q$ is the intersection with $W_3$ of the union of the sidelines and the Neuberg cubic of $ABC$ and a quartic which goes through the vertices of $ABC$.

**Proof.** From the proof of previous theorem we know the affix of $Q$ and since the affixes of $B_r$ and $C_r$ are $\tau_b/u$ and $\tau_c/w$, we can find the affix of the circumcentre $S_a$ of $B_r C_r Q$. The circumcentres $S_b$ and $S_c$ of $C_r A_r Q$ and $A_r B_r Q$ have analogous affixes. The triangles $ABC$ and $S_a S_b S_c$ are homologous if and only if $K M^{-2} N \mathbb{P} \delta_a n_a/(u L_a) = 0$, where $K$ and $L_a$ denote expressions $2 \tau_2 \bar{p}^3 + \mu^2 \sigma_2 \bar{p}^2 + (4 \tau^3 - \sigma^2 \tau^2 + +4 \mu \sigma^3 - 15 \mu \sigma \tau + 12 \mu^2) \bar{p}^2 \bar{p}^2 + (4 \mu - \sigma \tau)(p^3 + \mu \bar{p}^3) + (8 \mu \tau + 2 \mu \sigma^2 - -3 \sigma \tau^2) \bar{p}^2 \bar{p} + \mu (8 \mu \sigma + 2 \tau^2 - 3 \sigma^2 \tau) p \bar{p}^2 + (\sigma^2 \tau - \tau^2 - 2 \mu \sigma)(2 p^2 - -\sigma p) + (\sigma \tau^2 - \mu \sigma^2 - 2 \mu \tau)(2 \mu \bar{p}^2 - \tau \bar{p}) + 3 (\sigma^2 \tau^2 - 2 \mu \sigma^3 - 2 \tau^3 + 7 \mu \sigma \tau -$
\[-12 \mu^2) p \bar{p} + 2 (\mu \sigma^3 - 6 \mu \sigma \tau + 8 \mu^2) \text{ and } \tau_a p^2 + (\zeta_2 a u^2 - \zeta_a (\zeta_2 a - \mu_a) u + + \mu_a \zeta_2 a) p \bar{p} - u(\zeta_2 a + u \zeta_a) p + \mu_a \sigma_a \bar{p}^2 - \mu_a (\mu_a \zeta_2 a + u \zeta_a) \bar{p} + (u^2 + + \mu_c) \text{. Notice that } K = 0 \text{ is the equation of a quartic which goes through the vertices of } ABC \text{ while } L_a = 0 \text{ is the equation of the conic } \Delta_a. \Diamond

Instead of the homology with } ABC, \text{ the following result looks at the homology of } O_O O_\gamma \text{ with the triangle on circumcentres of } B_r C_r P, C_r A_r P, \text{ and } A_r B_r P. \text{ Let } W_{r0} \text{ denote the complement of the union of sidelines of triangles } ABC \text{ and } A_r B_r C_r \text{ in the plane. Theorem 3.5. The locus of all points } P \text{ in } W_{r0} \text{ such that the triangle } O_O O_\gamma \text{ on circumcentres of } BCP, CAP, \text{ and } ABP \text{ is homologous to the triangle on the circumcentres of the triangles } B_r C_r P, C_r A_r P, \text{ and } A_r B_r P \text{ is the intersection with } W_{r0} \text{ of the union of the circumcircle of } A_r B_r C_r, \text{ the circumcircle of } ABC, \text{ and the Neuberg cubic of } ABC. \text{ Proof. Since the affixes of } A_r, B_r, \text{ and } C_r \text{ are } \tau_a / u, \tau_b / v, \text{ and } \tau_c / w, \text{ the circumcentre } O^*_r \text{ of the triangle } B_r C_r P \text{ is } (\mu \tau_a p \bar{p} + 2 \mu_a (\mu_a - u^2) p - - \sigma_a \tau_b \tau_c ) B_r C_r (P)^{-1}, \text{ where } B_r C_r (P) \text{ denotes the value at } P \text{ of the equation of the line } B_r C_r. \text{ Now we can determine the line } O_O O^*_r \text{, find the lines } O_\beta O^*_r \text{ and } O_\gamma O^*_r \text{ using the usual substitutions, and discover that these lines are concurrent if and only if}

\[k(A_r, B_r, C_r)(P) MN P^{-1} \delta_a B_r C_r (P)^{-1} BC(P)^{-1} = 0,\]

where } k(A_r, B_r, C_r)(P) \text{ is the value at } P \text{ of the equation of the circumcircle of } A_r B_r C_r. \Diamond

Remark 1. We can get analogous results to the above two theorems by replacing } A_r B_r C_r \text{ with either } A_u B_v C_u \text{ or } A_v B_u C_v. \text{.

4. Homology of triangles — antipedal triangles

The common feature of results in this section is that the homology of the antipedal triangle of a variable point } P \text{ with triangles on circumcentres is used to recognise the Neuberg cubic. Theorem 4.1. The locus of all points } P \text{ in } W_0 \text{ such that the antipedal triangle } P^\alpha P^\beta P^\gamma \text{ of } P \text{ with respect to } ABC \text{ is homologous to the triangle on the circumcentres of } BCH_\alpha, CAH_\beta, \text{ and } ABH_\gamma, \text{ where } H_\alpha, H_\beta, \text{ and } H_\gamma \text{ are orthocentres of } BCP, CAP, \text{ and } ABP, \text{ is the intersection with } W_0 \text{ of the union of the circumcircle and the Neuberg cubic of } ABC. \text{ Proof. The point } H_\alpha \text{ has affix } (p^2 - \mu_a p \bar{p} + \mu_a \zeta_a \bar{p} - \zeta_2 a) / n_a \text{ so that the affix of the circumcentre } S_\alpha \text{ of } BCH_\alpha \text{ is } (\zeta_a p + \mu_a \zeta_a \bar{p} - \mu_a p \bar{p} - \zeta_2 a - - \mu_a) / n_a. \text{ The lines } P^\alpha S_b \text{ and } P^\alpha S_c \text{ are relatives of } P^\alpha S_a. \text{ These three lines are concurrent if and only if } 2 MN P \delta_a / (u n_a) = 0. \Diamond
Remark 2. Let $R_\alpha$ denote the reflection of a point $P$ at the sideline $BC$. The triangles $BCH_\alpha$ and $BCR_\alpha$ have the same circumcentre so that in the above theorem the orthocentres $H_\alpha$, $H_\beta$, and $H_\gamma$ could be replaced with the reflections $R_\alpha$, $R_\beta$, and $R_\gamma$ of $P$ at the sidelines of $ABC$.

Let $W_2$ be the complement in the plane of the union of the three sidelines of $ABC$ and the three circles with sides of $ABC$ as diameters.

**Theorem 4.2.** The locus of all points $P$ in $W_2$ such that the antipedal triangle $P^\alpha P^\beta P^\gamma$ of $P$ with respect to $ABC$ is homologous to the triangle on the circumcentres of $BCO_\alpha$, $CAO_\beta$, and $ABO_\gamma$, where $O_\alpha$, $O_\beta$, and $O_\gamma$ are circumcentres of $BCP$, $CAP$, and $ABP$, is the intersection with $W_2$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

**Proof.** The point $O_\alpha$ has affix $\mu_\alpha M/n_\alpha$ so that the affix of the circumcentre $S_\alpha$ of $BCO_\alpha$ is $\mu_\alpha (\mu_\alpha (p^2 \bar{p}^2 - \mu_\alpha \bar{p}^2 - 1) - p^2 + \zeta_2 - 2U)/(n_\alpha U)$, where $U = 2 \mu_\alpha p \bar{p} - \zeta_2 p - \mu_\alpha \zeta_2 \bar{p} + \zeta_2$ is the equation of the circle with $BC$ as diameter. The lines $P^\beta S_\beta$ and $P^\gamma S_\gamma$ are relatives of $P^\alpha S_\alpha$. These three lines concur if and only if $2 M N \bar{p} \delta_\alpha (p - u)(u \bar{p} - 1)/(U n_\alpha) = 0$.

5. Homology of triangles — orthocentres

Here we obtain the Neuberg cubic in homologies with triangles on the orthocentres of variable triangles. The last result also uses the centres of the nine-point circles.

**Theorem 5.1.** The locus of all points $P$ in $W_0$ such that $ABC$ is homologous to the triangle on the orthocentres of the triangles $OO_\beta O_\gamma$, $OO_\gamma O_\alpha$, and $OO_\alpha O_\beta$ is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

**Proof.** The orthocentre $H_\alpha$ of $OO_\beta O_\gamma$ has affix $u \zeta_\alpha M/(p - u)/(n_\beta n_\gamma)$. Of course, the other two orthocentres $H_b$ and $H_c$ have analogous affixes. Hence, the triangles $ABC$ and $H_a H_b H_c$ are homologous if and only if $M N \bar{p} \delta_\alpha/(u n_\alpha) = 0$.

**Remark 3.** We get an analogous result to the above theorem by replacing $ABC$ with the triangle $G_\alpha G_\beta G_\gamma$, on centroids of $BCP$, $CAP$, and $ABP$.

**Theorem 5.2.** The locus of all points $P$ in $W_0$ such that $ABC$ is homologous to the triangle on the orthocentres of the triangles $GG_\beta G_\gamma$, $GG_\gamma G_\alpha$, and $GG_\alpha G_\beta$ is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

**Proof.** The orthocentre $H_\alpha$ of $GG_\beta G_\gamma$ has affix $(2 \mu_\alpha p \bar{p} + u p + \mu \bar{p} - \tau - -\mu_\alpha)/(3 n_\alpha)$. The other two orthocentres $H_b$ and $H_c$ are relatives of $H_a$. It follows that the triangles $ABC$ and $H_a H_b H_c$ are homologous if and only if $\frac{8}{27} M N \bar{p} \delta_\alpha/(u n_\alpha) = 0$.
Remark 4. We get a similar result to the above theorem by replacing ABC with the triangle $O_\alpha O_\beta O_\gamma$ on circumcentres of BCP, CAP, and ABP.

Theorem 5.3. The Neuberg cubic of the triangle ABC is the locus of all points $P$ such that ABC is homologous to the triangle on either the centres of the nine-point circles or the orthocentres of $R_\beta R_\gamma P$, $R_\gamma R_\alpha P$, and $R_\alpha R_\beta P$, where $R_\alpha$, $R_\beta$, and $R_\gamma$ are reflections of $P$ at the sidelines $BC$, $CA$, and $AB$.

Proof. The orthocentre $H_a$ of $R_\beta R_\gamma P$ is $p + (\tau - \mu_a)\bar{p} + \zeta_a$. The orthocentres $H_b$ and $H_c$ of $R_\gamma R_\alpha P$, and $R_\alpha R_\beta P$ are relatives of $H_a$. The triangles ABC and $H_a H_b H_c$ are homologous if and only if $\mu_2^{-1} N \mathbb{P} \delta_a = 0$.

For the second part observe that the centre of nine-point circle of $R_\beta R_\gamma P$ is collinear with the points A and $H_a$. ◊

6. Homology of triangles — symmedian and isogonic points

The second theorem in this section is analogous to the following theorem which is an exercise on page 200 of [16]. It was restated as the Superior Locus Problem by J. Tabov in [24] and it was resolved by the author in [3] (see also [4]).

Theorem 6.1. The locus of all points $P$ in $W_0$ such that the Euler lines of the triangles ABP, CAP, and BCP are concurrent (at the point on the Euler line of ABC) is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of ABC.

Proof. We know the circumcentre $O_\alpha$ of the triangle BCP and since its centroid $G_\alpha$ is $(p + \zeta_a)/3$ it follows that the Euler line $G_\alpha O_\alpha$ of this triangle is

$$ [n_a^{-1} (p^2 - 2 \mu_a p \bar{p} + \mu_a \zeta_a \bar{p} + \mu_a - \zeta_{2a}), n_a^{-1} M (p - \mu_a \bar{p})]. $$

Hence, the Euler lines $G_\alpha O_\alpha$, $G_\beta O_\beta$, and $G_\gamma O_\gamma$ concur if and only if

$$ M N \mu^{-1} \mathbb{P} \delta_a n_a^{-1} = 0. $$

Notice that these lines intersect on the Euler line GO of ABC. ◊

Recall that the Brocard diameter or the Brocard axis are the names for the central line joining the circumcentre with the symmedian point (or the Grebe-Lemoine point) of a triangle.

Theorem 6.2. The locus of all points $P$ in $W_0$ such that the Brocard diameters of the triangles ABP, CAP, and BCP are concurrent (at the point on the Brocard axis of ABC) is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of ABC.
Proof. The affix of $K_a$ is $(\mu_a \zeta_a p \bar{p} + 2 (\mu_a - \zeta_{2a}) p - 2 \mu_{2a} \bar{p} + \mu_a \zeta_a)/U_a$, where the complex number $U_a$ is $((2 p - \zeta_a)(2 \mu_a \bar{p} - \zeta_a) - 3 \delta_a^2)/2$ and thus is never zero. The affix of $O_a$ is $\mu_a M/n_a$, so that the triangles $K_a K_\beta K_\gamma$ and $O_a O_\beta O_\gamma$ are homologous if and only if

$$8 M N \mathbb{P} \delta_a (p - u)(u \bar{p} - 1)/(n_a U_a) = 0.$$ Notice that the lines $O_a K_a$, $O_\beta K_\beta$, and $O_\gamma K_\gamma$ intersect on the Brocard axis $K O$ of $ABC$. \hfill \diamondsuit

The second theorem in this section is similar to the Th. 3.1. In it we replace the circumcentres with isogonic points. Let $W_5$ be the complement in the plane of the apexes $A_u$, $B_u$, and $C_u$ of equilateral triangles built towards inside on the sides of $ABC$. Theorem 6.3. The locus of all points $P$ in $W_5$ such that $ABC$ is homologous to the triangle $I_{u\alpha} I_{u\beta} I_{u\gamma}$ on the second isogonic points of $BCP$, $CAP$, and $ABP$ is the intersection with $W_5$ of the union of the equilateral hyperbola through $A_u$, $B_u$, and $C_u$ with the centre at the first isogonic point $I_v$ of $ABC$ and the Newbery cubic of $ABC$.

Proof. The point $I_{u\alpha}$ is $(U \eta + V)/(X \eta + Y)$, where $\eta$ denotes $-\frac{1}{2} + i \frac{\sqrt{3}}{2}$ (the cube root of unity), the letter $U$ is an abbreviation for $\mu_a \zeta_a p \bar{p} + 2 (\mu_a - \zeta_{2a}) p - 2 \mu_{2a} \bar{p} + \mu_a \zeta_a$, the letter $V$ for

$$\nu \mu_a p \bar{p} - \delta_a p^2 + (\mu_a - \zeta_{2a}) p - \mu_a (\mu_a - \zeta_{2a}) \bar{p} + \nu \mu_a + \delta_{3a},$$ and the letters $X$ and $Y$ for $2 \mu_a p \bar{p} - \zeta_a p - 2 \mu_a \zeta_a \bar{p} + 4 \mu_a - \zeta_{2a}$ and $\mu_a p \bar{p} - (\delta_a + \nu) p - \mu_a (\delta_a + \nu) \bar{p} + \nu^2 + 2 \nu \delta_a$. The other two second isogonic points $I_{u\beta}$ and $I_{u\gamma}$ are relatives of $I_{u\alpha}$. It follows that the triangles $ABC$ and $I_{u\alpha} I_{u\beta} I_{u\gamma}$ are homologous if and only if

$$H_u N \mathbb{P} \delta_a/(u^2 (X \eta + Y)) = 0,$$ where $H_u = 0$ is the equation (in $p$) of the hyperbola from the statement of the theorem. In order to see that $X \eta + Y = 0$ only when $p = m$, where $m$ is the affix of $A_u$, it suffices to note that the value of $X \eta + Y$ at $m + n$ is equal $(1 + 2 \eta) n \bar{n} \nu \bar{\nu}$. \hfill \diamondsuit

Remark 5. Of course, there is a dual result to the above theorem with first isogonic points of $BCP$, $CAP$, and $ABP$. The hyperbola of the locus has its centre at the second isogonic point of $ABC$.

7. Homology of triangles — reflections

In this section we use homology with triangles whose vertices are reflections in appropriate lines. Let $W_4$ be the complement in the plane of the vertices $A$, $B$, and $C$ of the triangle $ABC$. 
Theorem 7.1. The locus of all points P in W₄ such that ABC is homologous to the triangle on reflections in sidelines of ABC of inversions of A, B, and C with respect to the circles k(B, C, P), k(C, A, P), and k(A, B, P) is the intersection with W₄ of the union of the sidelines, the circumcircle, and the Neuberg cubic of ABC.

Proof. The inversion of A with respect to the circle k(B, C, P) is \((\tau_a M - u n_a)/(u M - n_a)\) and its reflection \(T_a\) in BC is
\[(\mu M - \tau_a n_a)/(\mu_a M - u n_a).\]
The other two reflections \(T_b\) and \(T_c\) are relatives of \(T_a\). The triangles ABC and \(T_aT_bT_c\) are homologous if and only if
\[MN \in \delta^3 a \cap (u^2 (u M - n_a)(\mu_a M - u n_a)) = 0.\]
From this our theorem follows immediately provided one observes that up to a constant \(\mu_a M - u n_a\) is a complex conjugate of \(u M - n_a\) and both are zero only at the affixes of B and C.

Theorem 7.2. The locus of all points P in W₄ such that ABC is homologous to the triangle on reflections in sidelines of the extriangle \(A_bB_cC_e\) of inversions of A, B, and C with respect to the circles k(B, C, P), k(C, A, P), and k(A, B, P) is the intersection with W₄ of the union of the circumcircle and the Neuberg cubic of ABC.

Proof. In this proof, in order to avoid the appearance of square roots, we shall assume that the vertices A, B, and C have affixes \(u^2\), \(v^2\), and \(w^2\) for some unimodular numbers \(u\), \(v\), and \(w\). The reflection \(T_a\) in \(B_cC_e\) of the inversion of A with respect to the circle k(B, C, P) is
\[(U M - u^4 c_2(n_a))/(c_2(\mu_a M - u n_a)),\]
where \(U = \mu_a (u^4 + (\mu_a - \zeta_{2a}) u^2 + \mu_{2a})\) and \(c_2\) performs the substitution \(u \rightarrow u^2\), \(v \rightarrow v^2\), \(w \rightarrow w^2\). The other two reflections \(T_b\) and \(T_c\) are relatives of \(T_a\). The triangles ABC and \(T_aT_bT_c\) are homologous if and only if \(M^4 c_2(\delta^2 a)/(u^2 c_2(u M - n_a) c_2(\mu_a M - u n_a)) = 0.\)

Theorem 7.3. If ABC has no right angle, then the locus of all points P in W₄ such that the tangential triangle \(A_tB_tC_t\) is homologous to the triangle on reflections in BC, CA, and AB of the second intersections of lines AP, BP, and CP with the circumcircle of ABC is the intersection with W₄ of the union of the sidelines and the Neuberg cubic of ABC.

Proof. Since the affix of \(A_t\) is 2 \(\mu_a/\zeta_a\), the affix of the second intersection \(S_a\) of AP with the circumcircle of ABC is \((u - p)/(u \bar{p} - 1)\), and the affix of the reflection \(T_a\) of \(S_a\) in BC is \((\zeta_a p + \mu \bar{p} - \tau)/(p - u)\), the triangles \(A_tB_tC_t\) and \(T_aT_bT_c\) are homologous if and only if
\[2N \in \delta a \cap (u \zeta_a (p - u)(u \bar{p} - 1)) = 0.\]
Theorem 7.4. The locus of all points $P$ in $W_4$ such that the pedal triangle $P_\alpha P_\beta P_\gamma$ of $P$ with respect to $ABC$ is homologous to the triangle on reflections in $P_\beta P_\gamma$, $P_\gamma P_\alpha$, and $P_\alpha P_\beta$ of $P$ is the intersection with $W_4$ of the union of the sidelines, the circumcircle, and the Neuberg cubic of $ABC$.

Proof. Since the affix of $P_\alpha$ is $(p - \mu_\alpha \bar{p} + \zeta_\alpha)/2$ and the affix of the reflection $T_\alpha$ of $P$ in $P_\beta P_\gamma$ is

$$(p^2 - u \zeta_\alpha p \bar{p} + \zeta_\alpha p - \mu u \bar{p}^2 + u (\tau + \mu_\alpha) \bar{p} - \zeta_\alpha \zeta_\gamma)/(2(p - u)),$$

the triangles $P_\alpha P_\beta P_\gamma$ and $T_\alpha T_\beta T_\gamma$ are homologous if and only if

$$\frac{1}{16} M^2 N \mathbb{P} \delta_a n_a/(u^2 (p - u)(u \bar{p} - 1)) = 0.\diamondsuit$$

Remark 6. Since the triangle $R_\alpha R_\beta R_\gamma$ on reflections of a point $P$ in sides of $ABC$ is homothetic to the pedal triangle $P_\alpha P_\beta P_\gamma$ from $P$, the above theorem holds also for $R_\alpha R_\beta R_\gamma$ in place of $P_\alpha P_\beta P_\gamma$.

Theorem 7.5. The locus of all points $P$ in $W_0$ such that the antipedal triangle $P^\alpha P^\beta P^\gamma$ of $P$ with respect to $ABC$ is homologous to the triangle on reflections in $P^\beta P^\gamma$, $P^\gamma P^\alpha$, and $P^\alpha P^\beta$ of $P$ is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

Proof. The affix of $P^\alpha$ is $(\mu_\alpha p \bar{p} - p^2 + \zeta_\alpha p - 2 \mu_\alpha)/n_a$ and the affix of the reflection $T_\alpha$ of $P$ in $P^\beta P^\gamma$ is $2(u - \bar{p})$, so that the triangles $P^\alpha P^\beta P^\gamma$ and $T_\alpha T_\beta T_\gamma$ are homologous if and only if $16 M N \mathbb{P} \delta_a/(u n_a) = 0.\diamondsuit$

8. Homology of triangles — isogonal conjugacy

Here we encounter the Neuberg cubic in homologies with triangles whose vertices are isogonal conjugates of various points with respect to appropriate variable triangles.

Theorem 8.1. The union of the circumcircle and the Neuberg cubic of the triangle $ABC$ is the locus of all points $P$ such that the pedal triangle $P_\alpha P_\beta P_\gamma$ of the point $P$ with respect to $ABC$ is homologous to the triangle on isogonal conjugates of $P_\alpha$, $P_\beta$, and $P_\gamma$ with respect to triangles $P P_\beta P_\gamma$, $P P_\gamma P_\alpha$, and $P P_\alpha P_\beta$.

Proof. The vertex $P_\alpha$ has affix $(p - \mu_\alpha \bar{p} + \zeta_\alpha)/2$ while the isogonal conjugate $T_\alpha$ of $P_\alpha$ with respect to the triangle $P_\beta P_\gamma$ has affix $(2p - u M)/2$. Hence, the triangles $P_\alpha P_\beta P_\gamma$ and $T_\alpha T_\beta T_\gamma$ are homologous if and only if $\frac{1}{16} M^2 N \mathbb{P} \delta_a/u^2 = 0.\diamondsuit$

Remark 7. The above theorem holds also for the triangle on reflections of a point $P$ in sides of $ABC$ instead of the pedal triangle $P_\alpha P_\beta P_\gamma$. 
Theorem 8.2. The locus of all points $P$ in $W_0$ such that the triangle $O_\alpha O_\beta O_\gamma$ on the circumcentres of triangles $BCP$, $CAP$ and $ABP$ is homologous to the triangle on the isogonal conjugates of $O_\alpha$, $O_\beta$, and $O_\gamma$ with respect to triangles $PO_\beta O_\gamma$, $PO_\gamma O_\alpha$ and $PO_\alpha O_\beta$ is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

**Proof.** The vertex $O_\alpha$ has affix $\mu_\alpha M/n_\alpha$ while the isogonal conjugate $T_\alpha$ of $O_\alpha$ with respect to the triangle $PO_\beta O_\gamma$ has affix $u M/(u \bar{p} - 1)$. It follows that the triangles $O_\alpha O_\beta O_\gamma$ and $T_\alpha T_\beta T_\gamma$ are homologous if and only if $M^4 \langle N \mathbb{P} \delta_a/n_a (p - u)(u \bar{p} - 1) \rangle = 0.\diamond$

Theorem 8.3. The locus of all points $P$ in $W_4$ such that the triangle $S_\alpha S_\beta S_\gamma$ on the second intersections of lines $AP$, $BP$, and $CP$ with the circumcircle of $ABC$ is homologous to the triangle on the isogonal conjugates of $S_\alpha$, $S_\beta$, and $S_\gamma$ with respect to triangles $PS_\beta S_\gamma$, $PS_\gamma S_\alpha$, and $PS_\alpha S_\beta$ is the intersection with $W_4$ of the union of the sidelines, the circumcircle, and the Neuberg cubic of $ABC$.

**Proof.** The complex number $(u - p)/(u \bar{p} - 1)$ is the affix of the vertex $S_\alpha$. On the other hand, $(\zeta_a p^2 \bar{p} + \mu_\alpha p \bar{p}^2 - p^2 - (\tau + \mu_\alpha) p \bar{p} + u p + \mu_\alpha)/(u - p)(u \bar{p} - 1)(u \bar{p} - 1)$ is the affix of the isogonal conjugate $T_\alpha$ of $S_\alpha$ with respect to the triangle $PS_\beta S_\gamma$. Hence, the triangles $S_\alpha S_\beta S_\gamma$ and $T_\alpha T_\beta T_\gamma$ are homologous if and only if $M^6 \langle N \mathbb{P} \delta_a n_a/((p - u)^3 (u \bar{p} - 1)^3) \rangle = 0.\diamond$

Theorem 8.4. The locus of all points $P$ in $W_1$ such that the triangle $H_\alpha H_\beta H_\gamma$ on the orthocentres of triangles $BCP$, $CAP$, and $ABP$ is homologous to the triangle on isogonal conjugates of $A$, $B$, and $C$ with respect to those triangles is the intersection with $W_1$ of the union of the sidelines and the Neuberg cubic of $ABC$.

**Proof.** Since $\text{ig}(A, BCP)$ is $(p^2 + u \zeta_a p \bar{p} - \sigma p - \mu_\alpha \bar{p} + \mu_\alpha)/((u M)$ and the vertex $H_\alpha$ has the affix $(p^2 - \mu_\alpha p \bar{p} + \mu_\alpha \zeta_a \bar{p} - \zeta_2)/n_a$ it is easy to check that the triangles $H_\alpha H_\beta H_\gamma$ and

$$\text{ig}(A, BCP) \text{ig}(B, CAP) \text{ig}(C, ABP)$$

are homologous iff $M^{-3} \langle N \mathbb{P} \delta_a n_a u^{-3} \rangle = 0.\diamond$

9. Concurrent parallels

Results in this section use the condition that three lines are concurrent. However, these lines are not lines joining corresponding vertices of two triangles as in previous sections but are parallels to lines.

Theorem 9.1. The Neuberg cubic of the triangle $ABC$ is the locus of all points $P$ such that the parallels through $A$, $B$, and $C$ to the Euler lines of triangles $PP_\beta P_\gamma$, $PP_\gamma P_\alpha$, and $PP_\alpha P_\beta$ formed by $P$ and the vertices of its pedal triangle are concurrent.
Proof. The parallel $pa(A, G_aO_a)$ with the Euler line $G_aO_a$ of $PP_\beta P_\gamma$ through the vertex $A$ is the line $[(u\zeta_\alpha \bar{p} - p - \sigma_a)/2, (\mu\sigma\bar{p} - \tau p + \zeta_a (u^2 - -\mu_a))/(2\mu)]$. The other two parallels $pa(B, G_bO_b)$ and $pa(C, G_cO_c)$ are relatives of $pa(A, G_aO_a)$. The condition for these lines to concur is $\frac{1}{8}N \cdot \delta_a u^2 = 0$.

Remark 8. The above theorem is true also when parallels to the Euler lines of $PP_\beta P_\gamma, PP_\gamma P_\alpha$, and $PP_\alpha P_\beta$ are drawn through vertices of either the pedal triangle of $P$ with respect to $ABC$ or the triangle on reflections of $P$ in sidelines of $ABC$.

Theorem 9.2. The locus of all points $P$ in $W_0$ such that the parallels through the vertices $P^\alpha, P^\beta,$ and $P^\gamma$ of its antipedal triangle to the Euler lines of triangles $PP^\beta P^\gamma, PP^\gamma P^\alpha,$ and $PP^\alpha P^\beta$ are concurrent is the intersection with $W_0$ of the union of the circumcircle and the Neuberg cubic of $ABC$.

Proof. As in the proof of the previous theorem, we first find the parallel $pa(P^\alpha, G_aO_a)$ with the Euler line $G_aO_a$ of $PP^\beta P^\gamma$ through the vertex $P^\alpha$. This line has a rather complicated polynomial of order five in $p$ and $\bar{p}$ as the second term. Of course, the other two parallels $pa(B, G_bO_b)$ and $pa(C, G_cO_c)$ are relatives of $pa(A, G_aO_a)$. These lines concur if and only if $48M^2N \cdot \delta_a (p - u)(u\bar{p} - 1)/n_a = 0$.

Remark 9. The above theorem remains true when parallels to the Euler lines of $PP^\beta P^\gamma, PP^\gamma P^\alpha,$ and $PP^\alpha P^\beta$ are drawn through vertices of the triangle on the second intersections of lines $AP, BP,$ and $CP$ with the circumcircle of $ABC$.

10. Characterisations with power

Neuberg [18] noticed the following theorem which requires the notion of the power of a point with respect to a circle that we recall now.

Let $P$ be a point and $k$ be a circle in the plane with the centre $S$ and the radius $r$. Then the power $\omega(P, k)$ of the point $P$ with respect to the circle $k$ is the number $|PS|^2 - r^2$. For points $X$ and $Y$ in the plane, let $k(X, Y)$ denote the circle with the centre at $X$ which passes through $Y$.

Theorem 10.1. The Neuberg cubic of $ABC$ is the locus of all points $P$ in the plane such that the product of powers of the point $P$ with respect to the circles $k(A, B), k(B, C),$ and $k(C, A)$ is equal to the product of powers of the point $P$ with respect to the circles $k(A, C), k(B, A),$ and $k(C, B)$.
Proof. Let $W = p\, \overline{v} - u^{-1} p - u\, \overline{v}$. Since $W + \mu_{\alpha}^{-1}(\zeta_{2\alpha} - \mu_{\alpha})$ and $W + \mu_{\beta}^{-1}(\zeta_{2\beta} - \mu_{\beta})$ are the powers $\omega(p, k(A, B))$ and $\omega(p, k(A, C))$, the difference $\mathbb{P}\, \omega(p, k(A, B)) - \mathbb{P}\, \omega(p, k(A, C))$ is equal to $\mu_{\alpha}^{-1} N\, \mathbb{P}\, \delta_{\alpha}$.

The above result uses circles determined by two points (the centre and a point on it). Much more interesting is to consider powers with respect to circles which are given by three points.

For a point $P$ and triangles $UVW$ and $XYZ$, let
$$\mathbb{P}\, \omega(U, k(P, Z, X)) - \mathbb{P}\, \omega(U, k(P, X, Y))$$
be $\nu(P, UVW, XYZ)$.

Theorem 10.2. The locus of all points $P$ in $W_{0}$ such that
$$\nu(P, ABC, O_{\alpha}O_{\beta}O_{\gamma}) = 0$$
is the intersection with $W_{0}$ of the union of the circumcircle and the Neuberg cubic of $ABC$, where $O_{\alpha}, O_{\beta},$ and $O_{\gamma}$ are circumcentres of triangles $BCP, CAP,$ and $ABP$.

Proof. Since the quotient $((\tau - \mu_{\beta})p\, \overline{v} - \mu_{\alpha}\zeta_{2\alpha}p - \zeta_{2\alpha}p + \mu_{\beta} + v^{2})(p - u)(u\, \overline{v} - 1)/(u\, n_{a}\, n_{c})$ is the power $\omega(A, k(P, O_{\gamma}, O_{\alpha}))$ and the power $\omega(A, k(P, O_{\alpha}, O_{\beta}))$ is the analogous quotient $((\tau - \mu_{\alpha})p\, \overline{v} - \mu_{\alpha}\zeta_{2\alpha}p - \zeta_{2\alpha}p + \mu_{\alpha} + w^{2})(p - u)(u\, \overline{v} - 1)/(u\, n_{a}\, n_{b})$ and all the remaining four powers which appear in $\nu(P, ABC, O_{\alpha}O_{\beta}O_{\gamma})$ are their relatives,
$$\nu(P, ABC, O_{\alpha}O_{\beta}O_{\gamma}) = 0$$
is true if and only if $MN\, \mathbb{P}\, \delta_{\alpha}(p - u)(u\, \overline{v} - 1)/(u\, n_{a}^{2}) = 0$.

Remark 10. The above theorem remains true when
$$\nu(P, ABC, O_{\alpha}O_{\beta}O_{\gamma}) = 0$$
is replaced by the equation $\nu(O, ABC, O_{\alpha}O_{\beta}O_{\gamma}) = 0$, where $O$ is the circumcentre of $ABC$.

Theorem 10.3. The Neuberg cubic of $ABC$ is the locus of all points $P$ in the plane such that $\nu(P, ABC, R_{\alpha}R_{\beta}R_{\gamma}) = 0$, where $R_{\alpha}, R_{\beta},$ and $R_{\gamma}$ are reflections of $P$ in the sidelines $BC, CA,$ and $AB$, respectively.

Proof. Since
$$(u\, p + v\, \mu_{\alpha}\, \overline{v} - \mu_{\alpha}p\, \overline{v} + \mu_{\alpha} - \zeta_{2\alpha})/\mu_{\alpha}, (u\, p + w\, \mu_{\beta}\, \overline{v} - \mu_{\beta}p\, \overline{v} + \mu_{\beta} - \zeta_{2\beta})/\mu_{\beta}$$
are powers $\omega(A, k(P, R_{\gamma}, R_{\alpha}))$ and $\omega(A, k(P, R_{\alpha}, R_{\beta}))$ and the other four powers which appear in $\nu(P, ABC, R_{\alpha}R_{\beta}R_{\gamma})$ are their relatives, $\nu(P, ABC, R_{\alpha}R_{\beta}R_{\gamma}) = 0$ is true if and only if $N\, \mathbb{P}\, \delta_{\alpha}u^{-2} = 0$.

Let $W_{6}$ be the complement of the union of the circumcircles of triangles $BCO, CAO,$ and $ABO$ in the plane. If $P$ is a point different from the circumcentre $O$ of $ABC$, let $R$ denote the inversion of $P$ with respect to the circumcircle of $ABC$. 
Theorem 10.4. The intersection of the Neuberg cubic of $ABC$ with $W_6$ is the locus of all points $P$ in $W_6$ such that $\nu(R, \ ABC, R_\alpha R_\beta R_\gamma) = 0$.

Proof. Since $(u p + v \mu_a \bar{p} - \mu_b - v^2)(p - u)(u \bar{p} - 1)/(\mu_c (\zeta_6 M - n_6))$ is $\omega(A, k(R, R_\gamma, R_\alpha))$ and $\omega(A, k(R, R_\alpha, R_\beta))$ is

$$(u p + w \mu_a \bar{p} - \mu_c - w^2)(p - u)(u \bar{p} - 1)/(\mu_b (\zeta_6 M - n_6))$$

and all the other four powers which appear in $\nu(R, \ ABC, R_\alpha R_\beta R_\gamma)$ are their relatives, it follows that $\nu(R, \ ABC, R_\alpha R_\beta R_\gamma) = 0$ is true if and only if $N \mathbb{P} \delta_a (p - u)(u \bar{p} - 1)/(u^2 (\zeta_6 M - n_6)) = 0$. From this our theorem follows immediately if we observe that $\zeta_6 M - n_6 = 0$ is the equation of the circle $k(B, C, O)$ (or the sideline $BC$ when the angle $A$ is right).

When the angle $A$ is right, let $k_a$ denote the sideline $BC$ of $ABC$.

Otherwise, we use $k_a$ for a circle which passes through the points $B$ and $C$ and which has the lines joining these points with the circumcentre of $ABC$ as tangents. The (lines) circles $k_b$ and $k_c$ are defined analogously. Let $W_7$ be the complement in $W_1$ of the union of $k_a$, $k_b$, and $k_c$. For a point $P$ outside the circumcircle of $ABC$, let $Q$ denote its isogonal conjugate with respect to $ABC$.

Theorem 10.5. The intersection of the union of the sidelines and the Neuberg cubic of $ABC$ with $W_7$ is the locus of all points $P$ in $W_7$ such that $\nu(Q, \ ABC, R_\alpha R_\beta R_\gamma) = 0$.

Proof. Since $((\tau - \mu_c) p \bar{p} - \zeta_a p - \mu_b \zeta_a \bar{p} + \mu_c + w^2)(p - u)(u \bar{p} - 1) n_a/\mu M (\zeta_6 M - 2 n_6)$ is the power $\omega(A, k(R, R_\gamma, R_\alpha))$ and the power $\omega(A, k(R, R_\alpha, R_\beta))$ is also the quotient $((\tau - \mu_b) p \bar{p} - \zeta_a p - \mu_c \zeta_a \bar{p} + +\mu_b + v^2)(p - u)(u \bar{p} - 1) n_a/\mu M (\zeta_6 M - 2 n_6)$ and the other powers which appear in $\nu(Q, \ ABC, R_\alpha R_\beta R_\gamma)$ are their relatives,

$\nu(Q, \ ABC, R_\alpha R_\beta R_\gamma) = 0$

if and only if $M^{-2} N \mathbb{P} \delta_a n_a (p - u)(u \bar{p} - 1)/(u^2 (\zeta_6 M - 2 n_6)) = 0$. From this our theorem follows provided one observes that $\zeta_6 M - 2 n_6 = 0$ is the equation of the circle (line) $k_a$.

Remark 11. Let

$\nu_0(P, U VW, XYZ) = \mathbb{P} \omega(P, k(V, W, Y)) - \mathbb{P} \omega(P, k(V, W, Z))$

for a point $P$ and triangles $U VW$ and $XYZ$. It is interesting that in all results in this section replacing the function $\nu$ with the above function $\nu_0$ the Neuberg cubic of $ABC$ will again appear. However, the exception sets are more complicated and the locus might include curves of higher order.

References