A RÉDEI TYPE FACTORIZATION RESULT FOR A SPECIAL 2-GROUP

Keresztély Corrádi

Department of Computer Sciences, Eötvös University Budapest, H-1088 Budapest, Hungary

Sándor Szabó

Department of General Sciences and Mathematics, College of Health Sciences, P.O.Box 12, Manama, State of Bahrain

Dedicated to Professor Ludwig Reich on his 60th birthday

Received: November 1999

MSC 2000: 20 K 01; 52 C 22

Keywords: Factorization of finite and infinite abelian groups, Hajós-Rédei theory.

Abstract: If a finite abelian 2-group is a direct product of two cyclic groups and also the group is a direct product of its subsets whose orders are either four or two, then at least one of these subsets must be periodic.

1. Introduction

Let \( G \) be a finite abelian group with identity element \( e \). If \( G \) is a direct product of cyclic subgroups of order \( t_1, \ldots, t_s \), then we express this fact saying that \( G \) is of type \( (t_1, \ldots, t_s) \). Let \( A_1, \ldots, A_n \) be subsets of \( G \). If each element \( g \) of \( G \) can be written uniquely in the form

\[
g = a_1 \cdots a_n, \quad a_1 \in A_1, \ldots, a_n \in A_n,
\]

then the product \( A_1, \ldots, A_n \) is direct and is equal to \( G \). We express this fact saying that \( G \) is factored into its subsets \( A_1, \ldots, A_n \) or simply

Research partly supported by National Science Foundation Grant No. T02925.
that $G = A_1 \ldots A_n$ is a factorization of $G$. If $e \in A_1, \ldots, e \in A_n$, then the factorization is called normed. The $n$ tuple $(|A_1|, \ldots, |A_n|)$ is called the type of the factorization. A subset $A$ of $G$ is defined to be periodic if there is an element $g$ of $G$ such that $Ag = A$ and $g \neq e$.

L. Rédei [3] proved that if a finite abelian group is factored into subsets with prime order, then at least one of the factors is periodic.

Rédei’s theorem suggests the following problem. Given a finite abelian group $G$ find the factorization types for which one of the factors must be periodic. In [2] this problem was considered for 2-groups. It was proved (Th. 3) that if $G$ is of type $(2^\lambda, 2^\mu)$ and the factorization is of type $(4, 2, \ldots, 2)$, then at least one of the factors must be periodic.

In this note we will prove the next extension of the above result. If $G$ is of type $(2^\lambda, 2^\mu)$ and the factorization is of type $(4, \ldots, 4, 2, \ldots, 2)$, then at least one of the factors is periodic. The proof heavily depends on a result of [1] about the size of an annihilator in a factorization of a $p$-group of type $(p^\lambda, p^\mu)$.

2. The result

If $A$ is a subset and $\chi$ is a character of a finite abelian group $G$, then $\chi(A)$ will denote the sum

$$\sum_{a \in A} \chi(a).$$

The set of all characters $\chi$ of $G$ satisfying $\chi(A) = 0$ is called the annihilator set of $A$ and will be denoted by $\text{Ann}(A)$.

**Theorem 1.** If $G = A_1 \ldots A_n$ is a normed factorization, where $G$ is a group of type $(2^\lambda, 2^\mu)$ and $|A_i| = 2$ or $|A_i| = 4$ for each $i$, $1 \leq i \leq n$, then at least one of the factors $A_1, \ldots, A_n$ is periodic.

**Proof.** Let $G$ be a group of type $(2^\lambda, 2^\mu)$ and consider a normed factorization $G = A_1 \ldots A_n$ of $G$ such that $|A_i| = 2$ or $|A_i| = 4$ for each $i$, $1 \leq i \leq n$. We may assume that $|A_1| = \cdots = |A_s| = 4, \quad |A_{s+1}| = \cdots = |A_n| = 2$ since this is only a matter of reindexing the factors. We proceed by induction on $s$.

If $s = 0$, then by Rédei’s theorem one of the factors is periodic. For the remaining part we assume that $s \geq 1$. Let $A_s = \{e, a, b, c\}$ and introduce the elements $d_a, d_b, d_c$ defined by the equations
$a = bcd_a, \quad b = acd_b, \quad c = abd_c.$

Note that if $d_c = e$, then $A_s = \{e, a, b, ab\} = \{e, a\} \{e, b\}$. Now $s$ decreases and by the inductive assumption one of the factors is periodic. Thus we may assume that $d_a \neq e$, $d_b \neq e$, $d_c \neq e$.

To the factor $AI_i$ of $G$ we assign the subgroup $K_i$ of $G$ defined by

$$K_i = \bigcap_{\chi \in \text{Ann}(A_i)} \text{Ker} \chi.$$

We claim that it may be assumed that $K_s \neq \{e\}$. In order to prove this claim we distinguish two cases depending on $A_s$ as whether it contains a second order element or not.

In the first case suppose that $a^2 = e$ and denote the subgroup $\langle d_b, d_c \rangle$ by $L$. Note that $L \neq \{e\}$. We will show that $L \subset \text{Ker} \chi$ for each $\chi \in \text{Ann}(A_s)$. Let us consider

$$(1) \quad 0 = \chi(A) = 1 + \chi(a) + \chi(b) + \chi(c).$$

If $\chi(a) = 1$, then from (1) it follows that $\chi(b) = \chi(c) = -1$. Hence

$$\chi(d_a) = \chi(b)\chi(a)^{-1}\chi(c)^{-1} = \chi(b)[\chi(a)]^{-1}[\chi(c)]^{-1} = 1,$$

$$\chi(d_c) = \chi(c)\chi(a)^{-1}\chi(b)^{-1} = \chi(c)[\chi(a)]^{-1}[\chi(b)]^{-1} = 1.$$

If $\chi(a) = -1$, then from (1) it follows that $\chi(b) = -\chi(c)$ and so

$$\chi(d_a) = \chi(b)\chi(a)^{-1}\chi(c)^{-1} = \chi(b)[\chi(a)]^{-1}[\chi(c)]^{-1} = 1,$$

$$\chi(d_c) = \chi(c)\chi(a)^{-1}\chi(b)^{-1} = \chi(c)[\chi(a)]^{-1}[\chi(b)]^{-1} = 1.$$

Let us turn to the second case when $A_s$ does not contain any element of order two. Since the group $G$ is of type $(2^\lambda, 2^\mu)$ it follows that the product of the subgroups $\langle d_a \rangle$, $\langle d_b \rangle$, $\langle d_c \rangle$ cannot be direct. Thus we may assume that after a suitable reordering of the elements $d_a, d_b, d_c$ the inequality $\langle d_a \rangle \cap \langle d_b, d_c \rangle \neq \{e\}$ holds. Let us denote the subgroup on the left hand side by $L$. Since $L \neq \{e\}$ it will be enough to show that $L \subset \text{Ker} \chi$ for each $\chi \in \text{Ann}(A_s)$. From (1) it follows that

$$(2) \quad (1 + \chi(a))(1 + \chi(b)) = (1 - \chi(d_c))\chi(ab)$$

$$(3) \quad (1 + \chi(a))(1 + \chi(c)) = (1 - \chi(d_b))\chi(ac)$$

$$(4) \quad (1 + \chi(b))(1 + \chi(c)) = (1 - \chi(d_a))\chi(bc).$$

Also from (1) it follows that at least one of $\chi(a), \chi(b), \chi(c)$ is equal to $-1$. If $\chi(a) = -1$, the by (2) and (3) we get $\chi(d_c) = \chi(d_b) = 1$. If $\chi(a) \neq -1$, then by (4) we get $\chi(d_a) = 1$. 
Summing up our argument we may assume that $K_s \neq \{e\}$. Similarly, we may assume that $K_i \neq \{e\}$ for each $i$, $1 \leq i \leq s$.

Let $A_n = \{e, a\}$. Note that in the $a^2 = e$ case $A_n$ is periodic, so we assume that $a^2 \neq e$. This in turn implies that

$$\{e\} \neq \langle a^2 \rangle \subset K_n.$$  

Similarly, we may assume that $K_i \neq \{e\}$ for each $i$, $s + 1 \leq i \leq n$. Therefore, $K_i \neq \{e\}$ may be assumed for each factor. By Th. 1 of [1] this is not possible. The contradiction completes the proof. ◇

References


[2] CORRÁDI, K. and SZABÓ, S.: Periodicity forcing factorization types for finite abelian 2-groups, Atti del Seminario Mathematico e Fisico dell' Univ. di Modena, (accepted)