ON ITERATIVE ROOTS OF A HOMEOMORPHISM OF THE CIRCLE WITH AN IRRATIONAL ROTATION NUMBER

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Abstract: The aim of this paper is to give the general construction of continuous solutions of the equation $G^n = F$, where $n \geq 2$ is a fixed integer and $F : S^1 \to S^1$ is a given homeomorphism. Our basic assumptions are that $F$ has no periodic points and the iterative kernel of $F$ has some algebraic property.

1. Introduction

Let $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle with the positive orientation. Assume that a homeomorphism $F : S^1 \to S^1$ is without periodic points. Then $\alpha(F)$, the rotation number of $F$, is irrational and $F$ preserves orientation (see [5]).

Denote by $L_F$ the set of all cluster points of the orbit $\{ F^n(z), n \in \mathbb{Z} \}$ for a $z \in S^1$. This set does not depend on $z \in S^1$ and $L_F$ either

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equals $S^1$ or is a nowhere dense perfect set (see [5]).

For every continuous mapping $F : S^1 \rightarrow S^1$ there exist a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and an integer $k$ such that $F(e^{2\pi i x}) = e^{2\pi i f(x)}$, $x \in \mathbb{R}$ and $f(x + 1) = f(x) + k$, $x \in \mathbb{R}$. The function $f$ is said to be the lift of $F$ and the integer $k$ is called the degree of $F$, and is denoted by $\deg F$.

**Proposition 1.** ([4]) Let $F, G : S^1 \rightarrow S^1$ be orientation-preserving homeomorphisms and suppose that there exists a continuous function $\Phi : S^1 \rightarrow S^1$ such that $\Phi(F(z)) = G(\Phi(z))$, $z \in S^1$. Then $\alpha(G) = \alpha(F) \deg \Phi \mod 1$.

**Proposition 2.** ([4], [9]) Let $F$ be a homeomorphism with no periodic points. Then the Schröder equation

$$\varphi(F(z)) = s\varphi(z), \quad z \in S^1,$$

where $s = e^{2\pi i \alpha(F)}$ has a unique continuous solution $\varphi : S^1 \rightarrow S^1$ such that $\varphi(1) = 1$. Moreover, $\deg \varphi = 1$ and $\varphi$ is invertible iff $L_F = S^1$.

If $L_F \neq S^1$, then the set

$$K_F := \varphi[S^1 \setminus L_F],$$

where $\varphi$ is the continuous solution of (1) such that $\varphi(1) = 1$, is said to be an iterative kernel of $F$ (see [10]).

It was proved by M. C. Zdun [10] that if $L_F = S^1$, then the homeomorphism $F$ has exactly $n$ iterative roots of $n$-th order that is continuous solutions of the functional equation

$$G^n(z) = F(z), \quad z \in S^1.$$  

However, if $L_F \neq S^1$, then $F$ has iterative roots of $n$-th order with the rotation number $\frac{1}{n}(\alpha(F) + m)$ if and only if

$$\left(\sqrt[n]{s}\right)_m K_F = K_F,$$

where

$$\left(\sqrt[n]{s}\right)_m = e^{2\pi i \frac{1}{n}(\alpha(F) + m)}, \quad m \in \{0, \ldots, n - 1\}.$$  

Moreover, in this case $F$ has infinitely many iterative roots depending on an arbitrary function. In [10] M. C. Zdun also gave the construction of iterative roots. The problem of existence of iterative roots of homeomorphisms of the circle has also been worked out by J. H. Mai in [8].
In this paper we give a construction of iterative roots. We do this on the strength of the method given by M. Kuczma in [6] (see also [7]), i.e. we find an extension of a function defined on the set $S^1 \setminus L_F$.

2. Preliminaries

Let $u, w, z \in S^1$, then there exist unique $t_1, t_2 \in [0, 1)$ such that $we^{2\pi it_1} = z$, and $we^{2\pi it_2} = u$. Define

$$u < w < z \iff 0 < t_1 < t_2$$

and

$$u \preceq w \preceq z \iff t_1 \leq t_2 \text{ or } t_2 = 0$$

(see [1]). The properties of these relations can be found in [3] (see also [2]). If $u, z \in S^1, u \neq z$, then there exist $t_u, t_z \in \mathbb{R}$ such that $t_u < t_z < t_u + 1$ and $e^{2\pi it_u} = u, e^{2\pi it_z} = z$. Put

$$\langle u, z \rangle = \{ e^{2\pi it} : t \in (t_u, t_z) \},$$

this set is said to be an open arc.

The following lemma is easy to check.

**Lemma 1.** Let $L_1, L_2, L_3 \subset S^1$ be pairwise disjoint open arcs and $u, w, z \in S^1$ be such that $u \in L_1$, $w \in L_2$, $z \in L_3$. If $u < w < z$, then $u_1 < w_1 < z_1$ for every $u_1 \in L_1$, $w_1 \in L_2$, $z_1 \in L_3$.

Let $A \subset S^1$ be such that $\text{card} A \geq 3$. We say that the function $\varphi : A \longrightarrow S^1$ is strictly increasing (respectively increasing) with respect to the cyclic order if for every $u, w, z$ belonging to $A$ such that $u < w < z$ we have $\varphi(u) < \varphi(w) < \varphi(z)$ (respectively $\varphi(u) \leq \varphi(w) \leq \varphi(z)$). It is easy to check that every strictly increasing mapping is an injection and if $F, G$ are strictly increasing, then so are the mappings $F^{-1}$ and $F \circ G$.

Moreover, a homeomorphism $F : S^1 \longrightarrow S^1$ preserves orientation if and only if $F$ is strictly increasing.

Let $F : S^1 \longrightarrow S^1$ be a homeomorphism without periodic points and $L_F \neq S^1$, then the set $S^1 \setminus L_F$ is a countable sum of pairwise disjoint open arcs. Denote the family of these arcs by $\mathcal{A}$. Let $\mathcal{M} := \{ c(I), I \in \mathcal{A} \}$, where $c(I)$ is the middle point of the arc $I \subset S^1$. Put $I_p := c^{-1}(p)$ for $p \in \mathcal{M}$. Thus we have the decomposition

$$S^1 \setminus L_F = \bigcup_{p \in \mathcal{M}} I_p \text{ and } p \in I_p \text{ for } p \in \mathcal{M}.$$  

**Lemma 2.** ([10], [3]) Let $F$ be a homeomorphism without periodic points, $L_F \neq S^1$ and $\varphi$ be a continuous solution of (1) such that $\varphi(1) =$
Then for every \( p \in \mathcal{M} \) \( \varphi \) is constant in \( I_p \) and there exists a \( q \in \mathcal{M} \) such that \( F[I_p] = I_q \). If, moreover, an integer \( n \geq 2 \) and an \( m \in \{0, \ldots, n - 1\} \) are such that (3) holds, then a function \( H : \mathcal{M} \to \mathcal{M} \) defined by

\[
H(p) := \Phi^{-1} \left( \left( \sqrt[n]{s} \right)_m \Phi(p) \right), \quad p \in \mathcal{M},
\]

where \( \Phi := \varphi|_{\mathcal{M}} \), is a strictly increasing bijection and \( F[I_p] = I_{H^n(p)} \) for \( p \in \mathcal{M} \).

Let us note that \( H^i(p) \neq p \) for \( i \neq 0 \), since \( \alpha(F) \notin \mathbb{Q} \).

We introduce the following relation on \( \mathcal{M} \)

\( p \sim q \) iff there exists an integer \( k \) such that \( q = H^k(p) \).

It is clear that \( \sim \) is an equivalence relation on \( \mathcal{M} \). Let \( E \) be an arbitrary subset of \( \mathcal{M} \) which has exactly one point in common with every equivalence class with respect to the relation \( \sim \).

### 3. Main result

**Theorem.** Let \( F : S^1 \to S^1 \) be a homeomorphism with no periodic points such that \( L_F \neq S^1 \). Assume that (3) holds for an integer \( n \geq 2 \) and an \( m \in \{0, \ldots, n - 1\} \). If \( g_{p,k} : I_{H^k(p)} \to I_{H^{k+1}(p)} \) for \( p \in E \) and \( k \in \{0, \ldots, n - 2\} \) are increasing homeomorphisms, then there exists a homeomorphism \( G : S^1 \to S^1 \) such that \( G^n = F \) and \( G|_{I_{H^k(p)}} = g_{p,k} \) for \( p \in E \) and \( k \in \{0, \ldots, n - 2\} \). Moreover,

\[
\alpha(G) = \frac{1}{n} \left( \alpha(F) + m \right).
\]

**Proof.** Let us construct an auxiliary function \( \hat{G} \), which we shall extend to the whole circle \( S^1 \). Fix a \( p \in E \) and define

\[
g_{p,n-1} := F \circ g_{p,0}^{-1} \circ g_{p,1}^{-1} \circ \ldots \circ g_{p,n-2}^{-1}.
\]

For every integer \( i \) there exist a unique \( l \in \mathbb{Z} \) and \( r \in \{0, \ldots, n - 1\} \) such that \( i = ln + r \). Hence for every \( i \in \mathbb{Z} \setminus \{0, \ldots, n - 1\} \) define

\[
g_{p,i} = g_{p,ln+r} := F^l \circ g_{p,r} \circ F_{[I_{H^i(p)}]}^{-l}.
\]

It follows by Lemma 2 that
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(8) \[ g_{p,i} \left[ I_{H^i(p)} \right] = I_{H^{i+1}(p)} \text{ for } i \in \mathbb{Z}. \]

In fact, \( F^{-l} \left[ I_{H^i(p)} \right] = F^{-l-1} \left[ I_{H^{i+n+1}(p)} \right] = I_{H^r(p)}, \) \( g_{p,r} \left[ I_{H^r(p)} \right] = I_{H^{r+1}(p)} \) and \( F^l \left[ I_{H^{i+1}(p)} \right] = I_{H^{i+n+1}(p)} = I_{H^{i+1}(p)}. \) Consequently,

(9) \[ g_{p,i} : I_{H^i(p)} \rightarrow I_{H^{i+1}(p)} \text{ for } p \in E, \ i \in \mathbb{Z} \text{ are increasing homeomorphisms}. \]

For every \( q \in \mathcal{M} \) there exist a unique \( p \in E \) and a unique integer \( i \) such that \( q = H^i(p) \). Define

(10) \[ \hat{G}(z) := g_{p,i}(z) \text{ for } z \in I_q, \ q \in \mathcal{M}. \]

It follows by (8) and (9) that the function \( \hat{G} : S^1 \setminus L_F \rightarrow S^1 \setminus L_F \) is a bijection. We shall show that \( \hat{G} \) satisfies the equation

\[ \hat{G}^n(z) = F(z), \quad z \in S^1 \setminus L_F. \]

For this purpose we are going to show that

(11) \[ g_{p,i} = F \circ g_{p,i+n+1}^{-1} \circ g_{p,i+2}^{-1} \circ \ldots \circ g_{p,i-1}^{-1} \]

for all integers \( i \geq n - 1 \) and

(12) \[ g_{p,i} = g_{p,i+n+1}^{-1} \circ g_{p,i+2}^{-1} \circ \ldots \circ g_{p,i+n-1}^{-1} \circ F \mid I_{H^i(p)} \]

for \( i < n - 1. \)

We prove this by induction. Obviously for \( i = n - 1 \) we get (11) by (6). Assuming (11) to hold for a \( k - 1 \geq n - 1 \), we get

(13) \[ F \mid I_{H^{k-n}(p)} = g_{p,k-1} \circ g_{p,k-2} \circ \ldots \circ g_{p,k-n}. \]

We may write the index \( k \) in the form \( k = nl + r \), where \( l \geq 1 \), and \( r \in \{0, 1, \ldots, n - 1\} \). By (7) we have

\[ g_{p,k} = g_{p,ln+r} = F \circ F^{-1} \circ g_{p,r} \circ F^{-l+1} \circ F^{-1} \mid I_{H^k(p)} \]

\[ = F \circ g_{p,(-l-1)n+r} \circ F^{-1} \mid I_{H^k(p)} = F \circ g_{p,k-n} \circ F^{-1} \mid I_{H^k(p)}. \]

Using (13) we see that

\[ g_{p,k} = F \circ g_{p,k-n+1}^{-1} \circ \ldots \circ g_{p,k-1}^{-1}. \]

Hence by induction (11) holds for \( i \geq n - 1. \) Moreover, we also have (13) for all \( k \geq n \). Fix an \( r \in \{0, 1, \ldots, n - 2\} \). From (13) for \( k = n + r \) we have
\[ g_{p,r} = g_{p,r} \circ F^{-1} \circ F_{\mid I_{H^r(p)}} = g_{p,r} \circ g_{p,r+1}^{-1} \circ \ldots \circ g_{p,n+r-1}^{-1} \circ F_{\mid I_{H^r(p)}} , \]
since \( g_{p,r} \) satisfy (12). The proof of this part is completed by showing that \( g_{p,i} \) satisfy (12) for \( i \leq -1 \). To do this note that by (7) and (6) we get
\[ g_{p,-1} = g_{p,-n+(n-1)} = F^{-1} \circ g_{p,n-1} \circ F_{\mid I_{H^{-1}(p)}} = g_{p,0} \circ g_{p,1}^{-1} \circ \ldots \circ g_{p,n-2}^{-1} \circ F_{\mid I_{H^{-1}(p)}}, \]
so \( g_{p,-1} \) satisfies (12). Fix a \( k \in \mathbb{Z} \) such that \( k \leq -2 \). Suppose that \( g_{p,k+1} \) fulfills (12). Hence
\[ F_{\mid I_{H^{k+1}(p)}} = g_{p,k+n} \circ g_{p,k+n-1} \circ \ldots \circ g_{p,k+1}. \]
We shall prove that \( g_{p,k} \) satisfies (12). There exist an \( l \in \mathbb{Z} \) and an \( r \in \{0,1,\ldots, n-1\} \) such that \( k = ln + r \). By (7) and (14) we have
\[ g_{p,k} = F^{-1} \circ F^{l+1} \circ g_{p,r} \circ F^{-l-1} \circ F_{\mid I_{H^k(p)}} = F^{-1} \circ g_{p,k+n} \circ F_{\mid I_{H^k(p)}} \]
\[ = g_{p,k+1} \circ \ldots \circ g_{p,k+n-1} \circ F_{\mid I_{H^k(p)}}, \]
which completes this part of the proof.

By (10) and (8) we have
\[ \hat{G}^n(z) = g_{p,i+n-1} \circ \ldots \circ g_{p,i+1} \circ g_{p,i}(z) \quad \text{for} \quad z \in I_{H^i(p)}. \]
Thus, for \( i < n-1 \) using (12) we obtain \( \hat{G}^n(z) = F(z) \) for \( z \in I_{H^i(p)}. \)
However, for \( i \geq n-1 \) it follows from (8) and (11) that
\[ g_{p,i} \circ g_{p,i-1} \circ \ldots \circ g_{p,i-n+1}(z) = F(z) \quad \text{for} \quad z \in I_{H^{i-n+1}(p)}. \]
Hence \( \hat{G}^n(z) = F(z) \) for \( z \in I_{H^j(p)}, \quad j \geq 0. \) Thus
\[ \hat{G}^n(z) = F(z) \quad \text{for} \quad z \in S^1 \setminus L_F. \]

Now we shall show that \( \hat{G} \) is strictly increasing. To do this take \( u, w, z \in S^1 \setminus L_F \) such that \( u \prec w \prec z \) and consider three cases

1° There exist a \( p \in E \) and an \( i \in \mathbb{Z} \) such that \( \{u, w, z\} \subset I_{H^i(p)}. \)

By (9) and (10) it is clear that \( \hat{G}(u) \prec \hat{G}(w) \prec \hat{G}(z). \)

2° There exist \( p, q \in E, \quad i, j \in \mathbb{Z} \) such that \( H^i(p) \neq H^j(q), \) and

one of the following conditions is fulfilled:

(a) \( \{u, z\} \subset I_{H^i(p)}, \quad w \in I_{H^j(q)}, \)
(b) \( \{u, w\} \subset I_{H^i(p)}, \quad z \in I_{H^j(q)}, \)
(c) \( \{z, w\} \subset I_{H^i(p)}, \quad u \in I_{H^j(q)}, \)
According to Lemma 2 in [3] it suffices to consider only case (a). Then \((z, u) \subset I_{H^i(p)}\), whence by (9),
\[
g_{p, i}([z, u]) = (g_{p, i}(z), g_{p, i}(u)) = (\hat{G}(z), \hat{G}(u)) \subset I_{H^{i+1}(p)}.
\]
Since \(\hat{G}(w) \in I_{H^{i+1}(q)}\) and \(I_{H^{i+1}(q)} \cap I_{H^{i+1}(p)} = \emptyset\), it follows that \(\hat{G}(u) \prec \hat{G}(w) \prec \hat{G}(z)\).

3° There exist \(p, q, t \in E, i, j, k \in \mathbb{Z}\) such that \(H^i(p) \neq H^j(q) \neq H^k(t)\) and \(u \in I_{H^i(p)}, w \in I_{H^j(q)}\) and \(z \in I_{H^k(t)}\). Hence by Lemma 1
\[
H^i(p) \prec H^j(q) \prec H^k(t)
\]
but \(H\) is strictly increasing so
\[
H^{i+1}(p) \prec H^{j+1}(q) \prec H^{k+1}(t).
\]
Using Lemma 1 once more we have
\[
\hat{G}(u) \prec \hat{G}(w) \prec \hat{G}(z).
\]
Thus, we have shown that \(\hat{G}\) is a strictly increasing bijection.

Since the set \(S^1 \setminus L_F\) is dense in \(S^1\), it follows (see [2]) that the function \(\hat{G}\) has a unique homeomorphic extension \(G : S^1 \to S^1\). Moreover, \(G\) satisfies (2). It remains to prove (5). Let \(z \in I_p, p \in \mathcal{M}\), then by (8) and (10), \(G(z) \in I_{H(p)}\). By Lemma 2, we have
\[
\varphi(G(z)) = \Phi(H(p)) = \left(\sqrt{[n]} s \right)_m \Phi(p) = \left(\sqrt{[n]} s \right)_m \varphi(z).
\]
Hence \(G\) and the rotation \(R(z) = \left(\sqrt{[n]} s \right)_m z\) are semi-conjugate. By Prop. 1 and Prop. 2 we get \(\alpha(R) = \alpha(G) \text{deg}\varphi(\text{mod } 1)\) and \(\text{deg}\varphi = 1\), thus \(G\) and \(R\) have the same rotation number, which is our claim. ◊

**Remark.** Under the assumptions of Theorem every solution of (2) satisfying (5) may be obtained in the manner described in the proof of Theorem.

**Proof.** If \(G\) is a solution of (2) satisfying (5), then by Lemma 1 in [10] \(L_F = L_G\), so
\[
S^1 \setminus L_G = S^1 \setminus L_F = \bigcup_{p \in \mathcal{M}} I_p.
\]
Define functions \(p \in E\) \(g_{p, i}\) for \(p \in E, i \in \mathbb{Z}\) by \(g_{p, i} = G|_{I_{H^i(p)}}\). It is clear that they are strictly increasing. Next we prove that
Let us observe that by Prop. 2 there exists a continuous solution of the equation
\[ \psi(G(z)) = \left( \sqrt{|n|} s \right)_m \psi(z), \quad z \in S^1 \]
such that \( \psi : S^1 \to S^1 \), \( \psi(1) = 1 \). Moreover,
\[ \psi(G^n(z)) = \left( \left( \sqrt{|n|} s \right)_m \right)^n \psi(z), \]
so \( \psi \) satisfies (1) and \( \psi = \varphi \), by the uniqueness of the solution of the equation (1). Fix a \( p \in M \). By Lemma 2 there is a \( q \in M \) such that \( G[I_p] = I_q \) and
\[ \{ \Phi(q) \} = \varphi[G[I_p]] = G[I_q] = \left( \sqrt{|n|} s \right)_m \psi[I_p] = \left( \sqrt{|n|} s \right)_m \varphi[I_p] = \left\{ \left( \sqrt{|n|} s \right)_m \Phi(p) \right\}, \]
so by (4), \( q = H(p) \), which gives (15). What is left is to show that \( g_{p,i} \) for \( p \in E \) satisfy (6) and (7). Observe that from (2) for \( z \in I_p \) we get
\[ g_{p,n-1} \circ \ldots \circ g_{p,1} \circ g_{p,0}(z) = F(z), \]
so
\[ g_{p,n-1}(z) = F \circ g_{p,0}^{-1} \circ \ldots \circ g_{p,n-2}(z), \quad z \in I_{H^n(p)} \]
and we have (6). We conclude from (2) that \( G \circ F = F \circ G \), hence
\[ G \circ F^l = F^l \circ G, \quad l \in \mathbb{Z}. \]
Fix \( z \in I_{H^r(p)} \), \( r \in \{0, 1, \ldots, n-1\} \). Thus from (16), Lemma 2 and the definition of \( g_{p,i} \), \( i \in \mathbb{Z} \) we have
\[ g_{p,ln+r} \circ F^l(z) = F^l \circ g_{p,r}(z), \]
which gives (7), and the proof is completed. \( \Diamond \)

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References

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