THE RADICALNESS OF POLYNOMIAL RINGS OVER NIL RINGS

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Dedicated to my teacher Professor R. Wiegandt on his 70-th birthday

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Abstract: The main purpose of this note is to give the exact upper bound of approximating Köthe’s Problem by radicals. We construct and characterize the smallest radical $\ell$ such that $A[x] \in \ell$ for every nil ring $A$ and show that this improves the approximation given in [1].

1. In this note associative rings and Kurosh–Amitsur radicals will be considered. As usual, $I \triangleleft A$ and $L \triangleleft_{\ell} A$ denote that $I$ is an ideal and $L$ is a left ideal in $A$, respectively.

A class $\mathcal{M}$ of rings is said to be regular, if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the upper radical operator $\mathcal{U}$ yields a radical class:

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\[ \mathcal{U} \mathcal{M} = \{ A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M} \} \].

For a radical class \( \gamma \) the semisimple operator \( S \) gives its semisimple class:

\[ S\gamma = \{ A \mid A \text{ has no nonzero ideal in } \gamma \} \).

**Köthe's Problem:** Is the sum of two nil left ideals nil?

It has been posed in 1930 at the genesis of radical theory [6]. This problem has many equivalent formulations. One of the most interesting one, which stimulated many further studies, is the following due to Krempa [7].

Does \( A \in \mathcal{N} \) imply that the polynomial ring \( A[x] \) in indeterminate \( x \) over \( A \) is in \( \mathcal{J} \), where \( \mathcal{N} \) and \( \mathcal{J} \) denote the classes of nil rings and Jacobson radical rings, respectively?

In [9] it has been proved that \( A \in \mathcal{N} \) implies \( A[x] \in \mathcal{G} \), where \( \mathcal{G} \) stands for the Brown–McCoy radical.

We consider two natural radicals:

- **The antiregular radical** \( \mathcal{U}\nu \). This is the upper radical determined by the class \( \nu \) of all von Neumann regular rings.

- **The uniformly strongly prime radical** \( \mathcal{U} \). A ring \( A \) is said to be uniformly strongly prime, if there exists a finite subset \( F \) of \( A \), called a uniform insulator, such that \( xFy \neq 0 \) whenever \( 0 \neq x, y \in A \). The uniformly strongly prime radical is the upper radical determined by the class of uniformly strongly prime rings [8].

In [2] it has been proved that \( A \in \mathcal{N} \) implies \( A[x] \in \mathcal{U}\nu \cap \mathcal{G} \cap \mathcal{U} \) (see [2, Cor. 3.5]).

We recall also some statements we shall need in the sequel.

The **upper radical** \( \mathcal{N}_s \) determined by the class of rings which contain no nonzero nil left ideals or, equivalently, no nonzero nil right ideals is called the lower strong radical determined by \( \mathcal{N} \) (see [1] and [2]).

The **Behrens radical** \( \mathcal{B} \) is the upper radical determined by the class of all subdirectly irreducible rings having a nonzero idempotent in their heart.

Recently, in [1] the following has been proved.

**Proposition 1.1.** \( A \in \mathcal{N}_s \) implies \( A[x] \in \mathcal{B} \).

**Proposition 1.2** [2, Th. 3.4]. \( \mathcal{N}_s \subseteq \mathcal{U} \).

We say that a ring \( A \) has **bounded index of nilpotency** if there is a positive integer \( m \) such that \( a^m = 0 \) for each nilpotent element \( a \) of \( A \) [4].
Proposition 1.3 [5, Th. 10.8.2]. Let $A$ be PI algebra of degree $d$. Let $A(1)$ be the sum of the nilpotent ideals of $A$, and $B$ any nil subalgebra of $A$. Then $B^m \subseteq A(1)$ where $m = [d/2]$.

Proposition 1.4 [3, Th. 6.53]. If in a ring $A$ there exists a fixed positive integer $n$ such that $x^n = 0$ for every $x \in A$, then $A$ is locally nilpotent.

The Baer radical $\beta$ is the upper radical determined by the class of prime rings. A prime ring $A$ is said to be $\ast$-ring if its every proper homomorphic image $A'$ is in $\beta$. We denote by $M(A)$ the infinite matrix ring which has only finitely many nonzero entries from $A$.

Proposition 1.5 [12, Lemma 7]. If $A$ is a $\ast$-ring, then $M(A)$ is a $\ast$-ring with trivial center.

A class $M$ of rings is said to be principally left hereditary if $a \in A \in M$, then $Aa \in M$.

Proposition 1.6 [13, Th. 5.1]. The Behrens radical $B$ is the largest principally left hereditary subclass of the Brown–McCoy radical class $G$ in fact, $MG = B$ where

$$
MG = \{ A \mid Aa \in G \text{ for every } a \in A \}.
$$

2. We set

$$
M = \left\{ A \mid \begin{array}{l}
A \text{ has no nonzero locally nilpotent ideals and } \\
\text{every nil subring } S \text{ of } A \text{ is locally nilpotent}
\end{array} \right\},
$$

$$
M_0 = \left\{ A \mid \begin{array}{l}
A \text{ has no nonzero nil ideals and } \\
\text{all nilpotent elements form a subring in } A
\end{array} \right\}.
$$

Lemma 2.1. $M$ and $M_0$ are

a) hereditary classes of rings;

b) both consist of semiprime rings;

c) both contain no nonzero nilrings.

Proof. Trivial. $\Diamond$

Recall that a radical $\sigma$ is said to be left strong if $\sigma(L) = L \triangleleft_l A$ implies $L \subseteq \sigma(A)$. Right strong radical is defined correspondingly.

Proposition 2.2. $\gamma = UM$ and $\delta = UM_0$ are left and right strong and so is $\gamma \cap \delta$.

Proof. Let $\gamma(L) = L \triangleleft_l A$, and $L \not\subseteq \gamma(A)$. Then we have
0 \neq \gamma \left( \frac{L + \gamma(A)}{\gamma(A)} \right) = \frac{L + \gamma(A)}{\gamma(A)} \cdot \frac{A}{\gamma(A)} \in S\gamma.

Hence, we can choose \( \gamma(A) = 0 \) and so \( B = L + LA \in S\gamma \). Therefore \( B \) has a nonzero homomorphic image \( B/I \) in \( \mathcal{M} \). Let \( \langle I \rangle \) be the ideal of \( A \), generated by \( I \). By Andrunakievich Lemma \( \langle I \rangle^3 \subseteq I \subseteq \langle I \rangle \) and so by Lemma 2.1 a) and b) \( \langle I \rangle = I \). Thus it follows that \( I \triangleleft A \). Hence \( L \not\subseteq I \). Again we can choose \( B \in \mathcal{M} \). By Lemma 2.1 c) \( \mathcal{N} \subseteq \gamma \) and so also the locally nilpotent radical \( \mathcal{L} \) is contained in \( \gamma \). Since \( \mathcal{L} \) is left strong, we have \( \mathcal{L}(L) \neq L \) and so \( 0 \neq L/\mathcal{L}(L) \in \gamma \). Hence \( L/\mathcal{L}(L) \) has a non-locally nilpotent and nil subring \( \overline{S} \). Let \( S/\mathcal{L}(L) = \overline{S} \), then \( S \) is a nil subring of \( B \) which is not locally nilpotent, contradicting \( B \in \mathcal{M} \). For \( \delta \) the proof is similar. \( \Diamond \)

**Corollary 2.3.** \( \mathcal{N}_s \subseteq \gamma \cap \delta \cap B \cap u \).

**Proof.** \( \mathcal{N}_s \subseteq B \cap u \) follows from Props. 1.1 and 1.2. Since \( N \subseteq \gamma \cap \delta \), by Prop. 2.2 we get \( \mathcal{N}_s \subseteq \gamma \cap \delta \). \( \Diamond \)

**Lemma 2.4.** If for a ring \( A \) the factor ring \( A[x]/I \) is a prime (semiprime) ring, then there exist a prime (semiprime) ring \( B \) and an ideal \( J \) of \( B[x] \) such that \( A[x]/I \cong B[x]/J \) and \( B \cap J = 0 \).

**Proof.** Let \( H = A \cap I \triangleleft A \). Since \( H^2[x] = (A \cap I)^2[x] \subseteq I \) and \( (H[x])^2 \subseteq H^2[x] \subseteq I \). We claim that \( H[x] \subseteq I \). Suppose that \( H[x] \not\subseteq I \). Then \( I \subset H[x] + I \) and \( H^2[x] \subseteq (H[x] + I)^2 \subseteq I \) by \( H^2[x] \subseteq I \). Since \( I \) is a semiprime ideal, we conclude \( H[x] \subseteq I \). So

\[
\frac{I}{H[x]} \triangleleft \frac{A[x]}{H[x]} \cong (A/H)[x],
\]

where \( f \) is an isomorphism of \( (A[x])/(H[x]) \) onto \( (A/H)[x] \) such that

\[
f \left( \sum_{i=0}^{n} a_i x^i + H[x] \right) = \sum_{i=0}^{n} (a_i + H)x^i, \quad \text{for } a_i \in A.
\]

Choose \( B = A/H \) and \( J = f(I/H[x]) \). Then we have

\[
\frac{B[x]}{J} \cong \frac{A[x]/H[x]}{I/H[x]} \cong \frac{A[x]}{I},
\]

and we claim that \( B \cap J = 0 \). If \( B \cap J \neq 0 \) then \( 0 \neq B \cap J = H_1/H \), and \( H \subset H_1 \triangleleft A \). Let \( 0 \neq h \in H \setminus H_1 \). Since \( H[x] \subseteq I \) and \( h + H[x] = f^{-1}(h + H) \in f^{-1}(J) = I + H[x] = I \). We get \( h \in I \) and so \( H_1 + H[x] \subseteq I \). Thus \( H_1 \subseteq I \), contradicting \( A \cap I = H \).

Now, we shall show that \( B \) is semiprime. If \( B \) is not semiprime then there exists an ideal \( H_1 \) of \( B \) such that \( H \subset H_1 \) and \( H_1^2 \subseteq H \).
Hence $H_1^2[x] \subseteq H[x]$. So $H_1^2[x] \subseteq I$, and as above we have $H_1[x] \subseteq I$. Hence it follows $I_1 \subseteq I$, and so $H_1 \subseteq I \cap A = H$ implying $H_1 = H$, a contradiction.

Let $A[x]/I$ be a prime ring. If $H \subseteq H_1 \triangleleft A$ and $H \subseteq H_2 \triangleleft A$ and $H_1H_2 \subseteq H$, then $(H_1 \cap H_2)^2 \subseteq H_1H_2 \subseteq H$. It follows again that $H_1 \cap H_2 \subseteq I$, and so $H_1 \cap H_2 \subseteq H$.

Put $\overline{H}_1 = H_1/H$ and $\overline{H}_2 = H_2/H$, then $\overline{H}_1 \cap \overline{H}_2 = 0$. We have

$$\frac{H_1[x]}{H[x]} \cong \frac{(H_1/H)[x]}{\overline{H}_1[x]} = \overline{H}_1[x] \triangleleft B[x]$$

and

$$\frac{H_2[x]}{H[x]} \cong \frac{H_2[x]}{H[x]} = \overline{H}_2[x] \triangleleft B[x].$$

and also $\overline{H}_1[x] \cap \overline{H}_2[x] = 0$.

Since $I$ is a prime ideal of $A[x]$ and

$$H_1[x]H_2[x] \subseteq H_1[x] \cap H_2[x] \subseteq I,$$

we conclude that either $H_1[x] \subseteq I$ or $H_2[x] \subseteq I$, and so either $H_1[x] \subseteq H[x]$ or $H_2[x] \subseteq H[x]$. Hence either $H_1 \subseteq H$ or $H_2 \subseteq H$, a contradiction. \hfill \Box

**Corollary 2.5.** Let $A$ and $B$ be rings as in Lemma 2.4. If $A$ is nil ring, then $B$ is nil ring. \hfill \Box

A ring $A$ is said to be an $n$-ring if $A$ is not a homomorphic image of the polynomial ring $B[x]$ for any nil subring $B$ of $A$.

Put $n(x) = \{ A \mid A$ has no nonzero accessible subring $B$ which is $n$-ring$\}$. Denote by $\ell$ the lower radical generated by the class $\{ A[x] \mid A$ is a nil ring$\}$.

**Theorem 2.6.** $Un(x) = \ell$.

**Proof.** $Un(x) \subseteq \ell$: Let $A \in Un(x)$. then every homomorphic image $A'$ has a nil subring $B \subseteq A'$, such that $B[x]/I \cong I_n \triangleleft \cdots \triangleleft A'$. Therefore $I_n \in \ell$. Hence $\ell(A') \neq 0$. If $Un(x) \not\subseteq \ell$, then there exists a nonzero ring $A \in Un(x) \cap St$. As above $\ell(A) \neq 0$, a contradiction.

$\ell \subseteq Un(x)$: Let $A \in \ell \setminus Un(x)$. Then $A$ has a nonzero homomorphic image $A'$ in $n(x)$. Since $A' \in \ell$, there exists an accessible subring $I_n \triangleleft \cdots \triangleleft A'$, which is a homomorphic image of $B[x]$, where $B$ is a nil ring. Suppose $I_n \cong B[x]/I$. By Lemma 2.1 $I_n \cong B[x]/I$ is semiprime ring.

By Cor. 2.5, there exists a nil ring $B'$ such that
\[ B[x]/I \cong B'[x]/J \quad \text{and} \quad B' \cap J = 0. \]

Since \( B' \cap J = 0 \), we have
\[
B' \cong \frac{B'}{B' \cap J} \cong \frac{B' + J}{J} \cong \frac{B'[x]}{J} \cong \frac{B[x]}{I} \cong I_n.
\]

So \( I_n \) contains a nil subring \( S \) which is isomorphic to \( B' \) and so \( S[x] \cong \cong B'[x] \). Hence \( I_n \) is a homomorphic image of \( S[x] \). Therefore \( I_n \notin n(x) \) and so \( A' \notin n(x) \), a contradiction. \( \diamond \)

**Corollary 2.7.** Let \( \sigma \) be a radical. If \( A \in \mathcal{N} \) imply \( A[x] \in \sigma \) then \( \ell \subseteq \sigma \).

**Lemma 2.8.** Let \( A \) be a semiprime commutative ring. Then every nil subring \( S \) of \( M(A) \) is locally nilpotent.

**Proof.** Since \( A \) is commutative, for any natural number \( n \) the standard polynomial \( S_{2n} \) actually is an identity of matrix ring \( M_n(A) \) (see [10,6.1.17]). By Prop. 1.3, \( M_n(A) \) has bounded index. Let \( m \) be the smallest among these indices.

Put
\[ nM(A) = \{(a_{ij}) \mid a_{ij} \in A \text{ and } a_{ij} = 0 \text{ for } j > n \}\]
and
\[ V = \{B \in nM(A) \mid a_{ij} = 0 \text{ for } i, j \leq n \}. \]

Clearly \( V \triangleleft nM(A) \) and \( nM(A)/V \cong M_n(A) \). Let \( B \in nM(A) \) be a nilpotent element, then \( B^m \subseteq V \). Since \( V^2 = 0 \), also \( B^{2m} = 0 \). Hence \( nM(A) \) is of bounded index. For any \( s \in S \), there exists natural number \( n \), such that \( s \in nM(A) \). Since \( nM(A) \) is a left ideal of \( M(A) \), also \( nM(A) \cap S \triangleleft S \). Therefore \( nM(A) \cap I_\ell \) is of bounded index nil ring. By Prop. 1.4, \( nM(A) \cap S \) is locally nilpotent. Since the locally nilpotent radical is left strong, \( S \) has a locally nilpotent ideal \( I_s \) of \( S \)which is \( s \in I_s \), and so \( S \) is locally nilpotent. \( \diamond \)

**Theorem 2.9.** \( \ell = Un(x) \subseteq B \cap u \cap \gamma \cap \delta \subset B \cap u \cap \delta \).

**Proof.** By Prop. 1.1 and Cor. 2.7, we get \( Un(x) \subseteq B \cap u \). Let \( A \in Un(x) \setminus \gamma \). Then there exists a nonzero homomorphic image \( A' \) of \( A \) in \( \mathcal{M} \). Since \( A' \in Un(x) \), \( A' \) has a nonzero accessible subring \( I \) such that \( I \cong B[x]/J \) and for a nil subring \( B \) of \( I \) by Lemma 2.4. Since \( \mathcal{M} \) is hereditary, \( I \in \mathcal{M} \). Hence \( B \) is locally nilpotent and so \( B[x]/J \). Therefore \( I \) is locally nilpotent, a contradiction. It follows \( Un(x) \subseteq \gamma \).

Let \( A \in Un(x) \setminus \delta \). As above, we get an accessible subring \( I \) of \( A' \in M_0 \) and so \( I \in M_0 \) and \( I \cong B[x]/J \). Since \( B \) is nil, for the semigroup \( \{ax^n \mid a \in B, 0 \leq n \in \mathbb{Z} \} \) every element is nilpotent. The subring \( B' \) of \( B[x]/J \) generated by the set \( \{ax^n + J \} \) is isomorphic to \( I \),
because \( \{ax^n + J\} \) are generators of \( B[x]/J \). Hence \( I \) is nil ring. Again a contradiction. Thus, it follows \( \mathcal{U}n(x) \subseteq \gamma \cap \delta \). Let us consider the ring

\[
A = \left\{ \frac{2x}{2y + 1} \mid x, y \in \mathbb{Z}, (2x, 2y + 1) = 1 \right\}.
\]

We know that \( A \) is a commutative *-ring (see [12]). We consider the ring \( M(A) \). Since \( M_\infty(A) \) is a Jacobson radical ring, one can easily check that also \( M(A) \) is a quasi-regular ring. Hence \( M(A) \in \mathcal{B} \). Let \( a_1, \ldots, a_s \in M(A) \). Then there exists \( n \in \mathbb{N} \), such that \( a_1, \ldots, a_s \in M_n(A) \). Let \( V \) be as in the proof of Lemma 2.8, then \( M_n(A) \cdot V = 0 \) and \( V \neq 0 \). Hence \( M(A) \) has no finite subset \( F \), such that \( xFy \neq 0 \forall x, y \neq 0, x, y \in M(A) \). By Prop. 1.4 \( M(A) \) is a *-ring. Hence \( M(A) \in \mathcal{U} \). Since \( M(A) \) is not nil, \( M(A) \) has no nonzero nil ideal.

Put \( (x)_{ij} = (x_{k\ell}) = \begin{cases} x & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} \).

Clearly \( (x)_{21} \) and \( (y)_{12} \) are nilpotent for any \( x, y \in A \). If \( x \neq 0 \neq y \), then \( (x)_{21}(y)_{12} \) is not nilpotent. Therefore, since \( M(A) \) is a *-ring, \( M(A) \in \mathcal{U} \). It follows \( M(A) \in \mathcal{B} \cap \mathcal{U} \). By Lemma 2.8 any nil subring \( S \) of \( M(A) \) is locally nilpotent and so \( M(A) \notin \gamma \). ◊

**Corollary 2.10.** The radical \( \ell \) gives the best approximation of Köthe’s Problem from above:

\[
A \in \mathcal{N} \Rightarrow A[x] \in \ell
\]

and this improves the approximation

\[
A \in \mathcal{N} \Rightarrow A[x] \in \mathcal{B} \cap \mathcal{U}.
\]

**Proof.** The first statement follows from Th. 2.6, the second one follows from Th. 2.9. ◊

**Remark.** Obviously \( \mathcal{N} \subseteq \mathcal{N}_s \) and \( \mathcal{N} \subseteq \ell \). If Köthe’s Problem has a positive solution, then \( \mathcal{N} = \mathcal{N}_s \) and \( \mathcal{N}_s \subseteq \ell \). However, \( \mathcal{N}_s \notin \ell \) would mean that there exists a nil semisimple ring having a nonzero one-sided nil ideal, that is, Köthe’s Problem has a negative solution.

We denote by \( \sigma \), the upper radical generated by the class

\[
\left\{ A \mid \begin{array}{l}
A \text{ has no nonzero locally nilpotent ideals and } \\
\text{all nilpotent elements have bounded nilpotency index.}
\end{array} \right\}
\]
Proposition 2.11. 1) $\mathcal{L} \subset \mathcal{N} \subset \mathcal{J} \cap \ell \subseteq \ell \subset \sigma$.

2) If $R \in \sigma$ is a PI ring, then $R$ is locally nilpotent.

Proof. 1) Since $M(A)$ is not of bounded nilpotency index $M(A) \in \sigma$ and $M(A) \notin \ell$ by Th. 2.9, and $\mathcal{N} \subset J \cap \ell$ follows from [11, Th. 8].

2) If $R$ is not locally nilpotent, then $R/\mathcal{L}(R) \neq 0$, where $\mathcal{L}(R)$ is locally nilpotent radical of $R$. Since $R$ is a PI-ring, we get that $R/\mathcal{L}(R)$ is a PI-ring and semiprime. By Prop. 1.3 $R/\mathcal{L}(R)$ is of bounded nilpotency index. Hence $R/\mathcal{L}(R) \in \sigma \cap S\sigma = 0$, a contradiction. \hfill \checkmark

A normal radical $r$ may be defined as left strong and principally left hereditary radical. In [13] it has been proved that here left strongness can be replaced by the weaker condition of principally left strongness (that is $r(L) = L \triangleleft L A$ and for any $a \in L$, $La \in \gamma \Rightarrow L \subseteq r(A)$).

An $N$-radical $r$ may be defined as a normal radical containing the Baer radical $\beta$.

Set \[ \ell^o = \{ A \in \ell \mid Aa \in \ell, \text{ for any } a \in A \} . \]

Proposition 2.12. $\mathcal{N} \subseteq \ell^o \subseteq B \cap U \cap \gamma \cap \delta$, where $\gamma \cap \delta$ is largest $N$-radical in $\gamma \cap \delta$.

Proof. Clearly $\mathcal{N} \subseteq \ell^o$, since $\mathcal{N}$ is left hereditary. Let $A \in \ell^o$, then $Aa \in \ell$, for any $a \in A$. By Prop. 1.6 $A \in B$. Since $\gamma \cap \delta$ is left-strong, $L \triangleleft L A$ implies $L \subseteq \ell$ and so $L \in \gamma \cap \delta$. By [14, Th. 15], $A \in \gamma \cap \delta$. \hfill \checkmark

Finally we give the position of the radicals considered in this note. If Köthe's Problem has a positive solution, then
\[ N = N_s \subset \ell \subset J. \]
Moreover, \( J \not\subset B \cap u \cap \gamma \cap \delta \), but if \( B \cap u \cap \gamma \cap \delta \subset J \) then \( N = N_s \) and Köthe's Problem has a positive solution. Köthe's Problem has a positive solution if and only if \( \ell(\mathcal{A}[x]) = J(\mathcal{A}[x]) \), for any ring \( A \).

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**References**


