ON THE STRUCTURE OF NON-SINGULAR ITERATION GROUPS ON THE CIRCLE

Krzysztof Ciepliński

Institute of mathematics, Pedagogical University, Podchorąży 2, PL-30-084 Kraków, Poland

Received: October 2002

MSC 2000: 39 B 12, 37 E 10, 20 F 38

Keywords: Disjoint, strictly disjoint, non-singular, singular, non-dense, dense iteration group, rotation number, limit set.

Abstract: The aim of this paper is to investigate the structure of non-singular iteration groups on the unit circle $S^1$, that is, families $\mathcal{F} = \{ F^v : S^1 \to S^1, v \in V \}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1 + v_2}, \quad v_1, v_2 \in V,$$

and at least one $F^v \in \mathcal{F}$ has no periodic point ($V$ is a linear space over $\mathbb{Q}$ with $\dim V \geq 1$). Our main result shows that iteration groups under study are direct sums of some special subgroups.

1. Introduction

Denote by $S^1$ the unit circle and let $V$ be a linear space over $\mathbb{Q}$ such that $\dim V \geq 1$.

Recall that a family $\mathcal{F} = \{ F^v : S^1 \to S^1, \quad v \in V \}$ of homeomorphisms for which

$$F^{v_1} \circ F^{v_2} = F^{v_1 + v_2}, \quad v_1, v_2 \in V$$

is called an iteration group or a flow (on $S^1$). An iteration group is said

E-mail address: kc@wsp.krakow.pl or smciepli@cyf-kr.edu.pl
to be \textit{disjoint} if every its element either is the identity mapping or has no fixed point.

Some special cases of such iteration groups under the assumption that \(V = \mathbb{R}\) have been investigated in [2] and [3]. A complete description of disjoint iteration groups \(\mathcal{F} = \{F^u : S^1 \to S^1, \ u \in V\}\) can be found in [6] and [kc6].

An iteration group \(\mathcal{F} = \{F^u : S^1 \to S^1, \ u \in V\}\) is said to be \textit{non-singular} if at least one its element has no periodic point, otherwise \(\mathcal{F}\) is called a \textit{singular} iteration group.

The aim of this paper is to investigate the structure of non-singular iteration groups which do not need to be disjoint. We shall show that every such group is a direct sum of two subgroups and one of these subgroups is a special disjoint iteration group. In order to do this we use some ideas from [11].

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper \(\mathbb{N}\) stands for the set of all positive integers and the set of all cluster points of the set \(A \subset S^1\) will be denoted by \(A^d\).

A set \(A \subset S^1\) is said to be an \textit{open arc} if there are distinct \(v, z \in S^1\) for which

\[
A = (v, z) := \{e^{2\pi i t}, \ t \in (t_v, t_z)\},
\]

where \(t_v, t_z \in \mathbb{R}\) are such that \(e^{2\pi i t_v} = v, e^{2\pi i t_z} = z\) and \(0 < t_z - t_v < 1\).

It is well-known (see for instance [1], [2] and [12]) that for every continuous mapping \(F : S^1 \to S^1\) there is a continuous function \(f : \mathbb{R} \to \mathbb{R}\), which is unique up to translation by an integer, and a unique integer \(k\) such that

\[
F(e^{2\pi i x}) = e^{2\pi i f(x)}, \quad x \in \mathbb{R}
\]

and

\[
f(x + 1) = f(x) + k, \quad x \in \mathbb{R}.
\]

The integer \(k\) is called the \textit{degree} of \(F\), and is denoted by \(\deg F\). If \(F : S^1 \to S^1\) is a homeomorphism, then so is \(f\). Furthermore, \(|\deg F| = 1\). We say that a homeomorphism \(F : S^1 \to S^1\) \textit{preserves orientation} if \(\deg F = 1\), which is clearly equivalent to the fact that \(f\) is increasing.
Moreover, $F$ preserves orientation if and only if for any $v, w, z \in \mathbb{S}^1$ such that $w \in (v, z)$ we have $F(w) \in (F(v), F(z))$ (see [5]). Recall also that every element of an iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ preserves orientation (see [6]).

For every orientation-preserving homeomorphism $F$ the number $\alpha(F) \in [0, 1)$ defined by

$$\alpha(F) := \lim_{n \to \infty} \frac{f^n(x)}{n} \mod 1, \quad x \in \mathbb{R}$$

is called the rotation number of $F$. This number always exists and does not depend on $x$ and $f$. Furthermore, $\alpha(F)$ is rational if and only if $F$ has a periodic point. If $\alpha(F) \notin \mathbb{Q}$, then the non-empty set

$$L_F := \{F^n(z), n \in \mathbb{Z}\}^d,$$

(the limit set of $F$) does not depend on $z \in \mathbb{S}^1$ (see for instance [9] and [10]).

By the limit set of a disjoint iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ we mean the set

$$L_{\mathcal{F}} := \{F^v(z), v \in V\}^d$$

which does not depend on $z \in \mathbb{S}^1$. By the limit set of a non-singular iteration group $\mathcal{F}$ we mean the set

$$L_{\mathcal{F}} := L_{F^v},$$

where $F^v \in \mathcal{F}$ is an arbitrary homeomorphism with $\alpha(F^v) \notin \mathbb{Q}$. This set does not depend on the choice of such a homeomorphism.

Although the above definitions are different, in the case when the iteration group is both disjoint and non-singular they determine the very same set.

A non-singular or disjoint iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ is called:

— dense, if $L_{\mathcal{F}} = \mathbb{S}^1$;
— non-dense, if $\emptyset \neq L_{\mathcal{F}} \neq \mathbb{S}^1$;
— discrete, if $L_{\mathcal{F}} = \emptyset$.

It is worth pointing out that every discrete iteration group is both disjoint and singular, and every dense iteration group is disjoint (see [6]). Therefore we shall investigate only non-dense non-singular iteration groups.

We now repeat the relevant, slightly modified, material from [6].

**Lemma 1** (see [6] and also [8]). If $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ is a dense or non-dense iteration group, then there exists a unique pair...
(\varphi_{\mathcal{F}}, c_{\mathcal{F}}) such that \varphi_{\mathcal{F}} : S^1 \rightarrow S^1 is a continuous function of degree 1 with \varphi_{\mathcal{F}}(1) = 1 and c_{\mathcal{F}} : V \rightarrow S^1 satisfying the following system of functional equations

\varphi_{\mathcal{F}}(F^v(z)) = c_{\mathcal{F}}(v)\varphi_{\mathcal{F}}(z), \quad z \in S^1, \; v \in V.

Regarding the structure of dense non-singular iteration groups we have the following theorem.

Theorem 1 (see [6]). If \mathcal{F} = \{F^v : S^1 \rightarrow S^1, \; v \in V\} is a dense iteration group, then there exists a unique orientation-preserving homeomorphism \varphi_{\mathcal{F}} : S^1 \rightarrow S^1 having fixed point 1 such that

F^v(z) = \varphi_{\mathcal{F}}^{-1}(e^{2\pi i \alpha(F^v)}\varphi_{\mathcal{F}}(z)), \quad z \in S^1, \; v \in V.

If \mathcal{F} = \{F^v : S^1 \rightarrow S^1, \; v \in V\} is a non-dense iteration group, then its limit set is a non-empty perfect and nowhere dense subset of \( S^1 \), and therefore we have the following decomposition

\( S^1 \setminus L_{\mathcal{F}} = \bigcup_{q \in \mathbb{Q}} I_q, \)

where \( I_q \) for \( q \in \mathbb{Q} \) are open pairwise disjoint arcs.

Lemma 2 (see [6]). If \( \mathcal{F} = \{F^v : S^1 \rightarrow S^1, \; v \in V\} \) is a non-dense iteration group, then:

(i) for every \( q \in \mathbb{Q} \) the mapping \( \varphi_{\mathcal{F}} \) is constant on \( I_q \),

(ii) for any distinct \( p, q \in \mathbb{Q} \), \( \varphi_{\mathcal{F}}[I_p] \cap \varphi_{\mathcal{F}}[I_q] = \emptyset \),

(iii) \( \varphi_{\mathcal{F}}[S^1 \setminus L_{\mathcal{F}}] \cdot \text{Im} c_{\mathcal{F}} = \varphi_{\mathcal{F}}[S^1 \setminus L_{\mathcal{F}}] \).

According to Lemma 2 we can correctly define

\( \{\Phi_{\mathcal{F}}(q)\} := \varphi_{\mathcal{F}}[I_q], \quad q \in \mathbb{Q} \)

and

\( T_{\mathcal{F}}(q, v) := \Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q)c_{\mathcal{F}}(v)), \quad q \in \mathbb{Q}, \; v \in V. \)

Lemma 3 (see [6]). If \( \mathcal{F} = \{F^v : S^1 \rightarrow S^1, \; v \in V\} \) is a non-dense iteration group, then:

(i) \( T_{\mathcal{F}}(T_{\mathcal{F}}(q, v_1), v_2) = T_{\mathcal{F}}(q, v_1 + v_2) \) for \( q \in \mathbb{Q}, \; v_1, \; v_2 \in V, \)

(ii) \( T_{\mathcal{F}}(q, 0) = q \) for \( q \in \mathbb{Q}, \)

(iii) \( F^v[I_q] = I_{T_{\mathcal{F}}(q, v)} \) for \( q \in \mathbb{Q}, \; v \in V. \)

3. Main results

We start with some auxiliary results which are valid without any assumption on the iteration group \( \mathcal{F} = \{F^v : S^1 \rightarrow S^1, \; v \in V\}. \)
It is easily seen that we have

**Remark 1.** Let $\mathcal{F} = \{F^v : S^1 \rightarrow S^1, v \in V\}$ be an iteration group. If $z_0 \in S^1$ is a fixed point of $F^v \in \mathcal{F}$, then so is $F^w(z_0)$ for $w \in V$.

**Lemma 4.** Assume that $\mathcal{F} = \{F^v : S^1 \rightarrow S^1, v \in V\}$ is an iteration group. Then:

(i) if $F^{v_1}, F^{v_2} \in \mathcal{F}$ have fixed points, then they have a common fixed point,

(ii) if $\alpha(F^{v_0}) \in \mathbb{Q}$ for a $v_0 \in V$, then $\alpha(F^{r v_0}) \in \mathbb{Q}$ for $r \in \mathbb{Q}$.

**Proof.** (i) Fix $v_1, v_2 \in V$ and assume that $z_1 \in S^1$ is a fixed point of $F^{v_1}$. If $F^{v_2}(z_1) = z_1$, then $z_1$ has the desired property. Now, assume that $F^{v_2}(z_1) \neq z_1$ and let $z_2 \in S^1$ be a fixed point of $F^{v_2}$. If $z_2$ is the unique fixed point of $F^{v_2}$, then from Remark 1 it follows that $F^{v_1}(z_2) = z_2$. Finally, we turn to the case when $F^{v_2}$ has at least two fixed points.

Denote by $(a_1, b_1)$ the maximal open arc without fixed points of $F^{v_2}$ such that $z_1 \in (a_1, b_1)$. Since $F^{v_2}(a_1) = a_1, F^{v_2}(b_1) = b_1$ and the homeomorphism $F^{v_2}$ preserves orientation, we have $F^{v_2}(z_1) \in (a_1, b_1)$.

This together with $F^{v_2}(z_1) \neq z_1$ shows that either $F^{v_2}(z_1) \in (a_1, z_1)$ or $F^{v_2}(z_1) \in (z_1, b_1)$. Assume, for instance, that $F^{v_2}(z_1) \in (a_1, z_1)$.

Then $F^{n v_2}(z_1) \in (a_1, F^{(n-1)v_2}(z_1))$ for $n \in \mathbb{N}$ and consequently

\[
F^{l v_2}(z_1) \in (F^{kv_2}(z_1), F^{j v_2}(z_1))
\]

for $j, l, k \in \mathbb{N} \cup \{0\}$ with $j < l < k$.

Suppose that there are subsequences

\[
(F^{m k v_2}(z_1))_{k \in \mathbb{N} \cup \{0\}}, \quad (F^{m k v_2}(z_1))_{k \in \mathbb{N} \cup \{0\}}
\]

of the sequence $(F^{n v_2}(z_1))_{n \in \mathbb{N} \cup \{0\}}$ for which

\[
\lim_{k \to \infty} F^{m k v_2}(z_1) = g_1 \neq g_2 = \lim_{k \to \infty} F^{m k v_2}(z_1),
\]

where

\[
g_1, g_2 \in (a_1, b_1) \cup \{a_1, b_1\}
\]

and let $O_{g_1}, O_{g_2}$ be neighbourhoods of $g_1$ and $g_2$, respectively, with $O_{g_1} \cap O_{g_2} = \emptyset$. Then, by (1), there exist non-negative integers $m_k, n_{k_1}, n_{k_2}$ such that $n_{k_2} < m_k < n_{k_1}$ and

\[
(F^{n_{k_1} v_2}(z_1), F^{n_{k_2} v_2}(z_1)) \subset O_{g_1}, \quad F^{m k v_2}(z_1) \in O_{g_2}.
\]

Therefore $F^{m k v_2}(z_1) \notin (F^{n_{k_1} v_2}(z_1), F^{n_{k_2} v_2}(z_1))$, contrary to (1).
We have thus shown that the sequence \((F^{nv_2}(z_1))_{n \in \mathbb{N}\cup\{0\}}\) is convergent. It is obvious that its limit, which will be denoted by \(g\), is a fixed point of \(F^{v_2}\). Moreover,

\[
F^{v_1}(g) = \lim_{n \to \infty} F^{v_1}(F^{nv_2}(z_1)) = \lim_{n \to \infty} F^{nv_2}(F^{v_1}(z_1)) = \lim_{n \to \infty} F^{nv_2}(z_1) = g.
\]

(ii) Let \(z_0 \in S^1\) and \(n \in \mathbb{Z}\setminus\{0\}\) be such that \(F^{nv_0}(z_0) = z_0\). Fix an \(r \in \mathbb{Q}\) and take \(k \in \mathbb{Z}\), \(l \in \mathbb{N}\) for which \(r = \frac{k}{l}\). Putting \(m := nl \in \mathbb{Z}\setminus\{0\}\) we get

\[
F^{mrv_0}(z_0) = (F^{nv_0})^k(z_0) = z_0,
\]

which gives \(\alpha(F^{rv_0}) \in \mathbb{Q}\). \(\diamondsuit\)

**Corollary 1.** If \(\mathcal{F} = \{F^w : S^1 \to S^1, w \in \mathbb{Q}\}\) is an iteration group, then either \(\alpha(F^w) \in \mathbb{Q}\) for \(w \in \mathbb{Q}\) or \(\alpha(F^w) \notin \mathbb{Q}\) for \(w \in \mathbb{Q}\setminus\{0\}\).

**Definition.** An iteration group \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is said to be strictly disjoint if the fact that \(F^v \in \mathcal{F}\) has a fixed point implies \(v = 0\).

**Lemma 5.** If \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is an iteration group, then the following conditions are equivalent:

1. \(\mathcal{F}\) is strictly disjoint,
2. \(\alpha(F^v) \notin \mathbb{Q}\) for \(v \in V \setminus \{0\}\),
3. for any \(z \in S^1\) the mapping \(V \ni v \mapsto F^v(z) \in S^1\) is an injection.

**Proof.** It is immediate that (ii) yields (i). Now, assume that (i) holds true and let \(\alpha(F^v) \in \mathbb{Q}\) for a \(v \in V\). Then \(F^{nv} \in \mathcal{F}\) has a fixed point for an \(n \in \mathbb{Z}\setminus\{0\}\), which together with (i) gives \(nv = 0\), and consequently \(v = 0\). To finish the proof it suffices to observe that conditions (i) and (iii) are also equivalent. \(\diamondsuit\)

Let us observe that every strictly disjoint iteration group \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is disjoint and, by Lemma 5, non-singular.

**Lemma 6.** If \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is an iteration group, then the set

\[
U_\mathcal{F} := \{u \in V : \alpha(F^u) \in \mathbb{Q}\}
\]

is a linear subspace of \(V\).

**Proof.** Since \(0 \in U_\mathcal{F}\), we have \(U_\mathcal{F} \neq \emptyset\). Fix \(u_1, u_2 \in U_\mathcal{F}\) and let \(z_1, z_2 \in S^1, n_1, n_2 \in \mathbb{Z}\setminus\{0\}\) be such that \(F^{n_1u_1}(z_1) = z_1\) and \(F^{n_2u_2}(z_2) = z_2\). By Lemma 4(i) there is a \(z_0 \in S^1\) for which \(F^{n_1u_1}(z_0) = z_0 = F^{n_2u_2}(z_0)\), and therefore
\[(F^{u_1+u_2})^{n_1n_2}(z_0) = F^{n_1n_2u_1}(F^{n_1n_2u_2}(z_0)) = z_0.\]

As \(n_1n_2 \in \mathbb{Z} \setminus \{0\}\), we get \(\alpha(F^{u_1+u_2}) \in \mathbb{Q}\), and consequently \(u_1 + u_2 \in U_\mathcal{F}\). To finish the proof it suffices to apply Lemma 4(ii). \(\Diamond\)

The following fact follows immediately from Lemma 5.
\begin{corollary}
Assume that \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is an iteration group and let \(W\) be a complementary subspace to \(U_\mathcal{F}\) in \(V\). Then
\[\mathcal{F}_W := \{F^w : S^1 \to S^1, w \in W\}\]
is a strictly disjoint iteration group if \(\dim W \geq 1\), whereas \(\mathcal{F}_W = \{\text{id}\}\) if \(W = \{0\}\).

It is easily seen that we also have
\begin{remark}
Assume that \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is an iteration group and let \(W\) be a complementary subspace to \(U_\mathcal{F}\) in \(V\). Then \(\mathcal{F}\) is non-singular if and only if \(\dim W \geq 1\).

From now on we shall make some assumptions on iteration groups under study.

We start with
\begin{lemma}
Let \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) be a non-singular (respectively, singular and disjoint) iteration group. If \(F^v \in \mathcal{F}\) has a fixed point, then
\[F^v(z) = z, \quad z \in L_\mathcal{F}.\]
\end{lemma}

\begin{proof}
Let \(v \in V\) and \(z_0 \in S^1\) be such that \(F^v(z_0) = z_0\). If the iteration group \(\mathcal{F}\) is singular and disjoint, then our assertion follows from Remark 1 and the continuity of \(F^v\). Next, assume that \(\mathcal{F}\) is non-singular and let \(w \in V\) be such that \(\alpha(F^w) \notin \mathbb{Q}\). Using the same arguments as before we see that
\[F^v(z) = z, \quad z \in \{F^{nw}(z_0), n \in \mathbb{Z}\}^d = L_{F^w} = L_\mathcal{F}. \quad \Box\]

Next, let us note that an immediate consequence of Remark 2 and Cor. 2 is
\begin{corollary}
Assume that \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is a non-singular iteration group and let \(W\) and \(\mathcal{F}_W\) be as in Cor. 2. Then
\[L_\mathcal{F} = L_{\mathcal{F}_W}.\]
\end{corollary}

\begin{lemma}
If \(\mathcal{F} = \{F^v : S^1 \to S^1, v \in V\}\) is a non-dense iteration group and \(F^u \in \mathcal{F}\) has a fixed point, then:
\begin{enumerate}
\item \(F^u[I_p] = I_p\) for \(p \in \mathbb{Q}\),
\item \(F^{u+v}[I_p] = I_{T_\mathcal{F}(p, v)}\) for \(p \in \mathbb{Q}\), \(v \in V\).
\end{enumerate}
\end{lemma}
Proof. (i) Fix a $p \in \mathbb{Q}$ and let $a_p, b_p \in \mathbb{S}^1$ be such that $I_p = (a_p, b_p)$. Since $a_p, b_p \in I_{\mathcal{F}}$, from Lemma 7 it follows that $F^u(a_p) = a_p$ and $F^u(b_p) = b_p$, which together with the fact that $F^u$ preserves orientation gives

$$F^u[I_p] = (F^u(a_p), F^u(b_p)) = I_p.$$ 

(ii) Fix $p \in \mathbb{Q}$, $v \in V$. Using (i) and Lemma 3(iii) we obtain

$$F^{u+v}[I_p] = F^u[F^v[I_p]] = F^u[I_{T_{\mathcal{F}}(p, v)}] = I_{T_{\mathcal{F}}(p, v)}. \quad \Diamond$$

Lemma 9. Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ is a non-dense non-singular iteration group and let $W$ be a complementary subspace to $U_{\mathcal{F}}$ in $V$. If $T_{\mathcal{F}}(p, w_1) = T_{\mathcal{F}}(p, w_2)$ for some $p \in \mathbb{Q}$, $w_1, w_2 \in W$, then $w_1 = w_2$.

Proof. Fix $p \in \mathbb{Q}$, $w_1, w_2 \in W$ for which $T_{\mathcal{F}}(p, w_1) = T_{\mathcal{F}}(p, w_2)$. Then, by Lemma 3(i) and (ii), we have

$$T_{\mathcal{F}}(p, w_1 - w_2) = T_{\mathcal{F}}(T_{\mathcal{F}}(p, w_1), -w_2) = T_{\mathcal{F}}(T_{\mathcal{F}}(p, w_2), -w_2) = T_{\mathcal{F}}(p, 0) = p$$

and Lemma 3(iii) now shows that

$$(2) \quad F^{w_1-w_2}[I_p] = I_p.$$ 

Let $a_p, b_p \in \mathbb{S}^1$ be such that $I_p = (a_p, b_p)$. Since $F^{w_1-w_2}$ is an orientation-preserving homeomorphism, from (2) it follows that $F^{w_1-w_2}(a_p) = a_p$. Therefore $\alpha(F^{w_1-w_2}) \in \mathbb{Q}$, and consequently $w_1 - w_2 \in U_{\mathcal{F}}$. But we also have $w_1 - w_2 \in W$, and $U_{\mathcal{F}} \cap W = \{0\}$ finally yields $w_1 = w_2. \quad \Diamond$

The following fact follows immediately from Lemma 9.

Corollary 4. Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ is a non-dense non-singular iteration group and let $W$ be a complementary subspace to $U_{\mathcal{F}}$ in $V$. Then the mapping $c_{\mathcal{F}} | W : W \rightarrow \mathbb{S}^1$ is an injection.

Lemma 10. Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \rightarrow \mathbb{S}^1, v \in V\}$ is a non-dense non-singular iteration group and let $W$ be a complementary subspace to $U_{\mathcal{F}}$ in $V$. Then

$$(3) \quad 1 \leq \dim W \leq \aleph_0.$$ 

Proof. According to Remark 2 it suffices to show that $\dim W \leq \aleph_0$. Let the iteration group $\mathcal{F}_W$ be as in Cor. 2. This group is non-singular and, by Cor. 3, non-dense. Therefore from Lemma 2 and Cor. 4 it
follows that \( \text{cardIm} \, c_{F_W} = \aleph_0 \) and the mapping \( c_{F_W} : W \to S^1 \) is an injection. Consequently,
\[
\text{dim} W \leq \text{card} W = \text{cardIm} c_{F_W} = \aleph_0.
\]

\textbf{Theorem 2.} If \( F = \{F^v : S^1 \to S^1, \ v \in V\} \) is a non-dense non-singular iteration group, then there is a linear subspace \( W \) of \( V \) satisfying condition (3) and a linear subspace \( U \) of \( V \) such that \( V = U \oplus W \) and
\[
F = F_U \oplus F_W,
\]
where \( F_W := \{F^w : S^1 \to S^1, \ w \in W\} \) is a strictly disjoint non-dense iteration group with \( L_{F_W} = L_F \) and \( F_U := \{F^u : S^1 \to S^1, \ u \in U\} \) is a singular iteration group if \( \text{dim} U \geq 1 \), whereas \( F_U = \{\text{id}\} \) if \( U = \{0\} \).

\textbf{Proof.} Put \( U := U_F \) and note that, by Lemma 6 \( U \) is a linear subspace of \( V \). Let \( W \) be a complementary subspace to \( U \) in \( V \). Since from Lemma 10 it follows that \( W \) satisfies (3), Corollaries 2 and 3 show that \( F_W := \{F^w : S^1 \to S^1, \ w \in W\} \) is a strictly disjoint non-dense iteration group for which \( L_F = L_{F_W} \). It is also obvious that \( F_U := \{F^u : S^1 \to S^1, \ u \in U\} \) is a singular iteration group if \( \text{dim} U \geq 1 \), whereas \( F_U = \{\text{id}\} \) if \( U = \{0\} \). Finally, \( F_U \) and \( F_W \) are subgroups of \( (\mathcal{F}, \circ) \) with \( F_U \cap F_W = \{\text{id}\} \) and
\[
F_U \circ F_W = \{F_1 \circ F_2, \ F_1 \in F_U, \ F_2 \in F_W\}
\]
\[
= \{F^u \circ F^w : S^1 \to S^1, \ u \in U, \ w \in W\}
\]
\[
= \{F^{u+w} : S^1 \to S^1, \ u \in U, \ w \in W\}
\]
\[
= \{F^v : S^1 \to S^1, \ v \in V\} = F.
\]

\textbf{References}


K. Ciepliński: On the structure of non-singular iteration groups on the circle


