COMMUTATIVITY OF THE TOPOLOGICAL SEQUENCE ENTROPY ON FINITE GRAPHS

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Abstract: Let $f, g : G \rightarrow G$ be two continuous maps defined on a finite graph $G$. Denote by $h_A(f)$ the topological sequence entropy of $f$ relative to the sequence of positive integers $A$. We prove for any sequence $A$ the formula $h_A(f \circ g) = h_A(g \circ f)$.

1. Introduction

Let $(X, d)$ be a compact metric space and consider maps $F : X \times X \rightarrow X \times X$ defined by $F(x, y) = (f(y), g(x))$, $(x, y) \in X \times X$, where $f, g : X \rightarrow X$ are continuous maps. These maps model economic phenomena called duopoly games (see [4], [12] or [11]). Notice that, for any $(x, y) \in X \times X$, it holds that

$$F^2(x, y) = F(F(x, y)) = (f \circ g(x), g \circ f(y)).$$

So, the dynamical behaviour of $F$ must be connected in some sense with

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the dynamical behaviour of the maps \( f \circ g \) and \( g \circ f \). Following this idea, when \( X = [0, 1] \), some dynamical properties of \( F \) were studied in [10].

In this setting, in order to avoid unnecessary work, it is interesting to study which is the relationship between the dynamical properties of \( f \circ g \) and \( g \circ f \). For instance, in the case of the topological entropy, it is well known that \( h(f \circ g) = h(g \circ f) \) (see [4] and [9]) and hence it is easy to see that \( h(F) = h(f \circ g) = h(g \circ f) \). It is natural to think that a similar situation is held for others topological invariants. However, it was proved in [2] that the topological sequence entropy does not satisfy this property: for the sequence \( A = (2^i)_{i=1}^{\infty} \) there are two continuous maps \( f, g : X \to X \), with \( X \) a Cantor type set, such that \( h_A(f \circ g) \neq h_A(g \circ f) \). This situation is impossible when one considers the spaces \( X = [0, 1] \) or \( X = S^1 \); for any pair of continuous interval or circle maps \( f, g \) the formula \( h_A(f \circ g) = h_A(g \circ f) \) holds for any increasing sequence of positive integers \( A \) (see [2]). In this paper we will extend this result for maps defined on finite graphs.

2. Preliminaries

Let \((X, d)\) be a compact metric space. Let us denote by \( C(X, X) \) and \( I \) the sets containing all the continuous maps \( f : X \to X \) and all the increasing sequences of positive integers, respectively. For all \( n \in \mathbb{N} \), \( f^n \) will denote the composition \( f \circ \ldots \circ f \) (\( f^0 \) will be the identity). Given an \( f \in C(X, X) \) and \( A = (a_i)_{i=1}^{\infty} \in I \), the topological sequence entropy (see [7]) is defined as follows. Let \( Z \subset X \) and let \( \varepsilon > 0 \). A set \( E \subset Z \) is said \((A, n, \varepsilon, Z, f)\)-separated if for any \( x, y \in E \), \( x \neq y \) there is a \( k \in \{1, 2, \ldots, n\} \) with \( d(f^{a_k}(x), f^{a_k}(y)) > \varepsilon \). Denote by \( s_n(A, \varepsilon, Z, f) \) the cardinality of any maximal \((A, n, \varepsilon, Z, f)\)-separated contained in \( Z \).

It is easy to see that if \( Z_1 \subset Z_2 \subset X \), then

\[
(1) \quad s_n(A, \varepsilon, Z_1, f) \leq s_n(A, \varepsilon, Z_2, f).
\]

It is also easy to check that for any \( Z_1, Z_2 \subset X \) it holds that

\[
(2) \quad s_n(A, \varepsilon, Z_1 \cup Z_2, f) \leq s_n(A, \varepsilon, Z_1, f) + s_n(A, \varepsilon, Z_2, f).
\]

Let

\[
s(A, \varepsilon, Z, f) := \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \varepsilon, Z, f).
\]

The topological sequence entropy of \( f \) in \( Y \) is defined as the number
The topological sequence entropy of $f$ is
\[ h_A(f, Y) := \lim_{\varepsilon \to 0} s(A, \varepsilon, Y, f) \]
and the topological sequence entropy of $f$ is
\[ h_A(f) := h_A(f, X). \]
Clearly, when $A = (i)_{i=0}^{\infty}$ this definition leads us to the classical topological entropy (see [3]). When one dimensional maps are consider, the topological sequence entropy is a useful tool to check if a continuous map is chaotic in the sense of Li–Yorke (see [6] and [8]).

Recall that a point $x \in X$ is said periodic if there exists a positive integer $n$ such that $f^n(x) = x$. The smallest positive integer satisfying this condition is called the period of $x$. A point $x$ is eventually periodic if there exists a positive integer $k$ such that $f^k(x)$ is periodic. Denote by $\text{Per}(f)$ and $\text{EPer}(f)$ the sets of periodic and eventually periodic points of $f$, respectively.

A finite graph (or simply a graph) $G$ is a connected Hausdorff space which has a finite subspace $V$ (points of $V$ are called vertices) such that $G \setminus V$ is a disjoint union of finite number of open subsets $e_1, e_2, \ldots, e_k$ (called edges), each of them homeomorphic to an open interval of the real line, and one or two vertices are attached at the boundary of each edge. A graph $G$ can be embedded in a closed ball of radius one, and hence $G$ is a compact metric space. As usual, denote by $d$ the metric on $G$. For any edge $e_i$, denote by $|e_i|$ its diameter. Since the number of edges of $G$ is finite, let
\[ \lambda = \lambda(G) = \min\{|e_i| : e_i \text{ is an edge of } G\}. \]
For $x, y \in e_i$, let $[x, y] \subseteq e_i$ be the arc of $G$ connecting $x$ and $y$. For complementary information on graphs and dynamic properties of continuous maps defined on graphs see for instance [1].

3. Proof of the commutativity formula

The commutativity formula for the topological sequence entropy is deeply connected with the surjectivity of the maps $f$ and $g$. More precisely, let $X$ be a compact metric space and let $f \in C(X, X)$. Let $Y = \bigcap_{n \geq 0} f^n(X)$. Then, we have the following result (see [2]).

**Theorem 1.** If $h_A(f|Y) = h_A(f)$ holds for any $f \in C(X, X)$ and any $A \in \mathcal{I}$, then
\[ h_A(f \circ g) = h_A(g \circ f) \]
for any $f, g \in C(X, X)$ and any $A \in \mathcal{I}$. 

So, given a finite graph $G$, in order to prove the commutativity formula for maps $f, g \in C(G, G)$ it suffices to prove that $h_{\mathcal{A}}(f|_{Y}) = h_{\mathcal{A}}(f)$ for any $f \in C(G, G)$ and any $\mathcal{A} \in \mathcal{I}$. Previously, we need some useful definitions and several easy lemmas.

Notice that $f^n(G)$ is a finite graph for all $n \in \mathbb{N}$. Denote by $V_n$ the set of vertices of $f^n(G)$ for all $n = 0, 1, 2, \ldots$ Since $f^{n+1}(G) \subseteq f^n(G)$ for all $n \in \mathbb{N}$, it is clear that if $v \in V_n$ and $v \notin V_{n+1}$, then $v \notin Y$. Denote by $V_{\infty}$ the set of vertices of $Y = \bigcap_{n \geq 0} f^n(G)$. Here, we will also consider as vertices of $Y$ those points obtained as limit points of sequences $(v_i)_{i=0}^{\infty}$ with $v_i \in V_i$. In order to illustrate this, consider the following example. Let $Y = \{ z \in \mathbb{C} : z^3 \in [0, 1] \}$. Denote by $B_1$ the branch of $Y$ with vertices 0 and 1, that is, $B_1 = [0, 1]$. Denote by $B_2$ and $B_3$ the others two branches of $Y$ with vertices $v_2$ and $v_3$. Define $f : Y \to Y$ as follows. If $x \in B_1$, then let $f(x) = x/2$, and define $f$ on $B_2 \cup B_3$ satisfying that $f(B_2 \cup B_3) = B_2 \cup B_3$ and continuous ($f(0) = 0$). Notice that $Y = \bigcap_{n \geq 0} f^n(Y) = B_2 \cup B_3$ and the vertices of $Y$ are $v_2, v_3$ and 0.

Let $\lambda$ be defined in (3). The following result is obvious.

**Lemma 2.** Let $0 < \varepsilon < \lambda/2$. Then, there is a positive integer $n_0$ such that each connected component of $f^{n_0}(G) \setminus Y$ has length smaller than $\varepsilon$. Hence, each connected component of $f^{n_0}(G) \setminus Y$ is homeomorphic to an interval of the real line.

**Proof.** It follows because $f$ is uniformly continuous and the sequence $(f^i(G))_{i=0}^{\infty}$ decreases to $Y$. ◊

For $n \in \mathbb{N}$, $n \geq n_0$, denote by $C_1^n, C_2^n, \ldots, C_r^n$ the connected components of $f^n(G) \setminus Y$. Notice that, for $1 \leq i \leq r$, the closure of $C_i^n$, $\text{Cl}(C_i^n) = [v, u]$ with $v \in V_\infty$ and $u \in V_n$. For any $v \in V_\infty$, let $i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, r\}$ be such that $v \in \text{Cl}(C_{i_j}^n)$, $1 \leq j \leq s$. Define $C_v^n = \bigcup_{j=1}^{s} \text{Cl}(C_{i_j}^n)$. Notice that it is possible that $C_v^n = \emptyset$ for some $v \in V_\infty$. It is also clear that $C_v^{n+1} \subseteq C_v^n$ for all $n \geq n_0$. We distinguish four types of vertices of $Y$ in the following lemma.

**Lemma 3.** Let $v \in V_\infty$. Under the conditions of Lemma 2, there is a $n_0 \in \mathbb{N}$ such that one and only one of the following possibilities holds:

(a) $C_v^{n_0} = \emptyset$.

(b) $C_v^{n_0} = \{v\}$.

(c) $C_v^{n_0}$ is infinite and $f^{n_0}(C_v^{n_0}) \subseteq Y$.

(d) $C_v^{n_0}$ is infinite and $f^n(C_v^n) \not\subseteq Y$ for all $n \geq n_0$.
Proof. By Lemma 2, there is an \( m_0 \in \mathbb{N} \) and some vertices \( u \in V_{m_0} \) such that \( C_{v}^{m_0} = \bigcup u[v, u] \). If \( C_{v}^{m_0} = \emptyset \) or \( C_{v}^{m_0} = \{v\} \), then there is nothing to prove. Assume that \( C_{v}^{m_0} \neq \emptyset \) and \( C_{v}^{m_0} \neq \{v\} \). Clearly, \( C_{v}^{m_0} \) must be infinite. Let \( V_{m_0} = \{v \in V_{\infty} : C_{v}^{m_0} \text{ is infinite}\} \). For any \( v \in V_{m_0} \) two possibilities hold: either \( f^k(C_{v}^{m_0}) \nsubseteq Y \) for all \( k \in \mathbb{N} \) or there is a \( k_v \in \mathbb{N} \) such that \( f^{k_v}(C_{v}^{m_0}) \subseteq Y \). Since \( V_{m_0} \) is finite, let \( k_{v_1}, \ldots, k_{v_j} \) be positive integers associated to \( v_i \in V_{m_0}^{\infty} \), \( 1 \leq i \leq j \), such that \( f^{k_{v_i}}(C_{v_i}^{m_0}) \subseteq Y \) for \( 1 \leq i \leq j \). Suppose also that if \( v \in V_{m_0}^{\infty} \setminus \{v_1, \ldots, v_j\} \), then \( f^k(C_{v}^{m_0}) \nsubseteq Y \) for all \( k \in \mathbb{N} \). Let \( n_0 = \max\{k_{v_1}, \ldots, k_{v_j}, m_0\} \). Notice that, since \( C_{v}^{m_0} \subseteq C_{v_i}^{m_0} \) for all \( v \in V_{\infty} \) and \( f(Y) = Y \), it holds that \( f^{n_0}(C_{v_i}^{m_0}) \subseteq Y \) for all \( i = 1, \ldots, j \). This concludes the proof. \( \diamond \)

Let \( V_{\infty}^i = \{v \in V_{\infty} : v \text{ satisfies condition (d) in Lemma 3}\} \).

Lemma 4. \( f(V_{\infty}^i) \subseteq V_{\infty}^i \). Moreover, since \( V_{\infty}^i \) is finite, each \( v \in V_{\infty}^i \) is periodic or eventually periodic.

Proof. Let \( v \in V_{\infty}^i \). Then, there is a sequence of vertices \( v_n \in V_{\infty} \), \( v_n \neq v \) for all \( n \in \mathbb{N} \), such that \( \lim_{n \to \infty} v_n = v \) and holding that, if \( n \) is big enough, then \( f^k[v, v_n] \nsubseteq Y \) for all \( k \in \mathbb{N} \). Since \( f \) is continuous, \( \lim_{n \to \infty} f(v_n) = f(v) \).

Notice that \( f(v) \notin Y \setminus V_{\infty} \). In the contrary case, by the continuity of \( f \), it must exist a \( k \in \mathbb{N} \) with \( f(C_{v}^k) \subseteq Y \setminus V_{\infty} \), and this leads to a contradiction. Using a similar argument it can be proved that \( f(v) \notin V_{\infty} \setminus V_{\infty}^i \). Finally, \( f(v) \notin G \setminus Y \) because \( f(v) \) has infinite preimages and any point in \( G \setminus Y \) has a finite number of preimages. This concludes the proof. \( \diamond \)

Now, a general lemma on topological sequence entropy previously proved in [2]. Let \( \sigma : \mathcal{I} \to \mathcal{I} \) be the shift map defined by \( \sigma(A) = \sigma((a_i)_{i=1}^{\infty}) = (a_{i+1})_{i=1}^{\infty} \) for all \( A \in \mathcal{I} \).

Lemma 5. Let \( (X, d) \) be a compact metric space and let \( f \in C(X, X) \). Then, for any \( A \in \mathcal{I} \) any \( \varepsilon > 0 \) and any \( k \in \mathbb{N} \) it holds that

\[
s(A, 2\varepsilon, X, f) \leq s(\sigma^k(A), \varepsilon, X, f) \leq s(A, \varepsilon, X, f).
\]

Now, we are ready to prove our main theorem.

Theorem 6. Let \( f : G \to G \) be continuous. Then, for any \( A \in \mathcal{I} \) it holds that

\[
h_A(f) = h_A(f, Y).
\]

Proof. Fix a positive real number \( \varepsilon < \lambda/2 \) (see (3)). Since \( V_{\infty} \) is finite, by Lemmas 2, 3 and 4, there is a positive integer \( n_0 \) satisfying the following conditions:
(C1) \( \text{diam}(C_v^{n_0}) < \varepsilon \) for all \( v \in V_{\infty} \).

(C2) If \( v \in V_{\infty} \setminus V_{\infty}^i \), then \( f^{n}(C_v^{n_0}) \subset Y \) for all \( n \geq n_0 \).

(C3) \( f(V_{\infty}^i) \subset V_{\infty}^i \subset \text{Per}(f) \cup \text{EPer}(f) \).

(C4) Let \( v \in V_{\infty} \). If \( f(v) = u \in V_{\infty} \), then there is a \( \delta > 0 \) such that if \( U \) is a neighborhood of diameter smaller than \( \delta \) of some \( w \in V_{\infty} \), \( v \neq u \), then \( f(C_v^{n_0}) \cap U = \emptyset \). We can clearly assume that \( \varepsilon \leq \delta \).

Let \( k \) be the first integer such that \( a_{k+1} > n_0 \). By Lemma 5, it holds that

\[
(4) \quad s(A, 4\varepsilon, G, f) \leq s(\sigma^k(A), 2\varepsilon, G, f).
\]

In what follows, we will work with \( \sigma^k(A) \) instead of \( A \).

Take a partition of \( f^{n_0}(G) \setminus Y \) by connected sets with diameter smaller than \( \varepsilon \) homeomorphic to intervals. Let \( \mathcal{P}_1 = \{P_1, P_2, \ldots, P_r\} \) be the partition covering \( f^{n_0}(G) \setminus Y \). Clearly, if \( P_i \in \mathcal{P}_1 \), then \( f^j(P_i) \cap P_j = \emptyset \) for any \( j > n_0 \). Let \( \mathcal{P}_2 = \{C_v^{n_0} : v \in V_{\infty}\} \). So, we can construct a partition of \( G \setminus Y \) by

\[
\mathcal{P} = \{P_1, P_2, \ldots, P_r\} \cup \{C_v^{n_0} : v \in V_{\infty}\}.
\]

Fix \( n \in \mathbb{N} \). Any \( x \in G \setminus Y \) has associated a code \( (C_1, C_2, \ldots, C_l) \), \( l \leq n \), as follows; let \( l \) be the first integer such that \( f^{a_{k+i+1}}(x) \in Y \). For \( 1 \leq i \leq l \), put \( C_i = C_v^{n_0} \) if \( f^{a_{k+i}}(x) \in C_v^{n_0} \). Notice that it is impossible that \( f^{a_{k+i}}(x) \in P_j \) for some \( 1 \leq j \leq r \). Let

\[
Z(C_1, C_2, \ldots, C_l) = \{x \in G \setminus Y \text{ with code } (C_1, C_2, \ldots, C_l)\}.
\]

Let \( E \) be an \( (\sigma^k(A), n, \varepsilon, Y, f) \)-separated set of maximal cardinality. We claim that

\[
(5) \quad s_n(\sigma^k(A), 2\varepsilon, Z(C_1, C_2, \ldots, C_l), f) \leq \text{Card}(E) = s_n(\sigma^k(A), \varepsilon, Y, f).
\]

In order to see this, let \( F \) be an \( (\sigma^k(A), n, 2\varepsilon, Z(C_1, C_2, \ldots, C_l), f) \)-separated set of maximal cardinality. Since \( f|_Y \) is surjective and \( E \) is maximal, any \( x \in F \) has associated a point \( y \in E \) such that \( d(f^{a_{k+i}}(x), f^{a_{k+i}}(y)) < \varepsilon \) for \( l < i \leq n \). Notice that different \( x_1, x_2 \in F \) have associated different points \( y_1, y_2 \in E \). This is due to the following fact: if \( x_1 \) and \( x_2 \) have associated the same \( y_1 \), then

\[
\begin{align*}
d(f^{a_{k+i}}(x_1), f^{a_{k+i}}(x_2)) &\leq d(f^{a_{k+i}}(x_1), f^{a_{k+i}}(y_1)) + d(f^{a_{k+i}}(y_1), f^{a_{k+i}}(x_2)) < \\
&< \varepsilon + \varepsilon = 2\varepsilon
\end{align*}
\]

for all \( l < i \leq n \). Since \( x \) and \( y \) have the same code \( (C_1, \ldots, C_l) \), \( d(f^{a_{k+i}}(x), f^{a_{k+i}}(y)) < 2\varepsilon \) for \( 1 \leq i \leq l \). Then, \( x_1, x_2 \) would not be
(σ^k(A), n, 2ε, Z(C_1, C_2, \ldots, C_l), f)\)-separated points. This proves our claim.

Let G_k = \{x ∈ G : f^{ak+l}(x) ∈ Y\}. Notice that G = G_k ∪ (∪_{l=1}^n ∪ U_{C_l}(z)Z(C_1, \ldots, C_l)). Notice also that

s(σ^k(A), 2ε, Y, f) = s(σ^k(A), 2ε, G_k, f).

By (2) and (5),

s_n(σ^k(A), 2ε, G, f) ≤

≤ s_n(σ^k(A), 2ε, Y, f) + ∑_{l=1}^n ∑_{C_l} s_n(σ^k(A), 2ε, Z(C_1, C_2, \ldots, C_l), f)

≤ s_n(σ^k(A), 2ε, Y, f)

\left(1 + ∑_{l=1}^n ∑_{C_l} \text{Card}\{(C_1, C_2, \ldots, C_l) : C_l ∈ \mathcal{P}\}\right).

So, we must compute the cardinality of \{(C_1, C_2, \ldots, C_l) : C_l ∈ \mathcal{P}\} for some 1 ≤ l ≤ n. First of all, notice that C_i ∈ \mathcal{P}_2 for all 1 ≤ i ≤ l. Then

\text{Card}\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}\} = \text{Card}\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}_2\}.

So, we will estimate Card\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}_2\}. Notice that if 1 < l, then, by (C2) and (C4), we must consider only codes C^{n_0}_v with v ∈ V^{i}_∞. Notice also that, by (C2) and (C4), if C_1 = C^{n_0}_v, then C_i = C^{n_0}_u with f^{ak+i}(u) = u ∈ V^{i}_∞. Then

\text{Card}\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}_2\} = \text{Card}(V^{i}_∞).

If 1 = l, then obviously Card\{(C_i)\} ≤ \text{Card}(V_∞). In any case

\text{Card}\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}\} ≤ \text{Card}(V_∞).

Hence

\sum_{l=1}^n \text{Card}\{(C_1, \ldots, C_l) : C_i ∈ \mathcal{P}\} ≤ \sum_{l=1}^n \text{Card}(V_∞) ≤ n \text{Card}(V_∞).

So

(6) \quad s_n(σ^k(A), 2ε, G, f) ≤ (1 + n \text{Card}(V_∞))s_n(σ^k(A), ε, Y, f).

Then, by Lemma 5 and (6), we have that

s(A, 4ε, G, f) ≤ s(σ^k(A), 2ε, G, f) =

= \lim_{n → ∞} \frac{1}{n} \log s_n(σ^k(A), 2ε, G, f) ≤

≤ \lim_{n → ∞} \frac{1}{n} \log ((1 + n \text{Card}(V_∞))s_n(σ^k(A), ε, Y, f)) =
\[ s(\sigma^k(A), \varepsilon, Y, f) \leq s(A, \varepsilon, Y, f). \]

Since obviously \( s(A, \varepsilon, Y, f) \leq s(A, \varepsilon, G, f) \), we obtain taking limits when \( \varepsilon \) tends to zero that
\[ h_A(f, Y) = h_A(f), \]
which ends the proof. \( \diamond \)

**Theorem 7.** Let \( f, g : G \to G \) be two continuous maps. Then for all \( A \in \mathcal{I} \) it follows
\[ h_A(f \circ g) = h_A(g \circ f). \]

**Proof.** It follows by Ths. 1 and 6. \( \diamond \)

**Final remarks**

It seems that the commutativity formula for the topological sequence entropy is a one dimensional property. As we have mentioned above, in [2] an example showing that Th. 7 does not hold for arbitrary compact metric spaces has been constructed. We also conjecture that Th. 7 does not hold in general in the case of two-dimensional maps, for example triangular maps \( (F : [0,1]^2 \to [0,1]^2 \text{ is said triangular if it has the form } F(x,y) = (f(x), g(x,y)), (x,y) \in [0,1]^2). \) Our conjecture is supported by the following result.

**Theorem 8.** There is a triangular map \( F \) and an increasing sequence of positive integers \( A \) such that \( h_A(F; Y) = 0 \) and \( h_A(F) > 0 \).

**Proof.** By [5], there is a triangular map \( F_\alpha(x, y) = (\alpha x, g(x, y)), \alpha \in (0,1) \) satisfying that:

(a) \( F \) is non-chaotic in the sense of Li–Yorke (see [5] for the definition).

(b) There is an increasing sequence of positive integers such that \( h_A(F) > 0 \).

It is easy to see that \( \bigcap_{n \geq 0} F^n([0,1]^2) \subset \{0\} \times [0,1] \). On the other hand, the map \( g_0 : [0,1] \to [0,1] \) given by \( g_0(y) = F(0, y) \) is non-chaotic (if \( g_0 \) was chaotic, then \( F \) would be also chaotic). By Franzová–Smítał Theorem (see [6]), \( h_A(g_0) = 0 \). Then, we conclude that
\[ h_A(F; Y) \leq h_A(g_0) = 0 < h_A(F), \]
which ends the proof. \( \diamond \)
References


