ON QUASI-CONTINUOUS ITERATION GROUPS ON THE UNIT CIRCLE

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Abstract: The aim of this paper is to give a characterization of iteration groups defined on the unit circle $S^1$, continuous with respect to the iterative parameter. Such groups are named quasi-continuous. The problem of the embeddability of a given function $T : S^1 \rightarrow S^1$ into quasi-continuous iteration groups is also considered.

Let $S^1 = \{x \in \mathbb{C} : |x| = 1\}$ be the unit circle. A set $L \subset S^1$ is said to be an open arc if

$$\overrightarrow{L} = (x_1, x_2) := \{e^{2\pi it} : t \in (t_1, t_2)\},$$

where $t_1, t_2 \in \mathbb{R}$ are such that $t_1 < t_2 \leq t_1 + 1$ and $x_1 = e^{2\pi it_1}$, $x_2 = e^{2\pi it_2}$. Similarly we define $(x_1, x_2)$, $[x_1, x_2)$, $[x_1, x_2]$, but with one different detail: $t_1 < t_2 < t_1 + 1$. Each of these four will be called an arc in this paper.

Let $L$ be an arc or $L = S^1$ or $L$ be a singleton. Let us introduce the following:

Definitions (see [3], also [6]). A family $\{T^t, t \in \mathbb{R}\}$ of functions $T^t : L \rightarrow L$ is said to be an iteration group on $L$ if

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$T^t \circ T^s = T^{t+s}$ for $t, s \in \mathbb{R}$.

If for every $x \in L$ the mapping $h_x : \mathbb{R} \rightarrow L$ given by $h_x(t) := T_t(x), t \in \mathbb{R}$ is continuous then the iteration group is said to be quasi-continuous.

If, moreover, all functions $T^t$ are continuous then the quasi-continuous iteration group will be called a continuous iteration group.

The general construction of quasi-continuous iteration groups of real functions is given in [6]. On the base of these results we give a construction of quasi-continuous iteration groups on the unit circle.

Given an iteration group $\{T^t, t \in \mathbb{R}\}$ on $S^1$ and $x \in S^1$ put

$$C(x) := \{T^t(x), t \in \mathbb{R}\},$$

$$B(x) := \{t \in \mathbb{R} : T^t(x) = x\},$$

and

$$p(x) := \inf\{t > 0 : T^t(x) = x\}, \quad (\inf \emptyset := \infty).$$

$p(x)$ is called the period of the point $x$.

For any mapping $T : S^1 \rightarrow S^1$ we also put

$$A_T := \{x \in S^1 : T(x) = x\}.$$

We begin with some elementary properties of iteration groups on the circle.

**Proposition 1** (see also [2]). Let $\{T^t, t \in \mathbb{R}\}$ be an iteration group on $S^1$, then

(i) for $x, y \in S^1$ we have $C(x) = C(y)$ or $C(x) \cap C(y) = \emptyset$,

(ii) $T_0^0[S^1] = T^t[S^1]$ for $t \in \mathbb{R}$,

(iii) $T_0^0[T_0^0[S^1]] = Id[T_0^0[S^1]]$.

**Proof.** To prove (i) fix $t \in \mathbb{R}$, $x, y \in S^1$ such that $x \neq y$. Suppose that $C(x) \cap C(y) \neq \emptyset$, i.e. there exist $t_1, t_2 \in \mathbb{R}$ such that $T^{t_1}(x) = T^{t_2}(y)$. Then

$$T^t(x) = T^{t-t_1+t_1}(x) = T^{t-t_1}(T^{t_1}(x)) = T^{t-t_1}(T^{t_2}(y)) = T^{t-t_1+t_2}(y) \in C(y).$$

Thus, $C(x) \subseteq C(y)$. In the same way we can show that $C(y) \subseteq C(x)$.

In order to prove (ii), fix a $t \in \mathbb{R}$. First, take an $x \in T_0^0[S^1]$ and let $y \in S^1$ be such that $x = T_0^0(y)$. Then

$$x = T_0^0(y) = T^{t-t_1}(y) = T^t(T^{t-t_1}(y)) \in T^t[S^1].$$

If $x \in T^t[S^1]$, then there exists a $y \in S^1$ such that $x = T^t(y)$, and consequently
\[ x = T^t(y) = T^{0+t}(y) = T^0(T^t(y)) \in T^0[S^1]. \]

The proof of (iii) is trivial. \(\diamondsuit\)

**Remark 1.** Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group on \( S^1 \). Then for every \( x \in S^1 \), \( C(x) \) is either a singleton or the circle or an arc.

**Proof.** Since for every \( x \in S^1 \), \( C(x) = h_x[\mathbb{R}] \) and the function \( h_x : \mathbb{R} \rightarrow S^1 \) is continuous, the set \( C(x) \) is connected, and our assertion follows. \(\diamondsuit\)

The following lemmas are similar to Th. 1.13 in [2] but these ones give more facts and work with another assumptions.

**Lemma 1.** Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group on \( S^1 \) and \( x \in S^1 \). Then

(i) the following three conditions are equivalent

(a) \( p(x) = 0 \),
(b) \( C(x) = \{x\} \),
(c) \( B(x) = \mathbb{R} \);

(ii) the following three conditions are equivalent

(a) \( 0 < p(x) < \infty \),
(b) \( C(x) = S^1 \),
(c) \( B(x) \) is a nontrivial cyclic subgroup of \( \mathbb{R} \).

**Proof.** First, note that if \( B(x) \neq \emptyset \) then \( B(x) \) is a closed additive subgroup of \( \mathbb{R} \), so it is either \( \mathbb{R} \) or a cyclic subgroup of \( \mathbb{R} \). This together with the definitions of \( C(x) \) and \( p(x) \) gives (i).

Next, we prove (ii). To do this, let us first assume that \( B(x) = \{nt_0, n \in \mathbb{Z}\} \) for a positive \( t_0 \). Then \( h_x|_{\langle 0, t_0 \rangle} \) is one-to-one. Indeed, assuming \( h_x(s) = h_x(p) \) for some \( p, s \in \langle 0, t_0 \rangle \) we get

\[ x = T^0(x) = T^{s-s}(x) = T^{-s}(T^s(x)) = T^{-s}(T^p(x)) = T^{p-s}(x), \]

since \( 0 \in B(x) \). Then \( p - s \in B(x) \), so \( p = s \). Next, note that \( h_x(0) = h_x(t_0) = x \). From this, Remark 1 and the fact that \( h_x|_{\langle 0, t_0 \rangle} \) is a continuous injection we have

\[ C(x) = h_x[\mathbb{R}] = h_x[\langle 0, t_0 \rangle] = S^1. \]

Conversely, assume that \( C(x) = S^1 \). Then \( B(x) \neq \mathbb{R} \), and there exists a \( t \in \mathbb{R} \) such that \( x = T^t(x) \). Therefore \( B(x) \neq \emptyset \). If \( B(x) = \{0\} \), then \( h_x \) is easily seen to be one-to-one, which contradicts the known fact that there does not exist a continuous injection from \( \mathbb{R} \) onto \( S^1 \) (see for instance [2]). Consequently, \( B(x) \) is a nontrivial cyclic subgroup of \( \mathbb{R} \). The rest of the proof is immediate. \(\diamondsuit\)
Lemma 2. Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group on \( S^1 \). If \( x \in S^1 \) and \( p(x) = \infty \) then one of the following conditions occurs:

1. \( B(T^0(x)) = \mathbb{R} \),
2. \( B(T^0(x)) = \{0\} \),
3. \( x = T^0(x) \) and \( B(x) = \{0\} \).

Proof. Let us assume that \( x \in S^1 \) and \( p(x) = \infty \). Then \( B(x) \cap \mathbb{R}^+ = \emptyset \).

Thus, \( B(x) = \emptyset \) or \( B(x) = \{0\} \). Assuming \( B(x) = \emptyset \) we have \( T^t(x) \neq x \) for every \( t \in \mathbb{R} \), so \( x \notin C(x) \). Note that for \( y = T^0(x) \), \( C(x) = C(y) \).

Obviously, \( T^0(y) = y \), so \( 0 \in B(y) \). If \( B(y) \) is a cyclic nontrivial subgroup of \( \mathbb{R} \), then by Lemma 1, \( S^1 = C(y) = C(x) \), contrary to \( x \notin C(x) \). Hence \( B(y) = \mathbb{R} \) or \( B(y) = \{0\} \). If \( B(x) = \{0\} \), then \( x = T^0(x) \).

Lemma 3. Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group on \( S^1 \). Let \( x \in S^1 \), then

(i) if (H1) then \( C(x) = \{T^0(x)\} \), \( p(x) = \infty \) and the function \( h_x \) is constant,

(ii) if (H2) then \( C(x) \) is an arc such that \( x \notin C(x) \) and \( p(x) = \infty \),

(iii) if (H3) then \( C(x) \) is an arc such that \( x \in C(x) \) and \( p(x) = \infty \).

Moreover, the following three conditions are equivalent

(a) (H2) or (H3) occurs,
(b) \( h_x \) is a homeomorphism,
(c) \( C(x) \) is an arc.

Proof. Fix an \( x \in S^1 \) and put \( y := T^0(x) \). Let us first assume that \( x \neq T^0(x) \). Then \( B(x) = \emptyset \) and, by Lemma 1, \( p(x) = \infty \). If \( B(y) = \mathbb{R} \)
then, by Lemma 1, \( \{T^0(x)\} = C(y) = C(x) \), and consequently \( h_x \) is also constant. If \( B(y) = \{0\} \), then \( h_x(s) = h_x(p) \) implies \( h_y(s) = h_y(p) \), and consequently \( s = p \). Therefore \( h_x \) is one-to-one, and Remark 1 now shows that \( C(x) \) is an arc with \( x \notin C(x) \). Next, assume that (H3) holds true. Then \( h_x \) is an injection, and consequently \( C(x) \) is an arc with \( x \in C(x) \). Moreover, Lemma 1 now shows that \( p(x) = \infty \).

From Lemmas 1, 2 and the proved part of Lemma 3 it follows that conditions (a) i (c) are equivalent. Moreover, it is obvious that (b) implies (c). To complete the proof let us assume that (c) holds true. Then there is an open arc \( L \) such that \( C(x) \subset L \). Let \( g \) be a homeomorphism from \( L \) onto \( \mathbb{R} \). Since \( h_x \) is one-to-one, the mapping \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by \( f := g \circ h_x \) is a continuous injection, and consequently \( f \) is a homeomorphism. Therefore so is \( h_x \).

Corollary 1. Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group on \( S^1 \) and \( x \in S^1 \). If \( C(x) \) is an arc, then it is an open arc.
From Lemma 1, Prop. 1 and Remark 1 we have

**Remark 2.** Let \( \{ T^t, t \in \mathbb{R} \} \) be a quasi-continuous iteration group on \( S^1 \). If there exists an \( x_0 \in S^1 \) such that \( 0 < p(x_0) < \infty \), then \( 0 < p(x) < \infty \) for every \( x \in S^1 \).

We can now prove the following

**Theorem 1** (see also [2]). If \( \{ T^t, t \in \mathbb{R} \} \) is a quasi-continuous iteration group on \( S^1 \), then for every \( x \in S^1 \), \( \{ T^t |_{C(x)}, t \in \mathbb{R} \} \) is a continuous iteration group on \( C(x) \).

**Proof.** Fix \( x \in S^1 \) and \( t \in \mathbb{R} \). First, suppose that \( 0 < p(x) < \infty \). By Lemma 1 we see that \( C(x) = S^1 \). Moreover, from Prop. 1(iii) we conclude that \( T^0 = Id_{S^1} \). Therefore, by Th. 1.19 in [2], we deduce that \( \{ T^t, t \in \mathbb{R} \} \) is a continuous iteration group on \( S^1 \).

Now, assume that \( p(x) = 0 \) or \( p(x) = \infty \). The proof is completed by showing that \( T^t |_{C(x)} \) is continuous. If the orbit contains only one point our assertion follows. By Lemma 1 we only need to show the continuity of \( T^t |_{C(x)} \) in the case when the orbit is an arc.

Since \( C(x) \) is a metric space, it is sufficient to show that for every sequence \( (y_n)_{n \in \mathbb{N}} \) of elements of \( C(x) \) such that \( y_n \to y \in C(x) \), we have \( T^t(y_n) \to T^t(y) \). Fix such a sequence and an \( n \in \mathbb{N} \). Since \( C(x) = C(y) \), we can find \( s, s_n \in \mathbb{R} \) such that \( y_n = T^{s_n}(y) \) and \( y = T^s(x) \). Thus

\[
y_n = T^{s_n}(y) = T^{s_n}(T^s(x)) = T^{s_n+s}(x).
\]

Since \( y_n \to y \), we have \( T^{s_n+s}(x) \to T^s(x) \), i.e. \( h_x(s_n + s) \to h_x(s) \). By Lemma 3 we see that \( h_x \) is a homeomorphism, so \( s_n \to 0 \). Hence and from the fact that \( h_y \) is continuous we obtain

\[
T^t(y_n) = T^t(T^{s_n}(y)) = T^{t+s_n}(y) \to T^t(y). \quad \Diamond
\]

The general form of continuous iteration groups on the unit circle is well known (see for instance [5]), but we will remind it. We first need to prove

**Theorem 2.** Let \( \{ T^t, t \in \mathbb{R} \} \) be a quasi-continuous iteration group on \( S^1 \). If there exists an \( x_0 \in S^1 \) such that \( 0 < p(x_0) < \infty \) then

(i) for every \( t \in \mathbb{R} \), either \( T^t \equiv Id_{S^1} \) or \( T^t(x) \neq x \) for \( x \in S^1 \),

(ii) \( T^0 \equiv Id_{S^1} \),

(iii) \( T^t \) is a homeomorphism for \( t \in \mathbb{R} \).

**Proof.** Assume that \( A_{T^a} \neq \emptyset \) for an \( a \neq 0 \). Fix an \( x' \in A_{T^a} \). Then \( T^a(x') = x' \). We claim that \( C(x') \subset A_{T^a} \). Indeed, let \( y \in C(x') \). Then there exists a \( u \in \mathbb{R} \) such that \( y = T^u(x') \). Thus
$$T^a(y) = T^a(T^u(x')) = T^u(T^a(x')) = T^u(x') = y,$$
and consequently $y \in A_{T^a}$. By Remark 2 we see that $0 < p(x') < \infty$
and Lemma 1(ii) now shows that $S^1 = C(x') \subset A_{T^a}$. Consequently, $A_{T^a} = S^1$.

Next, by Lemma 1(ii), $T^0[S^1] = S^1$, since $S^1 = C(x_0) \subset T^0[S^1]$. By Prop. 1(ii) and (iii), $T^0 \equiv I_{S^1}$ and $T^t[S^1] = S^1$ for every $t \in \mathbb{R}$. Hence $T^{-t} \circ T^t = I_{S^1}$, so $T^t$ is invertible. Consequently, by Th. 1, every $T^t$ is a homeomorphism from $S^1$ onto $S^1$. \(\diamondsuit\)

Th. 2 lets us to use Th. 2 in [5]. Thus, the general form of quasi-
continuous iteration groups $\{T^t, t \in \mathbb{R}\}$ on $S^1$ such that $0 < p(x_0) < \infty$
for an $x_0 \in S^1$ is given by

$$T^t = \Phi^{-1} \circ Q_a \circ \Phi, \quad t \in \mathbb{R},$$
where $\Phi: S^1 \to S^1$ is an orientation preserving homeomorphism, $a \in \mathbb{R}$
and

$$Q_a(x) := e^{2\pi i a} \cdot x, \quad x \in S^1.$$

From now on we assume that

(1) \hspace{1cm} p(x) = 0 \quad \text{or} \quad p(x) = \infty \quad \text{for an} \ x \in S^1.

Lemma 4. Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on $S^1$
satisfying condition (1). If there exists an $s \neq 0$ and an $x_0 \in S^1$ such
that $T^s(x_0) = x_0$, then $T^t(x_0) = x_0$ for every $t \in \mathbb{R}$.

Proof. From Remark 2, Lemmas 1, 2 and 3 it follows that either $C(x_0)$
is an arc or $C(x_0) = \{x_0\}$ or $C(x_0) = \{T^0(x_0)\}$. Clearly,

$$T^s(x_0) = T^{s+0}(x_0) = T^0(T^s(x_0)) = T^0(x_0),$$
since $T^s(x_0) = x_0$, so $h_{x_0}$ is not a homeomorphism. Thus, by Lemma 3,
$C(x_0)$ is not an arc. Finally, $C(x_0) = \{x_0\}$. \(\diamondsuit\)

Put

(2) \hspace{1cm} A := \{x \in S^1 : \forall t \in \mathbb{R} \ T^t(x) = x\}.

By Lemma 4 we have

Remark 3. Let $\{T^t, t \in \mathbb{R}\}$ be a quasi-continuous iteration group on
$S^1$ and let condition (1) hold true. Then for every $t \in \mathbb{R} \setminus \{0\}$, $A_{T^t} = A$.
Moreover, $A \subset T^0[S^1]$.

We can now formulate

Theorem 3. Let $\{T^t, t \in \mathbb{R}\}$ be an iteration group on $S^1$ satisfying
(1) and let $A$ be given by (2). Then $\{T^t, t \in \mathbb{R}\}$ is quasi-continuous if
and only if either $T^0[S^1] = A$ or there exists a family of open pairwise
disjoint arcs \( \{L_n : L_n \cap A = \emptyset, \quad n \in \mathcal{M} \} \), where \( \emptyset \neq \mathcal{M} \subset \mathbb{N} \), such that

\[
T^0[S^1] = \bigcup_{n \in \mathcal{M}} L_n \cup A
\]

and for every \( n \in \mathcal{M} \), \( \{T^t|_{L_n}, t \in \mathbb{R}\} \) is a continuous iteration group on \( L_n \) such that all \( T^t|_{L_n} : L_n \to L_n \) are bijections.

**Proof.** Suppose that \( \{T^t, t \in \mathbb{R}\} \) is a quasi-continuous iteration group. In view of Lemmas 1, 2, 3, Remark 2 and Cor. 1, condition (1) shows that \( C(x) \) is an open arc or a singleton for every \( x \in S^1 \). Assume that \( T^0[S^1] \neq A \) and fix \( x \in T^0[S^1] \setminus A \), \( t \in \mathbb{R} \). By Lemmas 1, 2, 3 and Prop. 1, \( C(x) \) is an arc with \( x \in C(x) \), and therefore \( h_x \) is a homeomorphism. Thus, for every \( y \in C(x) \) there exists an \( s \in \mathbb{R} \) such that \( s = h_x^{-1}(y) \). Moreover,

\[
T^t(y) = T^t(h_x(s)) = T^t(T^s(x)) = T^{t+s}(x) = h_x(t + s),
\]

and consequently \( T^t(y) = h_x(t + h_x^{-1}(y)) \). Hence we infer that \( T^t|_{C(x)} \) is continuous and one-to-one. Clearly, \( T^t[C(x)] = C(x) \). Consequently, putting

\[
\{L_n, \quad n \in \mathcal{M}\} := \{C(x), \quad x \in T^0[S^1] \setminus A\},
\]

we obtain, in view of Prop. 1(i), a family of open pairwise disjoint arcs such that for every \( n \in \mathcal{M}, A \cap L_n = \emptyset \) and (3) holds true.

Conversely, we show that \( h_x \) is continuous for every \( x \in S^1 \). Indeed, we see at once that this is true for \( x \in T^0[S^1] \). If \( x \in S^1 \setminus T^0[S^1] \) then we have \( h_x = h_y \) with \( y := T^0(x) \in T^0[S^1] \). ∎

Now, on the base of Th. 3 we give the general construction of quasi-continuous iteration groups on \( S^1 \) satisfying condition (1).

**Theorem 4.** The following construction gives the general form of quasi-continuous iteration groups on \( S^1 \) satisfying condition (1).

1° Let \( \{L_n, n \in \mathcal{M}\} \), where \( \mathcal{M} \subset \mathbb{N} \) (we admit \( \mathcal{M} = \emptyset \)) be a family of open pairwise disjoint arcs.

2° For every \( n \in \mathcal{M} \) let \( \{F^t_n, t \in \mathbb{R}\} \) be a continuous iteration group on \( L_n \) such that all functions \( F^t_n \) are one-to-one and \( F^0_n(x) = x \) for \( x \in L_n \). (Such groups are given by the formula:

\[
F^t_n(x) = h(t + h^{-1}(x)), \quad x \in L_n, \quad t \in \mathbb{R},
\]

where \( h : \mathbb{R} \to L_n \) is a homeomorphism (see [1], p. 248–9).)

3° Let \( A \) be an arbitrary (if \( \mathcal{M} = \emptyset \) then, moreover, non-empty) subset of \( S^1 \setminus \bigcup_{n \in \mathcal{M}} L_n \).
4° Put

\[ J := \bigcup_{n \in \mathcal{M}} L_n \cup A \]

and let \( a \) be an arbitrary function defined in \( S^1 \) such that \( a[S^1] = J \) and \( a(x) = x \) for \( x \in J \).

5° Define

\[ T^t(x) := \begin{cases} 
    a(x) & \text{for } x \in a^{-1}[A], \quad t \in \mathbb{R}, \\
    F^t_n(a(x)) & \text{for } x \in a^{-1}[L_n], \quad t \in \mathbb{R}, \quad n \in \mathcal{M}.
\end{cases} \]

**Proof.** It is easy to check that the family of functions \( T^t \) defined by (4) is a quasi-continuous iteration group on \( S^1 \) for which (1) holds.

Conversely, we will show that every quasi-continuous iteration group satisfying (1) can be obtained in the above manner. Assume that \( \{T^t, t \in \mathbb{R}\} \) is such a group and define \( A \) by (2) and \( a := T^0 \). From Th. 3 it follows that either \( A = T^0[S^1] \neq \emptyset \) or there are a non-empty set \( \mathcal{M} \subset \mathbb{N} \) and a family of open pairwise disjoint arcs \( \{L_n, n \in \mathcal{M}\} \) such that (3) holds true and \( A \subset S^1 \setminus \bigcup_{n \in \mathcal{M}} L_n \). If \( T^0[S^1] = A \), then \( T^t = T^0 \) for \( t \in \mathbb{R} \). Therefore (4) holds true with \( \mathcal{M} := \emptyset \). In the later case, we put \( F^t_n := T^t|_{L_n} \) for \( t \in \mathbb{R}, n \in \mathcal{M} \). Prop. 1(iii) and Th. 3 complete the proof. \( \Diamond \)

We can now consider the problem of the embeddability of a given function into quasi-continuous iteration group. Recall that a function \( T : L \to L \), for \( L \subset S^1 \), is said to be embeddable into a quasi-continuous (continuous) iteration group if there exists a quasi-continuous (continuous) iteration group, defined on \( L \), \( \{T^t, t \in \mathbb{R}\} \) with \( T^1 = T \).

**Theorem 5.** A function \( T : S^1 \to S^1 \) is embeddable into a quasi-continuous iteration group if and only if one of the following occurs

(i) \( T \) is an orientation – preserving homeomorphism and either \( T^m = \text{Id}_{S^1} \) for a positive integer \( m \) or the set \( \{T^n(x), n \in \mathbb{N}\} \) is dense in \( S^1 \) for every \( x \in S^1 \),

(ii) there exists a non-empty set \( \mathcal{M} \subset \mathbb{N} \) and a family of open pairwise disjoint arcs \( \{L_n : L_n \cap A_T = \emptyset, n \in \mathcal{M}\} \) such that

\[ T[S^1] = \bigcup_{n \in \mathcal{M}} L_n \cup A_T \]

and for every \( n \in \mathcal{M}, T|_{L_n} : L_n \to L_n \) is a continuous bijection,

(iii) \( T[S^1] = A_T \).

**Proof.** Let \( \{T^t, t \in \mathbb{R}\} \) be a quasi-continuous iteration group such that \( T^1 = T \). First, suppose that \( 0 < p(x_0) < \infty \) for an \( x_0 \in S^1 \). By Th. 2
we infer that the iteration group \( \{ T^t, t \in \mathbb{R} \} \) is continuous and \( T \) is either without fixed points or the identity mapping. Th. 3 in [5] now shows that (i) holds true. Next, assume that (1) is satisfied. By Prop. 1(ii), Remark 3 and Th. 3 we see that (ii) or (iii) holds true.

Conversely, assume first (i). Then, by Th. 3 in [5], \( T \) is embeddable into a continuous iteration group on \( S^1 \). Assume now that (ii) or (iii) occurs. In the first case fix, moreover, an \( n \in \mathcal{M} \) and note that since \( T|_{L_n} \) is a continuous bijection, \( T|_{L_n} \) is embeddable into a continuous iteration group \( \{ F^t_n, t \in \mathbb{R} \} \) such that all functions \( F^t_n \) are one-to-one (see [1], p. 248–9, and [4]). In the later case, we define \( \mathcal{M} := \emptyset \). Put \( J := T[S^1] \) and \( A := A_T \). Clearly, \( T|_J : J \rightarrow J \) is a bijection and, if (iii) occurs, \( A \) is non-empty. Defining \( a := (T|_J)^{-1} \circ T \) we see that \( a \) maps \( S^1 \) onto \( J \) and \( a(x) = x \) for \( x \in J \). By Th. 4 the family \( \{ T^t, t \in \mathbb{R} \} \) of functions \( T^t \) given by (4) is a quasi-continuous iteration group such that, as one can check, \( T^1 = T \). \( \diamond \)

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