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ON CERTAIN GENERALIZED CIRCULANT MATRICES

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Abstract: Let $h, n$ be positive integers, where $1 \leq h < n$, $k = (n, h)$ and $n = kn'$. We call $h$-generalized circulant a matrix $A$ of order $n$ which can be partitioned into $h$-circulant submatrices of type $n' \times n$. We determine a characterization of $h$-generalized circulant matrices and, using this result, we prove that $A = \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^{jh}$ is permutation similar to the direct sum of $k$ matrices coinciding with $\sum_{j=0}^{\lfloor \frac{n}{n'} \rfloor} a_j P_{n'}^j$, where $P_n$ denote the $(0,1)$-circulant matrix of order $n$ whose first row is null but the element in position $(1,2)$. This implies new results on the values of the permanent and also on the determination of the eigenvalues of $(0,1)$-circulant matrices. A partial proof of a conjecture on the maximum value of permanents is achieved.

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1. Introduction

Recall that a matrix $A$ of type $m \times n$ ($m \leq n$) is said $h$-circulant when each row other than the first one is obtained from the preceding row by shifting the elements cyclically $h$ columns to the right. In the case of $h = 1$ $A$ is said circulant.

Let $P_n$ denote the $(0, 1)$-circulant matrix of type $n \times n$ with first row $(010\ldots0)$. If there is not possibility of ambiguity we often drop the subscript $n$ and simply write $P_n$ as $P$.

If $(a_0, a_1, \ldots, a_{n-1})$ is the first row of a circulant matrix $A$ of order $n$, then $A = \sum_{i=0}^{n-1} a_i P^i$.

It is easy to see that a matrix $A$ of type $m \times n$ is $h$-circulant if and only if it satisfies the relation $A P_n^h = P_m A$.

For $i = 1, 2, \ldots, k$, let $A_i$ be a square matrix of order $n_i$. The block diagonal square matrix

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k \end{bmatrix}$$

of order $n_1 + n_2 + \ldots n_k$ is called the direct sum of the matrices $A_1, \ldots, A_k$. It is denoted as $A = \text{diag} \{ A_1, A_2, \ldots, A_k \}$.

Recall that the permanent of a $n \times n$ matrix $A = [a_{i,j}]$, denoted by $\text{per} A$, is defined as

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the sum extends over all permutations $\sigma$ of the symmetric group of all permutations of the first $n$ integers.

For every $1 \leq r \leq n$, we denote by $(r)$ and $[r]$ the $r$-row and the $r$-column respectively of a matrix of order $n$.

Definition 1. Let $h, n$ be positive integers, where $1 \leq h < n$, $k = (n, h)$ and $n = kn'$. A matrix $A$ of order $n$ is said $h$-generalized circulant when it is partitioned into $k$ submatrices of type $n' \times n'$, which are $h$-circulant.

In other words a matrix $A$ of order $n$ is $h$-generalized circulant when it can be partitioned in the form
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\[
A = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_k
\end{bmatrix}
\]

where \(A_i, 1 \leq i \leq k\) are \(h\)-circulant \(n' \times n\)-submatrices, i.e. they satisfy \(A_j P^n_h = P^n A_j\).

The main result of this paper is proving a characterization of the \(h\)-generalized circulant matrices (Th. 1). By using this result we are able to prove that the matrix \(A = \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P^n\) is permutation similar to the matrix \(B = \text{diag} \left\{ \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P^n, \ldots, \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_{n'} \right\}\), the direct sum of \(k\) matrices coinciding with \(\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P^n\). As these matrices have the same permanent we obtain new values for the permanent of \((0,1)\) circulant matrices. In particular we obtain \(\text{per} \left( \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} P^n \right) = \left( \text{per} \left( \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} P^n \right) \right)^k\); in the particular case of three ones for row

\[
\text{per} (I + P^n + P^{2h}) = \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n'} \left( \frac{1 - \sqrt{5}}{2} \right)^{n'} + 2 \right]^k.
\]

A partial proof of a conjecture by Codenotti, Crespi and Resta [1] on the maximum value for permanents of very sparse matrices is achieved; the computation of the permanent of this class of matrices was extensively studied also in [2], [3] and [4]. Results are also obtained in relation to the characteristic polynomials of \(A\) and \(B\).

2. Characterization

Let us consider a matrix \(A\) of order \(n\); we denote by \(A_j, 1 \leq j \leq k\), the submatrix of \(A\) of type \(n' \times n\) formed by the rows of \(A\)

\((1 + (j - 1)n'), (2 + (j - 1)n'), \ldots, (jn')\).

**Theorem 1.** A matrix \(A\) of order \(n\) is \(h\)-generalized circulant, where \((n, h) = k\) and \(n = kn'\), if and only if it satisfies the relation
(2) \[ AP^h = P' A \]

where \( P' \) is direct sum of \( k \) matrices coinciding with \( P_{n'} \), i.e. \( P' = \text{diag}\{ P_{n'}, \ldots, P_{n'} \} \).

**Proof.** Let us assume that a matrix \( A = [a_{i,j}] \) of order \( n \), where \( 1 \leq i, j \leq n \) satisfies (2). The matrix \( AP^h \) is obtained by shifting cyclically the columns of \( A \) of \( h \) positions to the right. Taking into account the partitioned form of \( A \) we have

\[
AP^h = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} P^h = \begin{bmatrix} A_1 P^h \\ A_2 P^h \\ \vdots \\ A_k P^h \end{bmatrix}.
\]

Hence \((AP^h)_j = A_j P^h\) for \( 1 \leq j \leq k \).

Now consider the product \( P'A \). From the definition of \( P' \) and the partitioned form of \( A \) we have

\[
\begin{bmatrix} P_{n'} \\ P_{n'} \\ \vdots \\ P_{n'} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} = \begin{bmatrix} P_{n'} A_1 \\ P_{n'} A_2 \\ \vdots \\ P_{n'} A_k \end{bmatrix},
\]

which shows that \((P'A)_j = P_{n'} A_j\) for \( 1 \leq j \leq k \).

The equality (2) implies the equality of the submatrices \((AP^h)_j\) and \((P'A)_j\), where \( 1 \leq j \leq k \). But we have seen that \((AP^h)_j = A_j P^h\). Hence from the assumption (2) it follows \( A_j P^h = P_{n'} A_j \) \( (1 \leq j \leq k) \), i.e. the submatrices \( A_j \) \( (1 \leq j \leq k) \) are \( h \)-circulant \( n' \times n \) matrices and therefore \( A \) is \( h \)-generalized circulant.

Conversely, assume that every \( A_j, 1 \leq j \leq k, \) is \( h \)-circulant, i.e. \( P_{n'} A_j = A_j P^h \) \( (1 \leq j \leq k) \). Then \((P'A)_j = P_{n'} A_j \) and \((AP^h)_j = A_j P^h\) for \( 1 \leq j \leq k \).

Denote \( Q = [q_{ij}] \) the permutation matrix of order \( n \), that rearranges the columns of an arbitrary \( m \times n \) matrix \( M \) by postmultiplication \( MQ \) according to the permutation \( \alpha \). Using Kronecker symbols \( \delta \) the entries \( q_{ij} \) can be written in the form

\[
q_{ij} = \delta_{\alpha(i),j} = \begin{cases} 1 & \text{if } j = \alpha(i) \\ 0 & \text{otherwise.} \end{cases}
\]

By the assumption that \( A_j \) is \( h \)-circulant it follows that the rows of \((P'A)_j\) and \((AP^h)_j\) coincide. Then \((P'A)_j = (AP^h)_j\) and \( A \) satisfies (2). \( \diamond \)
When $k = 1$ a matrix $A$ which satisfies (2) turns out to be a $h$-circulant matrix; thus this definition turns out to be a generalization of the notion of $h$-circulant matrix.

Now we will consider the particular case of $h$-generalized circulant permutation matrices.

**Proposition 1.** Let $n$ and $h$ be positive integers, where $1 \leq h \leq n$, $k = (n, h)$ and $n = kn'$. The function $\alpha : i \mapsto 1 + (i - 1)h + t$, where $1 + tn' \leq i \leq (t + 1)n'$, $0 \leq t \leq k - 1$ and the integers are taken mod $n$ is a permutation, whose representing matrix satisfies (2).

**Proof.** In order to prove that $\alpha$ is a permutation it is sufficient to prove it is injective. Let $i, j \in [1, n]$, $i < j$ and $h = kh'$, where $(h', n') = 1$. Assume that $\alpha(i) = \alpha(j)$, that is

$$1 + (i - 1)h + t = 1 + (j - 1)h + t'$$

where $0 \leq t, t' \leq k - 1$. Let us distinguish the cases of $t = t'$ or $t \neq t'$. The condition of $t = t'$ implies the relation $(j - i)h \equiv 0 \pmod{n}$, which is impossible because $j - i < n'$. In the case of $t$ and $t'$ distinct, without loss of generality we may assume $t' > t$ and represent $t' = t + r$, where $0 < r < k$. Thus we obtain $(i - j)h \equiv r \pmod{n}$. It implies that for a suitable integer $m$ we obtain $k((i - j)h' - mn') = r$, which is impossible by the assumption on $r$. Then $\alpha$ is a permutation. Denote by $Q$ the matrix which represents such a permutation. Then by the construction we have that the submatrices $Q_i$ formed by the rows $(1 + (t - 1)n'), \ldots, (tn')$, where $1 \leq t \leq k$ are $h$-circulant. Then $Q$ is $h$-generalized circulant.

Also the permutation $\alpha$ is said $h$-generalized circulant.

**Corollary 1.** Let $\alpha$ an $h$-generalized circulant permutation; $\alpha$ is uniquely determined when

$$\alpha(1), \alpha(1 + n'), \ldots, \alpha(1 + (k - 1)n')$$

are assigned.

**Proof.** When we assign $\alpha(1)$ then the first row and therefore the consecutive $k - 1$ rows of the matrix which represents $\alpha$ are assigned. This means that in the decomposition (1) the submatrix $A_1$ is given. Similar considerations hold for the remaining submatrices.

**Proposition 2.** The number of $h$-generalized circulant matrices of order $n$, where $k = (n, h)$ and $n = kn'$, is

$$n(n - n')(n - 2n') \ldots n'$$
Proof. From the above considerations, $\alpha$ is uniquely determined when we assign $\alpha(1 + jn')$, for all $0 \leq j \leq k - 1$. We see that $\alpha(1)$ may assume $n$ values. When $\alpha(1)$ is assigned, also the following $n' - 1$ rows are assigned; then $\alpha(n' + 1)$ may assume $n - n'$ values. By continuing in this way the result follows.

We call regular an $h$-generalized circulant permutation matrix $Q = [q_{i,j}]$ of order $n = kn'$, when

$$q_{1,1} = q_{1+n',2} = \cdots = q_{1+(k-1)n',k} = 1.$$ 

In other words a $h$-generalized matrix $A$, representing the permutation $\alpha$, is regular when $\alpha$ satisfies the conditions

$$\alpha(1) = 1, \alpha(n' + 1) = 2, \ldots, \alpha((k - 1)n' + 1) = k.$$ 

When we need to remember the parameter $h$ in relation to an $h$-generalized circulant permutation matrix $Q$, we write $Q(h)$.

Theorem 2. Let $A = a_0 I + a_1 P^h + \ldots + a_t P^{th}$ be a matrix of order $n$, where $1 < h < n$, $(n, h) = k$, $n = kn'$, $t = \lfloor \frac{n}{k} \rfloor$ and $a_i, 1 \leq i \leq n' - 1$, real numbers; moreover let $Q$ be the $h$-generalized regular permutation matrix of order $n$. Then the matrix $B = QAQ^T$ is direct sum of $k$ matrices coinciding with $\sum_{i=0}^{t} a_i P_{n'}^i$.

Proof. By (2) $QP^hQ^T = P'$; then $B = QAQ^T = a_0 I + a_1 P' + \ldots + a_t (P')^t$.

An immediate consequence is that the circulant matrix $A = I + P^h + \ldots + P^{sh}$ where $s \leq \lfloor \frac{n}{k} \rfloor$, is permutation similar to the circulant matrix $B = I + P + \ldots + P^s$. Another consequence is the following

Corollary 2. Let $A = \sum_{j=0}^{r} P_{n}^{jh}$ be a square matrix of order $n$, where $1 < h < n$, $(n, h) = k$, $n = kn'$, $t = \lfloor \frac{n}{k} \rfloor$ and $1 \leq r \leq t$. Then we have

$$\text{per} \left( \sum_{j=0}^{t} P_{n}^{jh} \right) = \left( \text{per} \left( \sum_{j=0}^{t} P_{n'}^{j} \right) \right)^k.$$ 

Proof. As the permanent is invariant with respect to permutation of rows or columns, the result follows from Th. 1.

As example of regular 3-generalized permutation matrix we may consider the following matrix of order 9:
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\[
Q(3) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Then in relation to the matrix of order 9 \( A = I + P^3 + P^6 \), we obtain that

\[
Q(3)AQ(3)^T = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

(3)

3. Very sparse matrices

In this section we consider the case of \((0, 1)\) circulant matrices with 3 ones in each row. First consider the matrix \( I + P^h + P^{2h} \), where \((n, h) = 1\) and \( P \) denotes the permutation matrix \( P_n \). By using a Minc’s formula for the permanent of \( I + P + P^2 \) [8] we have:

**Corollary 3.** Let \( P \) be of order \( n \), where \( 1 < h < n \), \((n, h) = k\), \( n = kn' \); then

\[
\text{per} (I + P^h + P^{2h}) = \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^{n'} + \left(\frac{1 - \sqrt{5}}{2}\right)^{n'} + 2 \right]^k.
\]

Now, consider the matrix \( D = I + P^m + P^{2m} \) of order \( n = 3m \). From Th. 2 we have that \( A \) is permutation similar to the direct sum of \( m \) submatrices coinciding with \( J_3 \), the matrix of all ones of order 3, then \( \text{per} mA = (3!)^m \). It is known that in the class of \( vk \times vk \) \((0, 1)\)-matrices
with row sums and column sums equal to \( k \) the permanent function takes its maximum on the direct sum of \( k \times k \) matrices of 1's. Thus the matrix \( D \) satisfies partially the conjecture by Codenotti, Crespa and Resta [1].

Now consider the case of a matrix \( A = I + P^h + P^j \), where \((n, h) = 1 \) and \( j \neq 2h \) (mod \( n \)).

**Proposition 3.** Let \( h, n \) be positive integers, such that \( 1 < h < n \), \((n, h) = 1 \) and \( Q \) is the regular \( h \)-generalized permutation matrix of order \( n \). Then \( QPQT = P^s \) where \( s \) is the unique solution, modulo \( n \), of the equation

\[
sh \equiv 1.
\]

**Proof.** Denote by \( \alpha, \pi \) and \( \beta \) the \( h \)-generalized regular circulant permutations represented by \( Q_h, P \) and \( Q_hPQT \), respectively. Then \( \beta(1) = \alpha^{-1}(\pi(\alpha(1))) = \alpha^{-1}(\pi(1)) = \alpha^{-1}(2) \). Denote by \( s \) the integer, \( 1 < s \leq n \), such that \( 1 + sh \equiv 2 \). Because \((n, h) = 1 \), it easy to see that the equation \( sh \equiv 1 \) has a unique solution. Thus, for every \( 1 < i \leq n \), we have

\[
\beta(i) = \alpha^{-1}(\pi(\alpha(i))) = \alpha^{-1}(\pi(1 + (i - 1)h)) = \\
= \alpha^{-1}(2 + (i - 1)h) = \alpha^{-1}(1 + (s + i - 1)h) = s + i.
\]

This implies that \( \beta = \pi^s \). \( \Box \)

**Proposition 4.** Let \( A = I + P^h + P^j \) be a square matrix of order \( n \), where \( 1 < h < j < n \), \( j \neq 2h \) (mod \( n \)), \((h, n) = 1 \) and \( Q \) is the regular \( h \)-generalized permutation matrix of order \( n \). Then \( QAQ^T = I + P + P^v \), where \( v \) is the unique solution of the equation \( vh \equiv j \) (mod \( n \)).

**Proof.** From Prop. 4 we have that \( Q_hP^jQ_h^T = (Q_hP_h^T)^j = P^{sj} \). Denoted \( v = sj \), from the equation \( sh = 1 \), we obtain \( vh = j \) (mod \( n \)). This implies \( Q_hA^T = I + P + P^v \). \( \Box \)

In the case when \((n, h) \neq 1 \), but \((n, j) = 1 \) or \((n, j - h) = 1 \), we have a similar situation by multiplying \( A \) by a suitable power of \( P \).

4. Eigenvalues

Recall that if \( A \) is a circulant matrix hose first row is \([a_0a_1 \ldots a_{n-1}]\), the polynomial \( p(\lambda) = \sum_{i=0}^{n} a_i \lambda^i \) is said the Hall polynomial of the matrix \( A \). If \( \omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \), then the eigenvalues of \( A \) are \( 1, p(\omega), p(\omega^2), \ldots, p(\omega^{n-1}) \).
Let \( p(\lambda) \) and \( q(\lambda) \) the Hall polynomials of the matrices \( A = I + P^h + \cdots + P^{rh} \) and \( B = I + P + \cdots + P^r \) respectively, where \( 0 < h < n, k = (n, h), n = kn' \) and \( 1 < r \leq \left[ \frac{n}{k} \right] \). Moreover let \( r(\lambda) = 1 + \lambda + \cdots + \lambda^{n'-1} \) be the Hall polynomial of the matrix \( C = I + P_{n'} + \cdots + P_{n'}^r \), and \( \alpha = \cos \frac{2\pi}{n'} + i \sin \frac{2\pi}{n'} \). From Th. 2 it follows the following

**Proposition 5.** The sets of eigenvalues of \( A = I + P^h + \cdots + P^{rh} \) and \( C = I + P_{n'} + \cdots + P_{n'}^r \) coincide, when \( k = 1 \). In the case of \( k > 1 \), the set of eigenvalues of \( A \) is the union of \( k \) sets coinciding with \( \{1, r(\alpha), \ldots, r(\alpha^{n'-1})\} \).

A consequence is that, when \( k > 1 \), every eigenvalue of \( A \) has multiplicity at least \( k \).

**References**


