EXPECTED UTILITY WITH PSEUDOTRANSITIVE PREFERENCES

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Abstract: Given a separable metric space \( Y \), and a \( \sigma \)-algebra \( \mathcal{B}(Y) \) of subsets of \( Y \), consider the space \( \mathcal{M}(Y) \) of all (countably additive) probability measures on the measurable space \( (Y, \mathcal{B}(Y)) \), endowed with the topology of weak convergence. Further, denote by \( \prec \) a preference relation on a \( \sigma \)-convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \). Necessary and sufficient conditions are presented for the existence of a pair of real continuous bounded functions \( u, v \) on \( Y \), such that, for every \( p, q \in \mathcal{P} \), \([ p \prec q \text{ if and only if } \int_Y u dp < \int_Y v dq \] ), where the real functionals \( p \mapsto \int_Y u dp \) and \( p \mapsto \int_Y v dp \) are utility functionals for two weak orders naturally associated to \( \prec \).

1. Introduction

Grandmont [8, Th. 3] proved a classical theorem in expected utility theory. Given a separable metric space \( Y \), a \( \sigma \)-algebra \( \mathcal{B}(Y) \) of subsets of \( Y \), and a weak order (i.e., an asymmetric and negatively transitive binary relation) \( \prec \) on a \( \sigma \)-convex subspace \( \mathcal{P} \) of the space \( \mathcal{M}(Y) \) of all (countably additive) probability measures on the measurable space \( (Y, \mathcal{B}(Y)) \), Grandmont presented necessary and sufficient

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conditions for the existence of a continuous bounded real function \( u \) on \( Y \), such that, for every \( p, q \in \mathcal{P} \),

\[
p < q \quad \text{if and only if} \quad \int_Y u \, dp < \int_Y u \, dq.
\]

In this case, \( u \) is said to be a (continuous) von Neumann–Morgenstern utility function for the weak order \(<\).

Several authors pointed out that indifference relations should not be transitive (see e.g. Armstrong [1], Bridges [4], Chateauneuf [5], Chipman [6], Fishburn [7], Luce [10]). While (semi)continuous representations of preferences with intransitive indifference seem to have received a considerable attention in literature (see e.g. Bridges [4], Chateauneuf [5] and Bosi et al. [3]), only a few authors were concerned with linear representations of preferences of this kind (see e.g., Fishburn [7], Vincke [13], and Nakamura [11]).

In this paper, given a preference relation \(<\) on a \( \sigma \)-convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \), we are concerned with the existence of a pair of continuous bounded real functions \( u, v \) on the consequence space \( Y \), such that, for every \( p, q \in \mathcal{P} \),

\[
p < q \quad \text{if and only if} \quad \int_Y u \, dp < \int_Y v \, dq.
\]

In such a representation, \( u \) and \( v \) are von Neumann–Morgenstern utility functions for two weak orders naturally associated to \(<\).

2. Notation and preliminaries

Denote by \( Y \) the set of all consequences, and let \( \mathcal{B}(Y) \) be a \( \sigma \)-algebra of subsets of \( Y \). It is assumed that \( Y \) is a separable metric space. Moreover, let \( \mathcal{M}(Y) \) be the space of all (countably additive) probability measures (lotteries) on the measurable space \( (Y, \mathcal{B}(Y)) \), endowed with the topology of weak convergence. We recall that a sequence \( \{p_n, n \geq 1\} \) of probability measures in \( \mathcal{M}(Y) \) converges weakly to a probability measure \( p \) if

\[
\lim \int_Y f \, dp_n = \int_Y f \, dp
\]

for every real bounded continuous function \( f \) on \( Y \) (see Parthasarathy [12]).

A subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \) is said to be
(i) convex if $\lambda p_1 + (1 - \lambda)p_2$ belongs to $\mathcal{P}$ for any $p_1, p_2$ in $\mathcal{P}$, and for any real number $\lambda$ in $[0, 1]$, 

(ii) $\sigma$-convex if $p_0 = \sum_1^{\infty} \lambda_n p_n$ belongs to $\mathcal{P}$ for any sequence \{\(p_n, n \geq 1\)\} of elements of $\mathcal{P}$, and for any sequence \{\(\lambda_n, n \geq 1\)\} of nonnegative real numbers such that $\sum_1^{\infty} \lambda_n = 1$.

A real functional $f$ on a convex ($\sigma$-convex) subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ is linear ($\sigma$-linear) if, for every $p, q$ in $\mathcal{P}$, and any real number $\lambda$ in $[0, 1]$, it is $f(\lambda p + (1 - \lambda)q) = \lambda f(p) + (1 - \lambda)f(q)$ (respectively, for any sequence \{\(p_n, n \geq 1\)\} of elements of $\mathcal{P}$, and for any sequence \{\(\lambda_n, n \geq 1\)\} of nonnegative real numbers such that $\sum_1^{\infty} \lambda_n = 1$, it is $f(\sum_1^{\infty} \lambda_n p_n) = \sum_1^{\infty} \lambda_n f(p_n)$.

Let $\prec$ be a preference relation (i.e. an asymmetric binary relation) on a subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. Denote by $\preceq$ and $\sim$ the preference-indifference relation, and respectively the indiference relation associated with $\prec$, namely, for $p, q \in \mathcal{P}$,

\[ p \preceq q \iff \text{not } (q \prec p), \]
\[ p \sim q \iff (p \preceq q) \text{ and } (q \preceq p). \]

A preference relation $\prec$ on $\mathcal{P}$ is said to be a weak order if $\prec$ is negatively transitive. If $\prec$ is a weak order, then the associated preference-indifference relation $\preceq$ is a complete preorder (i.e., $\preceq$ is transitive and complete).

The preference-indifference relation $\preceq$ associated with a preference relation $\prec$ on $\mathcal{P}$ is said to be pseudotransitive if, for every $p, p', q, q' \in \mathcal{P}$,

\[ p \prec p' \preceq q \prec q' \Rightarrow p \prec q. \]

We say that a preference relation $\preceq_c$ on $Y$ is induced by a preference relation $\prec$ on $\mathcal{M}(Y)$ if, for every $y, z \in Y$,

\[ y \prec_c z \iff p_y \prec p_z, \]

where, for every $y \in Y$, $p_y$ is the probability distribution concentrated at the point $y \in Y$. Denote by $D$ the subspace of $\mathcal{M}(Y)$ whose elements are the probability distributions which are concentrated, namely

\[ D = \{ p \in \mathcal{M}(Y) : \exists y \in Y, p = p_y \}. \]

A preference relation $\prec$ is represented by a utility functional $U$ on $\mathcal{P}$ if, for every $p, q \in \mathcal{P}$,

\[ p \prec q \iff U(p) < U(q). \]

If such a representation exists, then $\prec$ is a weak order. Grandmont [8] found necessary and sufficient conditions for the existence of
a continuous von Neumann–Morgenstern utility function representing a weak order \( \prec \) on a \( \sigma \)-convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \) containing \( D \). We recall that \( u \) is said to be a von Neumann–Morgenstern utility function for a preference relation \( \prec \) on \( \mathcal{P} \) if \( u \) is a real function on \( Y \) representing the preference relation \( \prec \) among sure consequences and, for every \( p, q \in \mathcal{P} \),

\[ p \prec q \iff \int_Y udp < \int_Y udq. \]

It is clear that, if \( \prec \) is induced by \( \prec \), and there exists a real function \( u \) on \( Y \) such that

\[ p \to U(p) = \int_Y udp \]

is a utility functional for \( \prec \), then \( u \) is a von Neumann–Morgenstern utility function for \( \prec \).

A preference relation \( \prec \) is represented by a pair of real functionals \( U, V \) on \( \mathcal{P} \) if, for every \( p, q \in \mathcal{P} \),

\[ p \prec q \iff U(p) < V(q). \]

If such a representation exists, then \( \leq \) is pseudotransitive.

A preference relation \( \prec \) on \( \mathcal{P} \) is continuous if \( \{ q \in \mathcal{P} : p \prec q \} \) and \( \{ q \in \mathcal{P} : q \prec p \} \) are open sets in \( \mathcal{P} \) for every \( p \in \mathcal{P} \).

To each preference relation \( \prec \) on \( \mathcal{P} \) we may associate the binary relations \( \prec^* \) and \( \prec^{**} \) defined as follows:

\[ p \prec^* q \iff \exists r \in \mathcal{P} : p \prec r \leq q, \]

\[ p \prec^{**} q \iff \exists s \in \mathcal{P} : p \leq s \prec q. \]

Fishburn [7] proved that, if \( \prec \) is a preference relation with pseudotransitive preference-indifference, then \( \prec^* \) and \( \prec^{**} \) are both weak orders. The indifference relations associated to \( \prec^* \) and \( \prec^{**} \) are denoted by \( \sim^* \) and \( \sim^{**} \), respectively.

3. Expected utility with pseudotransitive preferences

In the following lemma, we present a necessary and sufficient condition for the continuity of a linear utility functional on a convex subspace of \( \mathcal{M}(Y) \).

**Lemma 1.** Let \( Y \) be a separable metric space, and let \( \prec \) be a preference relation on a convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \). Assume that there exists a
linear utility functional \( U \) for \( \prec \). Then \( U \) is continuous if and only if \( \prec \) is continuous.

**Proof.** Let \( U \) be a linear utility functional for a preference relation \( \prec \) on a convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \). It is clear that, if \( U \) is continuous, then \( \prec \) is continuous. Conversely, assume that \( \prec \) is continuous. In order to show that \( U \) is upper semicontinuous, consider \( p \in \mathcal{P} \), and \( \beta \in \mathbb{R} \), such that \( U(p) < \beta \). If \( p \) is a maximal element relative to \( \prec \), then \( U(q) \leq U(p) \) for every \( q \in \mathcal{P} \), and therefore \( \mathcal{P} \) is an open set containing \( p \) such that \( U(q) < \beta \) for every \( q \in \mathcal{P} \). If \( p \) is not a maximal element relative to \( \prec \), then there exists \( q' \in \mathcal{P} \) such that \( p \prec q' \), and therefore \( U(p) < U(q') \). Since \( \alpha \to (1 - \alpha)U(p) + \alpha U(q') \) is a continuous function from the closed real interval \([0, 1]\) onto the closed real interval \([U(p), U(q')]\), there exists \( \tilde{\alpha} \in [0, 1] \) such that \( U(p) < (1 - \tilde{\alpha})U(p) + \tilde{\alpha}U(q') < \beta \). Define \( \tilde{q} = (1 - \tilde{\alpha})p + \tilde{\alpha}q' \). Since \( U \) is a linear utility functional for \( \prec \), it is \( p \prec \tilde{q} \), \( U(q) < \beta \). Since \( \prec \) is continuous, \( L(\tilde{q}) = \{ q \in \mathcal{P} : q \prec \tilde{q} \} \) is an open set containing \( p \), such that \( U(q) < \beta \) for every \( q \in L(\tilde{q}) \). Analogously, it can be shown that \( U \) is lower semicontinuous. \( \diamond \)

In the following proposition, necessary and sufficient conditions are given for the existence of an integral representation of a linear utility functional on a \( \sigma \)-convex subspace of \( \mathcal{M}(Y) \).

**Proposition 1.** Let \( \prec \) be a preference relation on a separable metric space \( Y \), and let \( \prec \) be a preference relation on a \( \sigma \)-convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \) containing \( D \). Assume that there is a linear utility functional \( U \) for \( \prec \). Then there exists a real bounded continuous function \( u \) on \( Y \), which is a utility function for \( \prec \), such that, for every \( p \in \mathcal{P} \), \( U(p) = \int_Y u dp \), if and only if \( \prec \) is continuous and \( \prec \) is induced by \( \prec \).

**Proof.** It is easily seen that, if \( u \) is a real bounded continuous function on \( Y \), and \( p \to U(p) = \int_Y u dp \) is a utility functional for \( \prec \), then \( \prec \) is continuous and \( \prec \) is induced by \( \prec \). Let us show that, if \( U \) is a linear utility functional for \( \prec \), \( \prec \) is continuous and \( \prec \) is induced by \( \prec \), then there exists a real bounded continuous function \( u \) on \( Y \), which is a utility function for \( \prec \), such that, for every \( p \in \mathcal{P} \), \( U(p) = \int_Y u dp \). First observe that \( U \) is continuous by Lemma 1. Define, for every \( y \in Y \), \( u(y) = U(p_y) \). Since \( U \) is linear and continuous, and \( \mathcal{P} \) is \( \sigma \)-convex, we have that \( u \) is bounded (see Grandmont [8, Lemma 2]). From Parthasarathy [12, Chap. 2, Lemma 6.1], \( u \) is continuous. Since \( \mathcal{P} \) is convex and contains \( p_y \) for every \( y \in Y \), any finite support probability distribution
$p$ in $\mathcal{M}(Y)$ belongs to $\mathcal{P}$. From Parthasarathy [12, Ths. 6.2 and 6.3], each element $p$ of $\mathcal{P}$ is the limit in the topology of weak convergence of a sequence $\{p_n, n \geq 1\} \subseteq \mathcal{P}$ of finite support probability measures. Since $U$ is linear, it is easily seen that $U(p_n) = \int_Y u d p_n$ for every $n \geq 1$. By continuity of $U$, $\lim U(p_n) = U(p)$, and therefore, using the fact that $u$ is continuous and bounded, $U(p) = \lim \int_Y u d p_n = \int_Y u d p$. ♦

Let us consider necessary and sufficient conditions for the existence of a pair $U, V$ of linear functionals representing a preference relation $\prec$ with pseudotransitive preference-indifference on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. In this axiomatization, $U$ and $V$ are utility functionals for the associated weak orders $\prec^*$ and $\prec^{**}$, respectively. It is assumed that there is not a maximal element relative to $\prec$. We recall that another axiomatization was presented by Nakamura [11, Th. 1]. The following theorem allows us to recover an integral representation of both $U$ and $V$, and this is the reason why we present it.

**Theorem 1.** Let $Y$ be a separable metric space, and let $\prec$ be a preference relation without a maximal element on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. There exists a pair $U, V$ of real continuous linear functionals on $\mathcal{P}$ representing $\prec$, such that $U$ and $V$ are utility functionals for $\prec^*$ and $\prec^{**}$, respectively, if and only if

\[
\begin{align*}
\text{A1. } \prec & \text{ is pseudotransitive,} \\
\text{A2. } p \sim^{**} q & \Rightarrow \lambda p + (1 - \lambda) r \sim^{**} \lambda q + (1 - \lambda) r \\
& \quad \forall p, q, r \in \mathcal{P}, \lambda \in [0, 1], \\
\text{A3. } \prec^* \text{ and } \prec^{**} & \text{ are both continuous,} \\
\text{A4. } \lambda p + (1 - \lambda) q & \prec r \Rightarrow \exists r_1, r_2 \in \mathcal{P} : \lambda r_1 + (1 - \lambda) r_2 \prec^{**} r, \\
& \quad \forall p, q, r \in \mathcal{P}, \lambda \in [0, 1], \\
\text{A5. } p & \prec q, r \prec s \Rightarrow \lambda p + (1 - \lambda) r \prec \lambda q + (1 - \lambda) s \\
& \quad \forall p, q, r, s \in \mathcal{P}, \lambda \in [0, 1].
\end{align*}
\]

If $U, V$ and $U', V'$ are two pairs of such real functionals, then there exist two real numbers $a > 0$ and $b$, such that $U' = aU + b$ and $V' = V = aV + b$.

**Proof.** It is easily seen that conditions (1) are necessary for the existence of a pair $U, V$ of real continuous linear functionals on $\mathcal{P}$ representing $\prec$, such that $U$ and $V$ are utility functionals for $\prec^*$ and $\prec^{**}$, respectively. So let us prove that axioms (1) are sufficient for the existence of such a representation. By axiom A1, $\prec^*$ and $\prec^{**}$ are both
weak orders. By axioms \textbf{A2} and \textbf{A3}, for any \( p, q, r \in \mathcal{P} \), the sets \( \{ \lambda \in [0, 1] : p \prec \star \lambda q + (1 - \lambda) r \} \) and \( \{ \lambda \in [0, 1] : \lambda p + (1 - \lambda) q \prec \star r \} \) are open (see the proof of Th. 2 in Grandmont [8]). According to Herstein and Milnor [9, Th. 8], there exists a real linear utility functional \( V \) on \( \mathcal{P} \) representing \( \prec \star \). Define, for every \( p \in \mathcal{P} \),

\[
U(p) = \inf \{ V(q) : p \prec q, q \in \mathcal{P} \}.
\]

Let us show that the pair \( U, V \) represents \( \prec \). Consider \( p, q \in \mathcal{P} \) such that \( p \prec q \). By axioms \textbf{A4} and \textbf{A5}, there exists \( p' \in \mathcal{P} \) with \( p \prec p' \prec \star q \). Since \( V(p') < V(q) \), it is \( U(p) < V(q) \) from the definition of \( U \). Conversely, assume that \( U(p) < V(q) \). Then there exists \( p' \in \mathcal{P} \) such that \( U(p) < V(p') < V(q) \), \( p \prec p' \). Hence \( p \prec p' \prec \star q \), and therefore \( p \prec q \) by axiom \textbf{A1}.

Let us prove that \( U \) is a utility functional for \( \prec \star \). If \( p \prec \star q \), then there exists \( q' \in \mathcal{P} \) such that \( p \prec q' \preceq q \). Then \( U(p) < V(q') \leq U(q) \), and therefore \( U(p) < U(q) \). Conversely, assume that \( U(p) < U(q) \). Then there exists \( q' \in \mathcal{P} \) such that \( U(p) < V(q') < U(q) \), \( p \prec q' \preceq q \), and therefore \( p \prec \star q \).

Now, let us show that \( U \) is linear. Assume that there exist \( p, q \in \mathcal{P} \), and \( \lambda \in [0, 1] \), such that \( \lambda U(p) + (1 - \lambda) U(q) < U(\lambda p + (1 - \lambda) q) \). From the definition of \( U \), and from linearity of \( V \), there exist \( r_1, r_2 \in \mathcal{P} \) with \( p \prec r_1 \), \( q \prec r_2 \), \( \lambda U(p) + (1 - \lambda) U(q) < V(\lambda r_1 + (1 - \lambda) r_2) < U(\lambda p + (1 - \lambda) q) \). By axiom \textbf{A5}, it is \( \lambda p + (1 - \lambda) q \prec \lambda r_1 + (1 - \lambda) r_2 \), and therefore \( V(\lambda r_1 + (1 - \lambda) r_2) < U(\lambda p + (1 - \lambda) q) \) is contradictory. Using similar considerations, it can be shown that for no \( p, q \in \mathcal{P} \), and \( \lambda \in [0, 1] \), it is \( U(\lambda p + (1 - \lambda) q) < \lambda U(p) + (1 - \lambda) U(q) \).

Since \( U \) and \( V \) are real linear utility functionals for \( \prec \star \) and \( \prec \star \), respectively, and \( \prec \star \) and \( \prec \star \) are both continuous by axiom \textbf{A3}, then \( U \) and \( V \) are continuous by Lemma 1.

Let \( U, V \) and \( U', V' \) be two pairs of real functionals both satisfying axioms (1). From Herstein and Milnor [9, Th. 8], there exist two real numbers \( a > 0 \) and \( b \), and two real numbers \( a' > 0 \) and \( b' \), such that \( U' = a U + b \) and \( V' = a' V + b' \). Assume that either \( a \neq a' \) or \( b \neq b' \), and consider \( p, q \in \mathcal{P} \), such that \( p \prec q \). Then it is both \( U(p) < V(q) \) and \( U(p) < 1/a (a' V(q) + b' - b) \). If \( 1/a (a' V(q) + b' - b) < V(q) \), then, using the fact that \( U \) is linear, it is easily seen that there exists \( p' \in \mathcal{P} \) such that \( 1/a (a' V(q) + b' - b) < U(p') < V(q) \), and this is impossible since \( U, V \) and \( U', V' \) are two representations of \( \prec \). Analogous considerations lead to a contradiction in the case when \( V(q) \leq 1/a (a' V(q) + b' - b) \).
So the proof is complete. ♦

**Remark 1.** Observe that axiom A5 is found in the axiomatization presented by Nakamura [11, Th. 1]. Axiom A4 is a continuity axiom involving the preference relation \( \prec \) and the associated weak order \( \prec^{**} \). ♦

Now we are able to present the main result of this section.

**Theorem 2.** Let \( \prec^c \) be a preference relation on a separable metric space \( Y \), and let \( \prec \) be a preference relation without maximal elements on a \( \sigma \)-convex subspace \( \mathcal{P} \) of \( \mathcal{M}(Y) \) containing \( D \). There exists a pair \( u, v \) of real continuous bounded functions on \( Y \), which are utility functions for \( \prec^* \) and \( \prec^{**} \), respectively, such that, for every \( p, q \in \mathcal{P} \),

\[
\begin{align*}
B1. \quad p \prec q & \iff \int_Y u dp < \int_Y v dq, \\
B2. \quad p \prec^* q & \iff \int_Y u dp < \int_Y v dq, \\
B3. \quad p \prec^{**} q & \iff \int_Y v dp < \int_Y u dq,
\end{align*}
\]

if and only if axioms (1) of Th. 1 hold, and

\[
\prec^c \text{ is induced by } \prec.
\]

If \( u, v \) and \( u', v' \) are two pairs of such real functions on \( Y \), then there exist two real numbers \( a > 0 \) and \( b \), such that \( u' = au + b \) and \( v' = av + b \).

**Proof.** It is easily seen that axioms (1) of Th. 1, and condition (3) are necessary for the existence of a pair \( u, v \) of real continuous bounded functions on \( Y \) satisfying conditions (2). So, let us prove the sufficiency part. From Th. 1, there exists a pair of real continuous linear functionals \( U, V \) on \( Y \), representing \( \prec^* \) and \( \prec^{**} \), respectively. From Prop. 1, since it is easily seen that, if \( \prec^c \) is induced by \( \prec \), then \( \prec^c \) and \( \prec^{**} \) are induced by \( \prec^* \) and \( \prec^{**} \), respectively, there exists a pair \( u, v \) of real bounded continuous functions on \( Y \), which are utility functions for \( \prec^c \) and \( \prec^{**} \), respectively, such that, for every \( p \in \mathcal{P} \), \( U(p) = \int_Y u dp \) and \( V(p) = \int_Y v dp \).

Finally, if \( u, v \) and \( u', v' \) are two pairs of such real functions on \( Y \), then, by Th. 1, there exist two real numbers \( a > 0 \) and \( b \), such that \( u' = au + b, v' = av + b \). So the proof is complete. ♦
References