ON THE QUASI-CONFORMAL CURVATURE TENSOR OF A KENMOTSU MANIFOLD

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Abstract: We consider quasi-conformally flat and quasi-conformally semisymmetric Kenmotsu manifolds. We show that the following three statements are equivalent: (a) $M$ is quasi-conformally flat, (b) $M$ is quasi-conformally semisymmetric and (c) $M$ is locally isometric to the hyperbolic space $H^n(-1)$.

1. Introduction

In [3], B. Y. Chen and K. Yano defined the notion of an $n$-dimensional Riemannian manifold $(M^n, g)$ of quasi-constant curvature as a conformally flat manifold with the curvature tensor $R$ satisfying the condition

\[ R(X, Y, Z, W) = (\alpha(X, W) - \alpha(Y, Z)) g(Y, Z) \]

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\[ R(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + \\
+ q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) + \\
+ g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)] \]

where \( R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W) \), \( \mathcal{R} \) is the curvature tensor of \( M \), \( p, q \) are scalar functions and \( T \) is a non-zero 1-form defined by

\[ g(X, U) = T(X), \]

where \( U \) is the unit vector field.

It can be easily seen that if the curvature tensor \( R \) is of the form (1.1), then the manifold is conformally flat. On the other hand, in [13], G. Vrânceanu defined the notion of almost constant curvature tensor by the same expression (1.1). Later in [8], A. L. Mocanu pointed out that the manifold introduced by Chen and Yano [3] and G. Vrânceanu [13] are the same. The notion of the quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki (see [9]). According to them a quasi-conformal curvature tensor is defined by

\[ \tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - \\
- g(X, Z)QY] - \frac{r}{n-1} \left[ a + 2b \right] [g(Y, Z)X - g(X, Z)Y], \]

where \( a \) and \( b \) are constants, \( S \) is the Ricci tensor, \( Q \) is the Ricci operator and \( r \) is the scalar curvature of the manifold \( M^n \).

A Riemannian manifold \((M^n, g) \) \((n > 3)\), is called quasi-conformally flat if the quasi-conformal curvature tensor \( \tilde{C} = 0 \). If \( a = 1 \) and \( b = \frac{1}{n-2} \), then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

A Riemannian manifold is said to be semi-symmetric (see [12]) if

\[ R(X, Y) \cdot R = 0, \]

where \( R \) is the Riemannian curvature tensor and \( R(X, Y) \) is considered as a derivation of the tensor algebra at each point of the manifold for tangent vector fields \( X, Y \). If a Riemannian manifold satisfies

\[ R(X, Y) \cdot \tilde{C} = 0, \]

where \( \tilde{C} \) is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.
2. Kenmotsu manifolds

Let $M$ be an almost contact metric manifold (see [1]) equipped with an almost contact metric structure $(\varphi, \xi, \eta, g)$ consisting of a $(1, 1)$ tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ and a compatible Riemannian metric $g$ satisfying

\[ \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \]

\[ g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \]

\[ g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X) \]

for all $X, Y \in TM$. An almost contact metric manifold is called a Kenmotsu manifold if it satisfies (see [6])

\[ (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad X, Y \in TM, \]

where $\nabla$ is Levi–Civita connection of the Riemannian metric. From the above equation it follows that

\[ \nabla_X \xi = X - \eta(X)\xi, \]

\[ (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \]

Moreover, the curvature tensor $R$, the Ricci tensor $S$, and the Ricci operator $Q$ satisfy (see [6])

\[ R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \]

\[ S(X, \xi) = (1 - n)\eta(X), \]

\[ Q\xi = (1 - n)\xi. \]

The equation (2.7) is equivalent to

\[ R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \]

which implies that

\[ R(\xi, X)\xi = X - \eta(X)\xi. \]

From (2.10) we have

\[ \eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X, Y). \]

Kenmotsu manifolds have been studied various authors. For example see [2], [4], [5], [7], [11].
A plane section $\Pi$ in $T_p M$ of an almost contact metric manifold $M$ is called a $\varphi$-section if $\Pi \perp \xi$ and $\varphi(\Pi) = \Pi$. If the sectional curvature $K(\Pi)$ does not depend on the choice of the $\varphi$-section $\Pi$ of $T_p M$, then $M$ is of pointwise constant $\varphi$-sectional curvature. A Kenmotsu manifold of pointwise constant $\varphi$-sectional curvature is called a Kenmotsu space form.

A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of $\varphi$ equals $-2d\eta \otimes \xi$) but not Sasakian. Moreover, it is also not compact since from the equation (2.5) we get $\text{div} \, \xi = n - 1$. In [6], K. Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval $I$ and a Kähler manifold $N$ with warping function $f(t) = se^t$, where $s$ is a nonzero constant; and (2) that a Kenmotsu manifold of constant $\varphi$-sectional curvature is a space of constant curvature $-1$, and so it is locally hyperbolic space. Examples of Kenmotsu manifolds of strictly pointwise constant $\varphi$-sectional curvature are not known so far and, according to D. Blair, one doubts that there are any, since the warped product structure of a Kenmotsu manifold involves a Kähler structure. Thus, one has to be careful for further study of Kenmotsu space forms with strictly pointwise constant $\varphi$-sectional curvature.

An almost contact metric manifold is said to be an $\eta$-Einstein if the Ricci tensor $S$ satisfies the condition

$$S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y)$$

where $a, b$ are certain scalars. If $b = 0$ then the manifold $M$ is an Einstein manifold.

3. Quasi-conformally flat Kenmotsu manifolds

Assume that $M^n$ is a quasi-conformally flat Kenmotsu manifold. Then from (1.1) we have

$$R(X, Y, Z, W) = \frac{b}{a} \left[ S(X, Z) g(Y, W) - S(Y, Z) g(X, W) + S(Y, W) g(X, Z) - S(X, W) g(Y, Z) \right] - \frac{r}{na} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z) g(X, W) - g(X, Z) g(Y, W)]$$

Putting $Z = \xi$ in (3.1) and using (2.3), (2.7) and (2.8) we obtain
\[ g(Y, W)\eta(X) - g(X, W)\eta(Y) = \frac{b}{a}[(1 - n)g(Y, W)\eta(X) - \]
\[ - (1 - n)g(X, W)\eta(Y) + S(Y, W)\eta(X) - S(X, W)\eta(Y)] + \]
\[ \frac{r}{na} \left[ \frac{a}{n - 1} + 2b \right] [g(X, W)\eta(Y) - g(Y, W)\eta(X)]. \]

Now putting \( Y = \xi \) in (3.2) and using (2.3), (2.7) and (2.8) it follows that

\[ \frac{b}{a} S(X, W) = Ag(X, W) + B\eta(X)\eta(W), \]

where

\[ A = \left[ 1 - \frac{b}{a}(1 - n) + \frac{r}{na} \left( \frac{a}{n - 1} + 2b \right) \right] \]

and

\[ B = \left[ -1 + 2(1 - n) \frac{b}{a} + \frac{r}{na} \left( \frac{a}{n - 1} + 2b \right) \right]. \]

Hence \( M^n \) is an \( \eta \)-Einstein manifold. By a contraction of the equation (3.2) we have

\[ r = nA + B. \]

In view of (3.4) and (3.5) we get

\[ \frac{b}{a} (2 - n) - 1 \left[ \frac{1}{n(n - 1)} r + 1 \right] = 0. \]

Hence either

\[ b = \frac{a}{2 - n} \]

or

\[ r = n(1 - n). \]

If \( b = \frac{a}{2 - n} \) then putting (3.8) into (3.7) we get

\[ \tilde{\mathcal{C}}(X, Y)Z = aC(X, Y)Z, \]

where \( C(X, Y)Z \) denotes the Weyl conformal curvature tensor. So the quasi conformally flatness and conformally flatness are equivalent in this case. But from [5] we know that a Kenmotsu manifold \( M^n \) is conformally flat if and only if it is locally isometric to the hyperbolic space \( H^n(-1) \). So in this case \( M^n \) is is locally isometric to the hyperbolic space \( H^n(-1) \).
If \( r = n(1 - n) \) then putting (3.9) into (3.4) and (3.5) the equation (3.3) turns into the form

\[
S(X, W) = (1 - n)g(X, W).
\]

This implies that \( M^n \) is an Einstein manifold. So putting (3.11) into (3.1) we obtain

\[
R(X, Y, Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W).
\]

Then \( M^n \) is of constant curvature \(-1\) and hence it is locally isometric to the hyperbolic space \( H^n(-1) \). If \( M^n \) is locally isometric to the hyperbolic space \( H^n(-1) \) then it is easy to see that \( M^n \) is quasi-conformally flat. This leads to the following theorem:

**Theorem 3.1.** Let \( (M^n, g) \) \((n > 3)\) be a Kenmotsu manifold. Then \( M^n \) is quasi-conformally flat if and only if \( M^n \) is locally isometric to the hyperbolic space \( H^n(-1) \).

### 4. Quasi conformally semi-symmetric Kenmotsu manifolds

Let us consider a quasi conformally semi-symmetric Kenmotsu manifold \((M^n, g), (n > 3)\). Then the condition

\[
R(X, Y) \cdot \tilde{C} = 0
\]

holds on \((M^n, g)\) for every vector fields \(X, Y\). Hence we have

\[
0 = (R(X, Y) \cdot \tilde{C})(U, V, W) = R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W - \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W.
\]

So for \(X = \xi\) we get

\[
0 = R(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(R(\xi, Y)U, V)W - \tilde{C}(U, R(\xi, Y)V)W - \tilde{C}(U, V)R(\xi, Y)W.
\]

In view of (2.10) the equation (4.1) can be written as

\[
0 = \eta(\tilde{C}(U, V)W)Y - \tilde{C}(U, V, W, Y)\xi - \eta(U)\tilde{C}(Y, V)W + g(Y, U)\tilde{C}(\xi, V)W - \eta(V)\tilde{C}(U, Y)W + g(Y, V)\tilde{C}(U, \xi)W - \eta(W)\tilde{C}(U, V)Y + g(Y, W)\tilde{C}(U, V)\xi,
\]

where \(\tilde{C}(U, V, W, Y) = g(\tilde{C}(U, V)W, Y)\). Taking the inner product of
(4.2) with \( \xi \) we have
\[
0 = \eta(\tilde{C}(U, V) W) \eta(Y) - \tilde{C}(U, V, W, Y) - \eta(U) \eta(\tilde{C}(Y, V) W) + \\
+ g(Y, U) \eta(\tilde{C}(\xi, V) W) - \eta(V) \eta(\tilde{C}(U, Y) W) + \\
+ g(Y, V) \eta(\tilde{C}(U, \xi) W) - \eta(W) \eta(\tilde{C}(U, V) Y).
\]
Putting \( Y = U \) the equation (4.3) turns into the form
\[
0 = - \tilde{C}(U, V, W, U) + g(U, U) \eta(\tilde{C}(\xi, V) W) + \\
+ g(U, V) \eta(\tilde{C}(U, \xi) W) - \eta(W) \eta(\tilde{C}(U, V) U).
\]
Let \( \{e_i\}, 1 \leq i \leq n, \) be an orthonormal basis of the tangent space at any point. Then in view of the equations (1.3), (2.7), (2.8), (2.10) and (2.12) the sum for \( U = e_i, 1 \leq i \leq n, \) of the relation (4.4) gives us
\[
S(V, W) = \left[ \frac{-br - b(n^2 - 1) + a(1 - n)}{a - b} \right] g(V, W) + \\
+ \left[ \frac{b[n(n - 1) + r]}{a - b} \right] \eta(V) \eta(W).
\]
So contracting the last equation we find the scalar curvature \( r \) of \( M^n \) as
\[
r = n(1 - n).
\]
Hence putting (4.6) into (4.5) we obtain
\[
(4.7) \quad S(V, W) = (1 - n)g(V, W).
\]
Then \( M^n \) is an Einstein manifold. So in view of (4.6), (4.7) and (1.3) the equation (4.2) is reduced to the form
\[
(4.8) \quad R(U, V, W, Y) = \left[ \frac{2nb - a}{a} \right] (g(U, V) g(Y, W) - g(U, W) g(V, Y)).
\]
Hence by a suitable contraction of the last equation we find
\[
(4.9) \quad S(V, W) = \left[ \frac{2nb - a}{a} \right] (n - 1)g(V, W).
\]
Comparing the right-hand sides of the equations (4.7) and (4.9) we obtain \( \frac{2nb - a}{a} = -1 \), which gives us \( b = 0 \). So the equation (4.8) turns into the form \( R(U, V, W, Y) = g(U, W) g(V, Y) - g(V, W) g(U, Y) \). Then \( M^n \) is locally isometric to hyperbolic space \( H^n(-1) \). Hence in view of Th. 3.1 we get that \( M^n \) is quasi-conformally flat. Then it is trivially quasi-conformally semisymmetric. So we have the following result:
Theorem 4.1. Let $(M^n, g) \ (n > 3)$ be a Kenmotsu manifold. Then $M^n$ is quasi conformally semisymmetric if and only if $M^n$ is locally isometric to the hyperbolic space $H^n(-1)$.

In view of Th. 3.1 and Th. 4.1 we have the following corollary:

Corollary 4.2. A Kenmotsu manifold $(M^n, g) \ (n > 3)$ is quasi conformally flat if and only if $M^n$ is quasi conformally semisymmetric.

References