INCLUSION THEOREMS FOR THE SPACES $\mathcal{F}_\alpha$

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Abstract. We give a new proof of one known inclusion theorem for the space $\mathcal{F}_\alpha$ that enables us to extend this theorem from the unit disc in $\mathbb{C}$ to the unit ball in $\mathbb{C}^n$, $n > 1$. We also improve an inclusion relation between Bergman spaces and the spaces $\mathcal{F}_\alpha$, $\alpha > 0$.

1. Introduction

Let $\Delta$ denote the unit disc in $\mathbb{C}$ with boundary $T$. We denote by $m$ the Lebesgue measure on $\Delta$ and by $\sigma$ the normalized Lebesgue measure on $T$.

Let $\mathcal{M}$ denote the set of complex valued Borel measures on $T$. For each $\alpha \geq 0$ let $\mathcal{F}_\alpha = \mathcal{F}_\alpha(\Delta)$ denote the family of functions on $\Delta$ having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T K_\alpha(z \xi) \, d\mu(\xi), \quad z \in \Delta,$$

where for $\alpha > 0$, $K_\alpha(w) = (1 - w)^{-\alpha}$, $w \in \Delta$, and $K_0(w) = 1 + \log \frac{1}{1 - w}$, $w \in \Delta$. In (1.1) and throughout this paper each logarithm means the principal branch. The family $\mathcal{F}_\alpha$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_\alpha} = \inf \|\mu\|$, where $\mu$ varies over all measures in $\mathcal{M}$ for which (1.1) holds and where $\|\mu\|$ denotes the total variation norm of $\mu$.

A function $f$ holomorphic in $\Delta$ (abbreviated $f \in H(\Delta)$) is said to belong to the Hardy space $H^p$, $0 < p < \infty$, if $\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty$ and to the weighted Bergman space $A^{p,\alpha}$, $0 < p < \infty$, $\alpha > -1$, if

$$\|f\|_{A^{p,\alpha}}^p = \int_0^1 (1 - r)^\alpha M_p^2(r, f) \, dr < \infty.$$ 

Here $M_p(r, f)$ is the $L^p(\{z \in \Delta : |z| = r\})$ “norm” of $f$.

Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be holomorphic in $\Delta$. We define the multiplier transformation $D^\beta g$ of $g$, where $\beta$ is a real number, by

$$D^\beta g(z) = \sum_{k=0}^{\infty} (k + 1)^\beta a_k z^k.$$ 

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A function $f \in H(\Delta)$ is said to belong to the space $H^p_\alpha$ if $\|D^p f\|_{H_\alpha} < \infty$ and to the space $A^p_\alpha$ if $\|D^p f\|_{A_\alpha} < \infty$.

For an arc $I \subset T$ and a function $g \in H^1$ let $MO(g, I)$ denote the mean oscillation of $g$ over $I$, i.e.

$$MO(g, I) = \frac{1}{\sigma(I)} \int_I [g(\xi) - \frac{1}{\sigma(I)} \int_I g(\eta) \, d\sigma(\eta)] \, d\sigma(\xi),$$

and let, as usual, $BMOA = BMOA(\Delta)$ be the space of functions $g \in H^1$ such that $\|g\|_{BMOA} = \sup MO(g, I) < \infty$, where the supremum runs over all arcs $I \subset T$.

We are now ready to state our first result

**Theorem 1.** $\mathcal{F}_0(\Delta) \subset BMOA(\Delta)$.

Theorem is known (see [5]). The argument given in [5] is limited to the one dimensional case. Our proof is different and may be easily extended to the corresponding spaces $\mathcal{F}_0(B^n)$ and $BMOA(B^n)$ of holomorphic functions on the unit ball $B^n$ in $\mathbb{C}^n$, $n > 1$. Thus we have

**Theorem 2.** $\mathcal{F}_0(B^n) \subset BMOA(B^n)$.

In [6] it is shown that if $\alpha > 0$, then $A^{1,\alpha-1}_1 \subset \mathcal{F}_0$ (Lemma 1 and Lemma 2, pp. 159–160). Since $A^{1,\alpha-1}_1 \subset H^{1-\alpha}_{1,\alpha}$ and $H^{1-\alpha}_{1,\alpha} \setminus A^{1,\alpha-1}_1$ is a non-empty set, the following theorem is an improvement of this inclusion.

**Theorem 3.** Let $\alpha > 0$. Then $H^{1-\alpha}_{1,\alpha} \subset \mathcal{F}_0$.

2. Proof of Theorem 1.

Let $f \in \mathcal{F}_0$. Then there exists $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_T \log \frac{1}{1 - z\xi} \, d\mu(\xi). \tag{2.1}$$

To show that $f \in BMOA$ it suffices to show that

$$\sup_{\Delta} \int_{\Delta} \frac{1 - |z|^2}{|1 - z\xi|^2} |f'(w)|^2 (1 - |w|^2) \, dm(w) < \infty,$$

(see [4], p. 240). Using (2.1) we find that

$$\int_{\Delta} \frac{1 - |z|^2}{|1 - z\xi|^2} |f'(w)|^2 (1 - |w|^2) \, dm(w) \leq C \int_T d|\mu|(\xi) \int_{\Delta} \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\xi|^2|1 - \bar{\xi}z|^2} \, dm(w).$$

Since $|\mu|(T) < \infty$, it is sufficient to show that

$$\int_{\Delta} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^2|1 - \bar{\xi}z|^2} \leq \frac{C}{1 - |z|^2}, \quad \text{for all } z \in \Delta \text{ and } \xi \in T. \tag{2.2}$$
In this note we follow the custom of using letters $C$, $C_1$, $C_2$, \ldots, to stand for positive constants which change their values from one occurrence to another while remaining independent of the important variables.

Given $z \in \Delta$ and $\xi \in T$ we consider the following partition of $\Delta$:

$$
\begin{align*}
\Omega_1 &= \{ w \in \Delta : |1 - zw| \leq \frac{1}{2}|1 - z\xi| \}, \\
\Omega_2 &= \{ w \in \Delta : |1 - w\xi| \leq \frac{1}{2}|1 - z\xi| \}, \\
\Omega_3 &= \{ w \in \Delta : \frac{1}{2}|1 - z\xi| < |1 - zw| \leq |1 - w\xi| \}, \\
\Omega_4 &= \{ w \in \Delta : \frac{1}{2}|1 - z\xi| < |1 - w\xi| \leq |1 - wz| \}.
\end{align*}
$$

With this notation we have

$$
\begin{align*}
|1 - wz| \leq C_1 |1 - z\xi| \leq C_2 |1 - w\xi|, & \quad w \in \Omega_1, \\
|1 - w\xi| \leq C_1 |1 - z\xi| \leq C_2 |1 - zw|, & \quad w \in \Omega_2, \\
|1 - z\xi| \leq C_1 |1 - zw| \leq C_2 |1 - w\xi|, & \quad w \in \Omega_3, \\
|1 - w\xi| \leq C_1 |1 - w\xi| \leq C_2 |1 - wz|, & \quad w \in \Omega_4.
\end{align*}
$$

Using this we find

$$
\int_{\Omega_1} \frac{(1 - |w|^2) \, dm(w)}{|1 - zw|^2 |1 - \xi\bar{w}|^2} \leq \frac{C}{1 - |z|} \int_{\Delta} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^4} \leq \frac{C}{1 - |z|}, \quad (2.3)
$$

$$
\int_{\Omega_2} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_2} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi| + |1 - w\xi|^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_2} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^2 + (1 - |w|)^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_2} \frac{\, dx}{(1 + x)^2} \leq \frac{C}{1 - |z|}, \quad (2.4)
$$

$$
\int_{\Omega_3} \frac{(1 - |w|^2) \, dm(w)}{|1 - zw| |1 - \xi\bar{w}|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_3} \frac{(1 - |w|^2) \, dm(w)}{|1 - zw|^4} \leq \frac{C}{1 - |z|}, \quad (2.5)
$$

$$
\int_{\Omega_4} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_4} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi| + |1 - w\xi|^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_4} \frac{(1 - |w|^2) \, dm(w)}{|1 - z\xi|^2 + (1 - |w|)^2 |1 - w\xi|^2} \leq \frac{C}{1 - |z|} \int_{\Omega_4} \frac{\, dx}{(1 + x)^2} \leq \frac{C}{1 - |z|}, \quad (2.6)
$$

Now (2.2) follows from (2.3), (2.4), (2.5) and (2.6). This finishes the proof of Theorem 1. \hfill \blacksquare
3. Proof of Theorem 3.

Let $\alpha$ be a positive real numbers. Define the function $H_\alpha$ by

$$H_\alpha(z) = \sum_{n=0}^{\infty} (n+1)^{\alpha-1} z^n, \quad \text{for } z \in \Delta.$$  

Let $\mathcal{H}_\alpha$ denote the family of functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \Delta$, having the property that there exists $\mu \in \mathcal{M}$ such that

$$a_n = (n+1)^{\alpha-1} \int_T \xi^n d\mu(\xi), \quad n = 0, 1, 2, \ldots$$  

(3.1)

Let $\|f\|_{\mathcal{H}_\alpha} = \inf \|\mu\|$, where $\mu$ varies over all members of $\mathcal{M}$ for which (3.1) holds. Then $\mathcal{H}_\alpha$ is a Banach space.

For the proof of Theorem 3 the following lemma is needed.

**Lemma 3.1.** If $\alpha > 0$ then $f \in \mathcal{F}_\alpha$ if and only if $f \in \mathcal{H}_\alpha$. There is a positive constant $C$ depending only on $\alpha$ such that if $f \in \mathcal{F}_\alpha$ then

$$C^{-1} \|f\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{H}_\alpha} \leq C \|f\|_{\mathcal{F}_\alpha}.$$  

**Proof.** Suppose that $f \in \mathcal{F}_\alpha$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \Delta$. Then there exists $\mu \in \mathcal{M}$ such that

$$a_n = A_n(\alpha) \int_T \xi^n d\mu(\xi), \quad n = 0, 1, 2, \ldots,$$  

(3.2)

where $A_n(\alpha) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$, $n = 0, 1, 2, \ldots$. From this it follows that

$$A_n(\alpha) = (n+1)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + B_n(\alpha) \right), \quad \text{for } n = 1, 2, \ldots,$$  

(3.3)

and there is a positive constant $B(\alpha)$ such that $|B_n(\alpha)| \leq B(\alpha)/n$, for $n = 1, 2, \ldots$. For $n = 1, 2, \ldots$ let

$$c_n(\alpha) = B_n(\alpha) \int_T \xi^n d\mu(\xi)$$

and define the function $g$ by $g(z) = \sum_{n=1}^{\infty} c_n(\alpha) z^n$, for $z \in \Delta$. Since $|c_n(\alpha)| \leq |B_n(\alpha)| \|\mu\| \leq \frac{B(\alpha)}{n} \|\mu\|$, $g \in H^2$. Therefore $g \in \mathcal{F}_1$ (see [2]) and hence there exists $\nu \in \mathcal{M}$ such that

$$g(z) = \int_T \frac{d\nu(\xi)}{1 - z\xi}.$$  

This implies that

$$c_n(\alpha) = \int_T \xi^n d\nu(\xi), \quad \text{for } n = 1, 2, \ldots.$$
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Thus

$$a_n = (n+1)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} \int_T \xi^n d\mu(\xi) + \int_T \xi^n d\nu(\xi) \right), \quad \text{for } n = 1, 2, \ldots$$

Let $\lambda = \frac{1}{\Gamma(\alpha)} \mu + \nu + b \sigma$, where $b = \mu(T)(\frac{1}{\Gamma(\alpha)} + 1) - \nu(T)$. Then

$$a_n = (n+1)^{\alpha-1} \int_T \xi^n d\lambda(\xi), \quad n = 0, 1, 2, \ldots$$

Since $\lambda \in \mathcal{M}$, $f \in \mathcal{H}_\alpha$.

The argument given above shows that

$$\|f\|_{\mathcal{N}_\alpha} \leq \frac{1}{\Gamma(\alpha)} \|\mu\| + \|\nu\| + |b| \leq C(\|\mu\| + \|g\|_{H^2}) \leq C\|\mu\|.$$

Here we have used again that $H^2 \subset H^1 \subset \mathcal{F}_1$. This inequality holds for every $\mu \in \mathcal{M}$ for which (3.2) holds. Hence $\|f\|_{\mathcal{N}_\alpha} \leq C\|f\|_{\mathcal{F}_\alpha}$.

The same argument shows that if $f \in \mathcal{H}_\alpha$ then $f \in \mathcal{F}_\alpha$ and $\|f\|_{\mathcal{F}_\alpha} \leq C\|f\|_{\mathcal{N}_\alpha}$.

Instead of (3.3) it should use the relation

$$(n+1)^{\alpha-1} = A_n(\alpha)[\Gamma(\alpha) + D_n(\alpha)], \quad \text{for } n = 1, 2, \ldots, \quad (3.4)$$

and $|D_n(\alpha)| \leq \frac{D(\alpha)}{\alpha}$, for some positive constant $D(\alpha)$. For (3.3) and (3.4), see [3] and [7].

**Proof of Theorem 3.** Let $f \in H^1_{\lambda_w}$ and $f(z) = \sum_{k=0}^n a_k z^k$, $z \in \Delta$. Then $D^{1-\alpha} f \in H^1$. Since $H^1 \subset \mathcal{F}_1$ we have $D^{1-\alpha} \in \mathcal{F}_1$. Hence there exists a measure $\mu \in \mathcal{M}$ such that

$$(n+1)^{\alpha-1} a_n = \int_T \xi^n d\mu(\xi), \quad n = 0, 1, 2, \ldots,$$

or equivalently

$$a_n = (n+1)^{\alpha-1} \int_T \xi^n d\mu(\xi), \quad n = 0, 1, 2, \ldots$$

Therefore, $f \in \mathcal{H}_\alpha$ and by Lemma 3.1 we have $f \in \mathcal{F}_\alpha$. Thus, $H^1_{\lambda_w} \subset \mathcal{F}_\alpha$. ■

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