BILINEAR EXPANSIONS OF THE KERNELS OF SOME NONSELFADJOINT INTEGRAL OPERATORS

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Abstract. Let $H$ and $S$ be integral operators on $L^2(0, 1)$ with continuous kernels, Suppose that $H > 0$ and let $A = H(I + S)$. It is shown that if the (nonselfadjoint) operator $S$ is small in a certain sense with respect to $H$, then the corresponding Fourier series of functions from $R(A)$ (or $R(A^*)$) converges uniformly on $[0, 1]$.

1. Introduction

Let $H$ and $S$ be integral operators on $L^2(0, 1)$ (with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$) with continuous kernels $H(x, y)$ and $S(x, y)$ on $[0, 1] \times [0, 1]$. Suppose that $H > 0$ and let

$$A = H(I + S).$$

Classical theorems (case $S = 0$, see [5]) state that the kernel $H$ can be expanded into a uniformly convergent (on $[0, 1] \times [0, 1]$) bilinear series.

A consequence of this is that every function $f \in R(H)$ has the uniformly convergent Fourier series with respect to the system of eigenfunctions of $H$. ($R(H)$ denotes the image of $H$ in $L^2(0, 1)$).

Similar results hold in some cases when $S \neq 0$. Namely, if $S = S^*$, it was proved in [1] and [2] that the corresponding variant of Mercer’s theorem holds. The proof was based on the spectral theorem for an operator on $L^2(0, 1)$ with the definite or indefinite inner product generated by the formula

$$[f, g] = \langle (I + S)f, g \rangle.$$

In [4], a series of nice results was obtained which were related to bilinear expansions of smooth Carleman’s kernels of Mercer type.

A natural question is about bilinear expansions when $S \neq S^*$. We shall show that if the operator $S$ is small in a certain sense with respect to $H$, then the corresponding Fourier series of functions from $R(A)$ (or $R(A^*)$) converges uniformly uniformly

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on $[0, 1]$. (Note that, because of the continuity of the kernel of $A$, we have $R(A) \subset C[0, 1]$).

In the sequel, $\mathcal{A}(\cdot, \cdot)$ will denote the kernel of the operator $A$ (defined by (1)).

## 2. The results

**Theorem.** Let, for the operators $H$ and $S$ from (1), there exists $\omega > 0$ such that $S^*S \leq \omega H^2$. If $s_k$ are singular values of $A$ and $f_k$ the normalized eigenvectors of $A^*A$ (i.e., $A^*A f_k = s_k^2 f_k$) and $g_k = (s_k)^{-1} A f_k$, then the series $\sum_{k \geq 1} s_k f_k(y) g_k(x)$ is absolutely convergent on $[0, 1]^2$ and uniformly convergent on $[0, 1]$ with respect to arbitrary variable and its sum is equal to $\mathcal{A}(x, y)$. Also, for every $f \in R(A)$ (resp. $g \in R(A^*)$) the series $\sum_{k \geq 1} \langle f, g_k \rangle g_k$ (resp. $\sum_{k \geq 1} \langle g, f_k \rangle f_k$) converges uniformly on $[0, 1]$ to $f$ (resp. $g$).

In the proof of this assertions we need the following two Lemmas:

**Lemma 1.** [5] If $T: L^2(0, 1) \rightarrow L^2(0, 1)$ is the linear operator defined by $Tf(x) = \int_0^1 M(x, y) f(y) dy$ and if $M \in C([0, 1]^2)$ and $\langle T f, f \rangle \geq 0$ for all $f \in L^2(0, 1)$, then $M(x, x) \geq 0$ for all $x \in [0, 1]$.

**Lemma 2.** If $A = H(I + S)$, $H > 0$, $S^*S \leq \omega H^2$, then there exists a constant $c > 0$ such that

$$\sqrt{AA^*} \leq c H, \quad \sqrt{A^*A^*} \leq c H.$$

**Proof.** Since

$$A^*A = H^2 + S^*H^2 + H^2S + S^*H^2S$$

we have to estimate $\langle S^*H^2 f, f \rangle$ and $\langle H^2Sf, f \rangle$.

The operator $H^2$ is positive and thus, by the Cauchy inequality, we have

$$\langle S^*H^2 f, f \rangle = \langle H^2 f, Sf \rangle \leq \langle H^2 f, f \rangle \langle H^2 Sf, Sf \rangle$$

$$= \|Hf\|^2 \|HSf\|^2 \leq \|Hf\|^2 \|H\|^2 \|Sf\|^2$$

$$= \|Hf\|^2 \|H\|^2 \langle S^*Sf, f \rangle \leq \|Hf\|^2 \|H\|^2 \omega \langle H^2 f, f \rangle$$

$$= \omega \|H\|^2 \|Hf\|^2.$$

Therefore

$$\langle S^*H^2 f, f \rangle \leq \sqrt{\omega} \|H\| \langle H^2 f, f \rangle$$

and hence we get

$$\langle H^2Sf, f \rangle \leq \sqrt{\omega} \|H\| \langle H^2 f, f \rangle.$$  

Since

$$\langle S^*H^2Sf, f \rangle = \|HSf\|^2 \leq \|H\|^2 \|Sf\|^2 = \|H\|^2 \langle S^*Sf, f \rangle \leq \omega \|H\|^2 \langle H^2 f, f \rangle,$$

from (2), (3), (4) it follows that $\langle A^*Af, f \rangle \leq (1 + \sqrt{\omega} \|H\|)^2 \langle H^2 f, f \rangle$, i.e. $A^*A \leq (1 + \sqrt{\omega} \|H\|)^2 H^2$. 

Bilinear expansions of the kernels of integral operators

Having in mind that the function \( \lambda \mapsto \sqrt{\lambda} \) is operator monotone, we get

\[
\sqrt{A^*A} \leq (1 + \sqrt{\omega} \|H\|)H.
\] (5)

From the equality \( A^* = (I + S^*)H \) we get \( \|A^*f\| \leq \|I + S^*\| \cdot \|Hf\| \ (f \in L^2(0,1)), \)

i.e.

\[
\sqrt{A^*A} \leq \|I + S^*\| \cdot H.
\] (6)

From (5) and (6) we obtain the assertion of the Lemma, with

\[ c = \max\{1 + \sqrt{\omega} \|H\|, \|I + S^*\|\}. \]

\[ \square \]

**Proof of the Theorem.** From \( A^*Af_k = s_k^2 f_k \) and \( Af_k = g_k \) it follows that \( f_k, g_k \in C[0,1] \) (because \( A \) has the continuous kernel) and \( A \) has the following singular (see [3]) expansion

\[ A = \sum_{k \geq 1} s_k \langle \cdot, f_k \rangle g_k. \]

Also, there holds

\[
\sqrt{A^*A} f = \sum_{k \geq 1} s_k \langle f, f_k \rangle g_k
\]

\[
\sqrt{A^*A} f = \sum_{k \geq 1} s_k \langle f, g_k \rangle g_k, \quad f \in L^2(0,1).
\] (7)

The series on the right-hand side of the previous equalities converge in the norm of \( L^2(0,1) \).

Consider the operators (on \( L^2(0,1) \)) \( S_n^* \), \( S_n \) defined in the following way:

\[
S_n^* = cH - \sum_{k=1}^n \langle \cdot, f_k \rangle f_k, \quad S_n = cH - \sum_{k=1}^n \langle \cdot, g_k \rangle g_k.
\]

Since, by (7) and Lemma 2, \( \langle S_n^*, f \rangle \geq 0, \langle S_n, f \rangle \geq 0, f \in L^2(0,1) \) and since the operators \( S_n^*, S_n \) have continuous kernels, we get from Lemma 1

\[
cH(x, x) \geq \sum_{k=1}^n s_k |f_k(x)|^2, \quad cH(x, x) \geq \sum_{k=1}^n s_k |g_k(x)|^2, \quad n \in \mathbb{N}.
\]

Since \( H \in C([0,1]^2) \), there exists \( M_0 < +\infty \) such that

\[
\sum_{k=1}^n s_k |f_k(x)|^2 \leq M_0, \quad \sum_{k=1}^n s_k |g_k(x)|^2 \leq M_0.
\] (8)

From (8) it follows that the series

\[
\sum_{k \geq 1} s_k f_k(y) g_k(x)
\]

is absolutely convergent for all \( x, y \in [0,1] \). Let \( S(x, y) \) denote its sum. Observe that form (8) it follows that the partial sums of the previous series are bounded by \( M_0 \).

Fix \( x \in [0,1] \). Then we have

\[
\left| \sum_{k=p}^q s_k f_k(y) g_k(x) \right|^2 \leq \sum_{k=p}^q s_k |f_k(y)|^2 \sum_{k=p}^q s_k |g_k(x)|^2
\]

\[
\leq M_0 \sum_{k=p}^q s_k |g_k(x)|^2 \to 0 \quad (p, q \to \infty)
\]

\[ \square \]
and the series \( \sum_{k \geq 1} s_k f_k(y) g_k(x) \) converges uniformly with respect to \( y \) on \([0, 1]\), for every fixed \( x \) and hence its sum is a continuous function with respect to \( y \).

Let \( f \in C[0, 1] \) be a fixed function. Then (because of the uniform convergence with respect to \( y \))

\[
\int_0^1 S(x, y) f(y) \, dy = \sum_{k \geq 1} s_k g_k(x) \int_0^1 f(y) f_k(y) \, dy = \sum_{k \geq 1} s_k g_k(x) (f, f_k).
\] (9)

(The series on the right-hand side of (9) converges not only for every \( x \) but also uniformly with respect to \( x \) because

\[
\left| \sum_{k=p}^q s_k g_k(x) (f, f_k) \right|^2 \leq \sum_{k=p}^q s_k |g_k(x)|^2 \sum_{k=p}^q s_k (f, f_k)^2 \\
\leq M_0 s_p \| f \|^2 \to 0 \quad (p, q \to \infty) \)

On the other hand, from the singular expansion of \( A \) we get

\[
\int_0^1 A(x, y) f(y) \, dy = \sum_{k \geq 1} s_k g_k(x) (f, f_k) 
\] (10)

(the series converges in the norm of \( L^2(0, 1) \)).

Thus, from (9), (10) it follows that for every \( f \in C[0, 1] \) we have

\[
\int_0^1 (A(x, y) - S(x, y)) f(y) \, dy = 0.
\]

Putting \( f(y) = \overline{S(x, y)} - A(x, y) \) \(( \in C[0, 1] \)) we get

\[
\int_0^1 |A(x, y) - S(x, y)|^2 \, dy = 0
\]

and hence \( A(x, y) = S(x, y) \) for every \( y \in [0, 1] \). Since \( x \in [0, 1] \) was arbitrary, we have \( A(x, y) = S(x, y) \), \( x, y \in [0, 1] \). So

\[
A(x, y) = \sum_{k \geq 1} s_k f_k(y) g_k(x)
\]

for every \( x, y \in [0, 1] \).

Let now \( f \in R(A) \). Then

\[
f(x) = \int_0^1 A(x, y) \varphi(y) \, dy, \quad \varphi \in L^2(0, 1)
\]

and thus, by the Lebesgue dominated convergence theorem, we have \( A_n(x, y) = \sum_{k=1}^n s_k f_k(y) g_k(x)) \)

\[
f(x) = \int_0^1 \lim_{n \to \infty} A_n(x, y) \varphi(y) \, dy = \lim_{n \to \infty} \int_0^1 A_n(x, y) \varphi(y) \, dy \\
= \lim_{n \to \infty} \sum_{k=1}^n s_k g_k(x) (\varphi, f_k) = \sum_{k \geq 1} s_k g_k(x) (\varphi, f_k).
\]
Since the preceding series converges uniformly with the respect to $x \in [0,1]$ and since $\{g_k\}$ is an orthonormal system in $L^2(0,1)$ (see [3]), we have $s_k(\varphi,f_k) = \langle f, g_k \rangle$ and, finally, we get
\[ f(x) = \sum_{k \geq 1} \langle f, g_k \rangle g_k(x) \]
(the series converges uniformly on $[0,1]$).

The assertion for the function $g \in R(A^*)$ can be proved in a similar way. ■

REMARK. The second part of the Theorem was proved in [5] in a different way. The proof presented here is a consequence of the previously established bilinear expansion of the function $A(x, y)$.

REFERENCES


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