A MULTIVALUED FIXED POINT THEOREM IN ULMETRIC SPACES

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Abstract. The purpose of this paper is to prove that a class of generalized contractive multivalued mappings on a spherically complete ultrametric space has a fixed point.

Let \((X,d)\) be a metric space. If the metric \(d\) satisfies strong triangle inequality: for all \(x, y, z \in X\)
\[d(x, y) \leq \max\{d(x, z), d(z, y)\},\]
it is called \textit{ultrametric} on \(X\) [4]. Pair \((X,d)\) is now an \textit{ultrametric space}.

Remark. If \(X \neq 0\), then the so called discrete metric \(d\) defined on \(X\) by
\[d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}\]
is an ultrametric.

Example. For \(a \in \mathbb{R}\) let \([a]\) be the entire part of \(a\). By
\[d(x, y) = \inf\{2^{-n} : n \in \mathbb{Z}, [2^n(x - e)] = [2^n(y - e)]\}\]
(here \(e\) is any irrational number) an ultrametric \(d\) on \(\mathbb{Q}\) is defined which determines the usual topology on \(\mathbb{Q}\) [4].

An ultrametric space \((X,d)\) is said to be \textit{spherically complete} if every shrinking collection of balls in \(X\) has a nonempty intersection.

In [3] authors proved a fixed point theorem for contractive function on spherically complete ultrametric space \(X\). Let us recall: \(T : X \to X\) is said to be contractive if for every \(x, y \in X, x \neq y, d(Tx, Ty) < d(x, y)\). This result is generalized in [2] for multivalued mappings \(T : X \to 2^X\) (\(2^X\) is the space of all nonempty compact subsets in \(X\) with Hausdorff metric \(H\)).

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On the other side, the result from [3] is generalized for a class of functions $T: X \rightarrow X$ such that for every $x, y \in X, x \neq y$

$$d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$  

Now we are going to prove the related result for multivalued mappings.

**Theorem.** Let $(X, d)$ be a spherically complete ultrametric space. If $T: X \rightarrow 2^X$ is such that for any $x, y \in X, x \neq y$,

$$H(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$  

(1) then $T$ has a fixed point (i.e., there exists $x \in X$ such that $x \in Tx$).

**Proof.** Let $B_a = B[a; d(a, Ta)]$ denote the closed ball centered at $a$ with radius $d(a, Ta) = \inf_{z \in Ta} d(a, z)$, and let $A$ be the collection of these spheres for all $a \in X$. The relation

$$B_a \subseteq B_b \text{ iff } B_a \subseteq B_b$$

is a partial order on $A$. Let $A_1$ be a totally ordered subfamily of $A$. Since $X$ is spherically complete, $\bigcup_{B_a \in A_1} B_a = B \neq \emptyset$. Let $b \in B$ and $B_a \in A_1$. Obviously, $b \in B_a$, so $d(b, a) \leq d(a, Ta)$.

Take $u \in T(a)$ such that $d(a, u) = d(a, Ta)$ (it is possible since $Ta$ is a nonempty compact set). Then

$$d(b, Tb) \leq \inf_{c \in Tb} d(b, c) \leq \max\{d(b, a), d(a, u), \inf_{c \in Tb} d(u, c)\}$$

$$\leq \max\{d(a, Ta), H(Ta, Tb)\} \leq \max\{d(a, Ta), d(b, Tb)\}$$

$$= \max\{d(a, Ta), d(b, Tb)\}$$

which is possible only for $d(b, Tb) < d(a, Ta)$. Now, for any $x \in B_b$,

$$d(x, b) \leq d(b, Tb) < d(a, Ta),$$

$$d(x, a) \leq \max\{d(x, b), d(b, a)\} < d(a, Ta),$$

so $x \in B_a$. We have just proved that $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus $B_b$ is an upper bound in $A$ for the family $A_1$. By Zorn’s lemma there is a maximal element in $A$, say $B_z$. We shall prove that $z \in Tz$.

In opposite case, $z \notin Tz$, there exists $\tilde{z} \in Tz, \tilde{z} \neq z$, such that $d(z, \tilde{z}) = d(z, Tz)$. Let us prove that $B_{\tilde{z}} \subseteq B_z$.

$$d(\tilde{z}, T\tilde{z}) \leq H(T\tilde{z}, T\tilde{z}) \leq \max\{d(\tilde{z}, \tilde{z}), d(z, Tz), d(\tilde{z}, T\tilde{z})\}$$

$$= \max\{d(z, Tz), d(\tilde{z}, T\tilde{z})\},$$

which is possible only for $d(\tilde{z}, T\tilde{z}) < d(z, Tz)$. Now, for any $y \in B_{\tilde{z}}$,

$$d(y, \tilde{z}) \leq d(\tilde{z}, T\tilde{z}) < d(z, Tz),$$

$$d(y, z) \leq \max\{d(y, \tilde{z}), d(\tilde{z}, z)\} \leq d(z, Tz),$$
which means that $y \in B_z$, so $B_z \subseteq B_z$. But $d(z, z) = d(z, Tz) > d(z, Tz)$, hence $z \notin B_z$, so $B_z \subsetneq B_z$. This fact contradicts the maximality of $B_z$. So we have proved that $T$ has a fixed point. ■

REFERENCES


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