AN $L_p$ ESTIMATE FOR THE DIFFERENCE OF DERIVATIVES OF SPECTRAL EXPANSIONS ARISING BY ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Abstract. We prove the estimate
\[ ||\sigma_\mu(x,f) - \check{\sigma}_\mu(x,f)||_{L_p(G)} \leq C||f||_{BV(G)} \cdot \mu^{1-1/p}, \]
where $2 \leq p < +\infty$, and $\sigma_\mu(x,f), \check{\sigma}_\mu(x,f)$ are the partial sums of spectral expansions of a function $f(x) \in BV(G)$, corresponding to arbitrary non-negative self-adjoint extensions of the operators $\mathcal{L}u = -u'' + q(x)u$, $\check{\mathcal{L}}u = -u'' + \check{q}(x)u$ $(x \in G)$ respectively; the operators are defined on an arbitrary bounded interval $G \subset \mathbb{R}$.

1. Introduction

Let $G = (a, b)$ be an arbitrary bounded interval, and let the operators
\[ \mathcal{L}u = -u'' + q(x)u, \quad \check{\mathcal{L}}u = -u'' + \check{q}(x)u \quad (1) \]
be defined on $G$, with potentials $q(x), \check{q}(x) \in L_\infty(G), 1 < s \leq 2$. Denote by $L, \check{L}$ arbitrary non-negative self-adjoint extensions, with discrete spectrum, of the operators (1) respectively (see §17, [1]). Let \( \{u_n(x)\}_{n=1}^\infty, \{\check{u}_n(x)\}_{n=1}^\infty \) be complete (in $L_2(G)$) and orthonormal systems of eigenfunctions of those extensions, and \( \{\lambda_n\}_{n=1}^\infty, \{\check{\lambda}_n\}_{n=1}^\infty \) the corresponding systems of non-negative eigenvalues, enumerated in non-decreasing order. If $f(x) \in L_1(G)$ and $\mu \geq 2$, we can form the partial sums of order $\mu$:
\[ \sigma_\mu(x,f) = \sum_{\sqrt{\lambda_n} < \mu} f_n u_n(x), \quad \check{\sigma}_\mu(x,f) = \sum_{\sqrt{\check{\lambda}_n} < \mu} \check{f}_n \check{u}_n(x), \]
where $f_n = \int_a^b f(x)u_n(x) \, dx$, $\check{f}_n = \int_a^b f(x)\check{u}_n(x) \, dx$. Let $AC(G)$ be the set of absolutely continuous functions on the closed interval $\overline{G}$. Denote by $BV(G)$ the Banach space of functions having bounded variation on $\overline{G}$, with the norm $||f||_{BV(G)} = \sup_{x \in \overline{G}} |f(x)| + V^s_a(f)$, where $V^s_a(f)$ stands for the total variation of $f(x)$ on $\overline{G}$.

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The problem of behavior of function $\sigma_\mu(x, f)$ (and its derivatives) on subsets of $G$, as $\mu \to +\infty$, is the classical one. One of the most fruitful approaches to the problem is so-called “equiconvergence approach”: one studies the behavior of the difference $\sigma_\mu(x, f) - S_\mu(x, f)$, as $\mu \to +\infty$, where $S_\mu(x, f)$ is the corresponding partial sum of the trigonometrical Fourier series of function $f$ (for a review see [2]). It seems that the first results concerning the equiconvergence rate estimates, in the case of arbitrary self-adjoint Sturm-Liouville operators, were obtained by V.A. Il’in and I. Joo in [3]. They obtained the following estimate:

If $q(x), \tilde{q}(x) \in L_s(G)$ ($s > 1$), $f(x) \in AC(G)$, and $K \subset G$ is an arbitrary compact set, then there exists a constant $C(K, f) > 0$ such that

$$\max_{x \in K} |\sigma_\mu(x, f) - \tilde{\sigma}_\mu(x, f)| \leq C(K, f) \cdot \frac{1}{\mu}, \quad \mu \geq 2; \quad (2)$$

$C(K, f)$ does not depend on $\mu$. The estimate is exact in order with respect to $\mu$.

In order to “globalize” the estimate (2), I.S. Lomov has considered the $L_p$ metric instead of the uniform one; in paper [4] he proved the following assertion: If $q(x), \tilde{q}(x) \in L_s(G)$ ($s > 1$), $f(x) \in BV(G)$, and $2 \leq p < +\infty$, then the estimate

$$||\sigma_\mu(x, f) - \tilde{\sigma}_\mu(x, f)||_{L_p(G)} \leq C||f||_{BV(G)} \cdot \frac{1}{\mu^{1/p}}, \quad \mu \geq 3, \quad (3)$$

holds, where $C > 0$ does not depend on $f$ and $\mu$. (Note that in earlier paper [5] Lomov obtained estimate (3) with $\mu^{-1/p} \ln \mu$ instead of $\mu^{-1/p}$.)

A local uniform estimate for the difference of the first derivatives $\sigma'_\mu(x, f)$, $\tilde{\sigma}'_\mu(x, f)$ was obtained by I. Joo and N. Lažetić in paper [6]. They proved: If $q(x)$ and $\tilde{q}(x)$ belong to $L_s(G)$ ($1 < s \leq 2$), $f(x) \in AC(G)$, and $K \subset G$ is an arbitrary compact set, then the estimate

$$\max_{x \in K} |\sigma'_\mu(x, f) - \tilde{\sigma}'_\mu(x, f)| \leq C(K, f), \quad \mu \geq 2; \quad (4)$$

holds, where $C(K, f) > 0$ is independent of $\mu$. This estimate is exact in order with respect to the spectral parameter $\mu$.

Recently, the estimate (4) has been extended on the set $BV(G)$. Namely, the authors of this paper have proved ([7]) that for every function $f(x) \in BV(G)$ and every compact set $K \subset G$ the following estimate is valid:

$$\max_{x \in K} |\sigma'_\mu(x, f) - \tilde{\sigma}'_\mu(x, f)| \leq C(K)||f||_{BV(G)}, \quad \mu \geq 2. \quad (5)$$

It is supposed that $q(x), \tilde{q}(x) \in L_s(G)(s > 1)$.

In this paper we propose an $L_p$ estimate for the difference mentioned above. That estimate “globalizes” (5), and shows how the estimate (3) is affected by the operation of differentiation (compare with estimates (8)-(9) below). Hence, our result is the following assertion.

**Theorem.** Suppose $q(x), \tilde{q}(x) \in L_s(G)$ ($1 < s \leq 2$), $f(x) \in BV(G)$, $p \in [2, +\infty)$, and $\mu \geq 2$. There exists a constant $C > 0$, independent of $f$ and $\mu$, such that the following estimate holds:

$$||\sigma'_\mu(x, f) - \tilde{\sigma}'_\mu(x, f)||_{L_p(G)} \leq C||f||_{BV(G)} \cdot \mu^{1-1/p}. \quad (6)$$
2. Auxiliary results

Proof of the theorem is based on estimate (5). But we will also use a variety of known results listed below.

Let $q(x) \in L_1(G)$. Then for systems of eigenfunctions and eigenvalues of an arbitrary non-negative self-adjoint extension $L$ of the operator $\mathcal{L}$ the following estimates are valid:

$$\sum_{\nu = \frac{1}{n} - \mu < 1} 1 \leq A, \quad \mu > 0,$$

where $A > 0$ does not depend on $\mu$ (see [8] and [9]);

$$\sup_{x \in G} |u_n(x)| \leq C(G),$$

where $C(G) > 0$ is independent of $n \in \mathbb{N}$ ([8]);

$$\sup_{x \in G} |u'_n(x)| \leq C_1(G)(\sqrt{\lambda_n} + 1),$$

with $C_1(G) > 0$ non-depending on $n \in \mathbb{N}$ ([10]).

If $f(x) \in BV(G)$, then for its Fourier coefficients $f_n$ (with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$) the estimate

$$|f_n| \leq \frac{C}{\sqrt{\lambda_n}} \cdot ||f||_{BV(G)}$$

holds, where $C > 0$ does not depend on $n \in \mathbb{N}$ (see [5]).

We will also use so-called “mean value formula” for the derivatives $u'_n(x)$ ([10]):

If $x \in G$ and $t > 0$ are such that $x \pm t \in G$, then

$$u'_n(x + t) - u'_n(x - t) = -2\sqrt{\lambda_n}u_n(x) \sin \sqrt{\lambda_n} t +$$

$$+ \int_{x-t}^{x+t} q(\xi)u_n(\xi) \cos \sqrt{\lambda_n}(|x| - \xi - t) d\xi.$$ (11)

(Nota: a function $u_\lambda(x)$ is called an eigenfunction corresponding to an eigenvalue $\lambda$ of the operator $L$ if $u_\lambda(x), u'_\lambda(x) \in AC(G)$ and the equality

$$-u''_\lambda(x) + q(x)u_\lambda(x) = \lambda u_\lambda(x)$$

holds a.e. on $G$.)

Finally, recall the “second part” of the known Riesz theorem ([11]): Let $\{v_n(x)\}_{n=1}^{\infty}$ be an orthogonal system of functions defined on a bounded interval $G$, and such that $\sup_{x \in G} |v_n(x)| \leq M$, where $M > 0$ is independent on $n \in \mathbb{N}$. If $1 < r \leq 2$ and $1/r + 1/p = 1$, then for every sequence of (complex) numbers $\{g_n\}_{n=1}^{\infty}$, satisfying $\left(\sum_{n=1}^{\infty} |g_n|^r\right)^{1/r} < +\infty$, there exists a function $g(x) \in L_p(G)$ such that $g_n = \int_{-u}^{u} g(y)v_n(y) dy$ and

$$\|g\|_{L_p(G)} \leq M^{2/r-1} \left(\sum_{n=1}^{\infty} |g_n|^r\right)^{1/r}.$$ (12)

Note that in proving the estimate (5) we have used all the results (7)-(12).
3. Proof of the theorem

The first step of the proof is the same as the one in the proof of Lemma 2 [3]. Let $K = [c, d] \subset G$ be an arbitrary fixed closed interval. Then we have

$$||\sigma'_\mu(x, f) - \bar{\sigma}'_\mu(x, f)||_{L^p(G)} = ||(\cdot)||_{L^p([a, c])} + ||(\cdot)||_{L^p(K)} + ||(\cdot)||_{L^p([d, b])}.$$  \hspace{1cm} (13)

In estimating the members on the right-hand side, we will assume, with no loss of generality, that $\lambda_n \geq 1$ ($n \in \mathbb{N}$). (This assumption is based on the equation $-u''_n(z) + [q(x) + 1]u_n(x) = (\lambda_n + 1)u_n(x).$) Set $\mu_n \overset{\text{def}}{=} \sqrt{\lambda_n}$.

Let us consider the first member. Introducing a new variable $z = x + h$, with $h \in (0, (d - c)/2)$ fixed, we obtain

$$||\sigma'_\mu(x, f) - \bar{\sigma}'_\mu(x, f)||_{L^p([a, c])} = \int_{K_1} ||\sigma'_\mu(z - h, f) - \bar{\sigma}'_\mu(z - h, f)||_{L^p} \, dz,$$  \hspace{1cm} (14)

where $K_1 = [a + h, c + h] \subset G$. By the mean value formula (11), we can write

$$\sigma'_\mu(z - h, f) = \sum_{\mu_n < \mu} f_n u'_n(z + h) + \sum_{\mu_n < \mu} 2\mu_n f_n u_n(z) \sin \mu_n h -$$

$$- \sum_{\mu_n < \mu} f_n \int_z^{z + h} q(\xi) u_n(\xi) \cos \mu_n (\xi - z - h) \, d\xi +$$

$$+ \sum_{\mu_n < \mu} f_n \int_{z - h}^z q(\xi) u_n(\xi) \cos \mu_n (\xi - z - h) \, d\xi.$$

Analogous equality can be written for $\bar{\sigma}'_\mu(z - h, f)$. Therefore, the following equality holds on $K_1$:

$$\sigma'_\mu(z - h, f) - \bar{\sigma}'_\mu(z - h, f) = \sum_{\mu_n < \mu} f_n u'_n(z + h) - \sum_{\mu_n < \mu} f_n u'_n(z + h) +$$

$$+ \sum_{\mu_n < \mu} 2\mu_n f_n u_n(z) \sin \mu_n h - \sum_{\mu_n < \mu} 2\mu_n f_n u'_n(z) \sin \bar{\mu}_n h -$$

$$- \sum_{\mu_n < \mu} f_n \int_z^{z + h} q(\xi) u_n(\xi) \cos \mu_n (\xi - z - h) \, d\xi +$$

$$+ \sum_{\mu_n < \mu} f_n \int_{z - h}^z q(\xi) u_n(\xi) \cos \bar{\mu}_n (\xi - z - h) \, d\xi +$$

$$+ \sum_{\mu_n < \mu} f_n \int_{z - h}^z q(\xi) u_n(\xi) \cos \bar{\mu}_n (\xi - z - h) \, d\xi -$$

$$- \sum_{\bar{\mu}_n < \mu} f_n \int_{z - h}^z q(\xi) u_n(\xi) \cos \bar{\mu}_n (\xi - z - h) \, d\xi.$$

That is why we have the inequality

$$||\sigma'_\mu(z - h, f) - \bar{\sigma}'_\mu(z - h, f)||_{L^p(K_1)} \leq C_p||\sigma'_\mu(z + h, f) - \bar{\sigma}'_\mu(z + h, f)||_{L^p(K_1)} +$$
An $L_p$ estimate for the difference of derivatives ...

\[ + C_p \left\| \sum_{\mu_n < \mu} 2\mu_n f_n u_n(z) \sin \mu_n h \right\|_{L_p(K_1)}^p + \]
\[ + C_p \left\| \sum_{\mu_n < \mu} 2\bar{\mu}_n \bar{f}_n \bar{u}_n(z) \sin \bar{\mu}_n h \right\|_{L_p(K_1)}^p + \]
\[ + C_p \int_{K_1} \left| \sum_{\mu_n < \mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - z - h) d\xi \right|^p dz + \]
\[ + C_p \int_{K_1} \left| \sum_{\mu_n < \mu} \bar{f}_n \int_z^{z+h} \bar{q}(\xi) \bar{u}_n(\xi) \cos \bar{\mu}_n(\xi - x - h) d\xi \right|^p dz + \]
\[ + C_p \int_{K_1} \left| \sum_{\mu_n < \mu} f_n \int_{z-h}^{z} q(\xi) u_n(\xi) \cos \mu_n(x - \xi - h) d\xi \right|^p dz + \]
\[ + C_p \int_{K_1} \left| \sum_{\mu_n < \mu} \bar{f}_n \int_{z-h}^{z} \bar{q}(\xi) \bar{u}_n(\xi) \cos \bar{\mu}_n(x - \xi - h) d\xi \right|^p dz. \quad (15) \]

Here and further, we denote by $C_p$ not necessarily equal positive constants.

In order to estimate the first member on the right-hand side of (15), we will use the estimate (5). Having in mind that $z + h \in K_2$ if $z \in K_1$, where $K_2 = [a + 2h, c + 2h] \subset G$, we have the inequalities

\[ \left| \sigma_\mu'(z + h, f) - \sigma_\mu'(z + h, f) \right|_{L_p(K_1)}^p \leq (c - a)C(1) \left\| f \right\|_{BV(G)} \]
\[ \leq C_p \left\| f \right\|_{BV(G)} \cdot \mu^{(1-1/p)p}. \quad (16) \]

The next two members have the same "structure", and they will be estimated by the Riesz theorem. First we introduce a new function:

\[ g(z) = \sum_{\mu_n < \mu} (2\mu_n f_n \sin \mu_n h) u_n(z), \quad z \in G. \]

It belongs to $L_p(G) \subset L_2(G)$, and its Fourier coefficients (with respect to the system $\{u_n(z)\}_{n=1}^\infty$) are given by

\[ g_n = \begin{cases} 2\mu_n f_n \sin \mu_n h & \text{if } \mu_n < \mu, \\ 0 & \text{if } \mu_n \geq \mu. \end{cases} \]

Let $r \in (1, 2]$ be a number such that $1/r + 1/2 = 1$. By estimates (7) and (10), we obtain

\[ \left( \sum_{n=1}^\infty |g_n|^r \right)^{1/r} \leq C \left\| f \right\|_{BV(G)} \left( \sum_{\mu_n < \mu} 1 \right)^{1/r} \]
\[ \leq C \left\| f \right\|_{BV(G)} \left( \sum_{k=1}^|A| \left( \sum_{k \leq \mu_n < k+1} 1 \right) \right)^{1/r} \leq 2^{1/r} C A^{1/r} \left\| f \right\|_{BV(G)}^{1/r}. \quad (17) \]

Hence, we can use the second part of the Riesz theorem: from estimate (12) it follows that the inequalities

\[ \|g\|_{L_p(K_1)} \leq \|g\|_{L_p(G)} \leq (C(G))^{2/r-1} \left( \sum_{n=1}^\infty |g_n|^r \right)^{1/r} \]
are valid. That is why we can conclude, by (17), that for the second member it holds:
\[
\left\| \sum_{\mu_n < \mu} 2\mu_n f_n u_n(z) \sin \mu_n h \right\|_{L_p(K_1)}^p \leq C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (18)

The same estimate holds for the third member:
\[
\left\| \sum_{\mu_n < \mu} 2\mu_n f_n \bar{u}_n(z) \sin \mu_n h \right\|_{L_p(K_1)}^p \leq \tilde{C}_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (19)

In the case of the fourth member, using estimates (7)-(8), (10), and the Hölder inequality, we obtain
\[
\int_{K_1} \left\| \sum_{\mu_n < \mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h) \, d\xi \right\|_{L_p}^p \, dz \leq
\]
\[
\int_{K_1} \left( \sum_{\mu_n < \mu} |\mu_n f_n| \left| \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h) \, d\xi \right| \right)^p \leq
\]
\[
\left( \sum_{\mu_n < \mu} |\mu_n f_n|^r \right)^{p/r} \int_{K_1} \left( \sum_{\mu_n < \mu} \left| \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h) \, d\xi \right| \right)^p \, dz \leq
\]
\[
C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p} \| g \|_{L_p(K_1)}^p \left( \sum_{\mu_n < \mu} \frac{1}{\mu_n^p} \right) \leq C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\]

Here $1/p + 1/r = 1$. Also we have in mind that
\[
\sum_{\mu_n < \mu} \frac{1}{\mu_n^p} \leq \sum_{k=1}^{\infty} \left( \sum_{\mu_n < k+1} \frac{1}{\mu_n^p} \right) \leq A \sum_{k=1}^{\infty} \frac{1}{k^{p'}}
\]

Therefore, the following estimate holds:
\[
\int_{K_1} \left\| \sum_{\mu_n < \mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h) \, d\xi \right\|_{L_p}^p \, dz \leq
\]
\[
\leq C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (20)

The estimates of the same form, with possibly different constants $C_p$, are valid for the last three members on the right-hand side of (15). So we get, by (14)-(16) and (18)-(20), the final estimate
\[
\| \sigma'_\mu(x, f) - \sigma'_\mu(x, f) \|_{L_p([a, c])} \leq C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (21)

Using the analogous argument, one can prove the estimate
\[
\| \sigma'_\mu(x, f) - \sigma'_\mu(x, f) \|_{L_p([d, b])} \leq C_p \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (22)

Finally, by estimate (5), we obtain
\[
\| \sigma'_\mu(x, f) - \sigma'_\mu(x, f) \|_{L_p(K)} \leq (b - a) C(K) \| f \|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.
\] (23)

Now, the estimate (6) follows from (13) and (21)-(23). The theorem is proved.
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