DISTRIBUTIONS GENERATED BY BOUNDARY VALUES OF FUNCTIONS OF THE NEVANLINNA CLASS N

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Abstract. In this work necessary and sufficient conditions are given for a regular distribution in $D'$ to be distribution generated by the boundary function of some function from the Nevanlinna class $N$.

1. Introduction

1.1. Denotations which will be used in the paper

Let $U$ denote the open unit disk in $C$, i.e., $U = \{z \in C : |z| < 1\}$, $T = \partial U$ and $\Pi^+$ denote the upper half-plane, i.e., $\Pi^+ = \{z \in C : \text{Im} z > 0\}$. For a given function $f$ which is analytic on some region $\Omega$ we will write $f \in H(\Omega)$.

For a function $f$, $f: \Omega \rightarrow C^n$, $\Omega \subseteq R^n$, $x \in \Omega$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \in N \cup \{0\}$, $D^\alpha f = D^\alpha f(x)$ denotes

$$D^\alpha f = \frac{\partial^{\alpha}_|f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

$L^p(\Omega)$ is the space of locally integrable functions on $\Omega$, i.e., $f(x) \in L^p_{\text{loc}}(\Omega)$ if $f(x) \in L^p(\Omega)$, for every bounded subregion $\Omega$ of $\Omega$.

1.2. The Nevanlinna class $N$ defined on $U$ and on $\Pi^+$ and some properties of $N$

The Nevanlinna class, $N(U)$, consists of all $f \in H(U)$ whose characteristic function

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

is bounded for $0 \leq r < 1$.

It is known [4] that a function $f \in H(U)$ belongs to the class $N(U)$ if and only if it is the quotient of two bounded analytic functions. It is also known [4] that for

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each function $f \in N(U)$ the nontangential limit $f^*(e^{i\theta})$ exists almost everywhere on $T$ and $\log |f(e^{i\theta})|$ is integrable over $T$, unless $f \equiv 0$.

For a function $f \in H(U)$, $\log(1 + |f|)$ is subharmonic, so the integrals

$$L(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(re^{i\theta})|) \, d\theta$$

increase with $r$. Thus the (possibly infinite) limit $||f|| = \lim_{r \to 1} L(r,f)$ exists, and the inequalities

$$\log^+ x \leq \log(1 + x) \leq \log 2 + \log^+ x, \quad (x > 0)$$

show that $f$ belongs to $N(U)$ if and only if $||f|| < \infty$

In the case of the upper half-plane $\Pi^+$, $N(\Pi^+)$ consists of all $f \in H(\Pi^+)$, for which

$$\sup_{0 < y < \infty} \int_{-\infty}^{+\infty} \log(1 + |f(x + iy)|) \, dx < \infty.$$ 

Note. From now on, we will write $N$ instead of $N(\Pi^+)$.

1.3. Some notions of distributions

$C^\infty(R^n)$ denotes the space of all complex valued infinitely differentiable functions on $R^n$ and $C^\infty_0(R^n)$ denotes the subspace of $C^\infty(R^n)$ that consists of those functions of $C^\infty(R^n)$ which have compact support. Support of a continuous function $f$, denoted by $\text{supp}(f)$, is the closure of $\{x|f(x) \neq 0\}$ in $R^n$.

$D = D(R^n)$ denotes the space of $C^\infty_0(R^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in $D$ as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset R^n$ such that $\text{supp}(\varphi_\lambda) \subseteq K$ for each $\lambda$, $\text{supp}(\varphi) \subseteq K$ and for every $n$-tuple $\alpha$ of nonnegative integers the sequence $\{D_\alpha^p \varphi_\lambda(t)\}$ converges to $D_\alpha^p \varphi(t)$ uniformly on $K$ as $\lambda \to \lambda_0$.

$D' = D'(R^n)$ is the space of all continuous, linear functionals on $D$, where continuity means that $\varphi_\lambda \to \varphi$ in $D$ as $\lambda \to \lambda_0$, implies $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$, as $\lambda \to \lambda_0$, $T \in D'$. $D'$ is called the space of distributions.

Note. $\langle T, \varphi \rangle$ denotes the value of the functional $T$, when it acts on the function $\varphi$.

Let $\varphi \in D$ and let $f(x) \in L^1_{loc}(R^n)$. Then the functional $T_f$ from $D$ to $C$, defined by:

$$\langle T_f, \varphi \rangle = \int_{R^n} f(t) \varphi(t) \, dt, \quad \varphi \in D$$

is a distribution on $D$ called regular distribution generated with $f$.

2. Main results

The idea for Theorem 1 and Theorem 2 comes from the following theorem, that is given in [7].
Theorem. Necessary and sufficient condition for a measurable function \( \varphi(e^{i\theta}) \), defined on \( T \) to coincide almost everywhere on \( T \) with boundary value \( f^*(e^{i\theta}) \) of some function \( f(z) \) of the Nevanlinna class \( N(U) \), is the existence of a sequence of polynomials \( \{P_n(z)\} \) such that:

(1) \( \lim_{n \to \infty} \int_0^{2\pi} \log^+ |P_n(e^{i\theta})| d\theta < \infty. \)

Theorem 1. Let \( T_{f^*} \) be the distribution in \( D' \) generated with the boundary value \( f^*(x) \) of some function \( f(z) \) from the space \( N \). Then there exist a sequence of polynomials \( \{P_n(z)\} \), \( z \in \Pi^+ \) and a respective sequence of distributions \( \{T_n\} \), \( T_n \in D' \) generated with the boundary values \( P_n^*(x) \) of \( P_n(z) \), satisfying \( T_n \to T_{f^*} \) in \( D' \):

(i) \( T_n \to T_{f^*} \) in \( D' \),

(ii) \( \lim_{n \to \infty} \int_{-\infty}^{\infty} \log(1 + |P_n^*(x)|)|\varphi(x)| dx < \infty, \quad \forall \varphi \in D. \)

Proof. Let the conditions of Theorem be satisfied. Since \( f \in N \), it follows that \( f \in H(\Pi^+) \) and there exists a constant \( C > 0 \), such that

\[ \int_{-\infty}^{\infty} \log(1 + |f(x + iy)|) dx \leq C, \quad \text{for all} \quad x + iy \in \Pi^+. \quad (1) \]

Let \( \{y_n\} \) be a sequence of positive real numbers such that \( \lim_{n \to \infty} y_n = 0. \)

We consider the sequence of functions \( \{F_n(z)\} \), defined by \( F_n(z) = f(z + iy_n) \). Then \( F_n(z) \) are analytic functions on \( \Pi^+ \cup R \). Using the theorem of Mergelyan we get that for a compact subset \( K \) of \( \Pi^+ \cup R \), whose complement is connected, and for the function \( F_n(z) \) there exists a polynomial \( P_n(z) \), such that \( |F_n(z) - P_n(z)| < \varepsilon_n \), for \( z \in K \), where \( \varepsilon_n > 0 \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). Now we will prove (i) and (ii).

Let \( \varphi \in D \) and let \( K \subset R \) be a compact set that contains \( \text{supp}(\varphi) \) and whose complement (in \( C \)) is connected. (It is possible to be \( K = \text{supp}(\varphi) \)).

(i) We have:

\[ |\langle T_n, \varphi \rangle - \langle T_{f^*}, \varphi \rangle| = \left| \int_{-\infty}^{\infty} P_n^*(x)\varphi(x) dx - \int_{-\infty}^{\infty} f^*(x)\varphi(x) dx \right| = \right| \int_{-\infty}^{\infty} [P_n^*(x) - f^*(x)]\varphi(x) dx \right| \leq \int_{K} |P_n^*(x) - f^*(x)||\varphi(x)| dx \leq M \left( \int_{K} |P_n^*(x) - f^*(x)| dx \right) \leq M \varepsilon_n m(K) \to 0 \quad \text{as} \quad n \to \infty \]

where \( m(K) \) is the Lebesgue measure of the set \( K \), \( M \) is positive real number and \( \varepsilon_n = \varepsilon_n + |f^*(x) - F_n(x)| \). Clearly, \( \varepsilon_n \to 0 \) as \( n \to \infty \). From the above computations we conclude that \( \langle T_n, \varphi \rangle \to \langle T_{f^*}, \varphi \rangle \) as \( n \to \infty \), for every \( \varphi \in D. \)
\[
\int_{-\infty}^{+\infty} \log(1 + |P_n^*(x)|)|\varphi(x)| \, dx \\
= \int_{-\infty}^{+\infty} \log(1 + |P_n^*(x) - F_n(x) + F_n(x)|)|\varphi(x)| \, dx \\
\leq \int_{-\infty}^{+\infty} \log(1 + |P_n^*(x) - F_n(x)| + |F_n(x)|)|\varphi(x)| \, dx \\
= \int_{K} \log(1 + |F_n(x)| + |P_n^*(x) - F_n(x)|)|\varphi(x)| \, dx \\
\leq \int_{K} \log(1 + |F_n(x)|)|\varphi(x)| \, dx + \int_{K} |P_n^*(x) - F_n(x)||\varphi(x)| \, dx \\
\leq M \int_{K} \log(1 + |F_n(x)|) \, dx + M \int_{K} |P_n^*(x) - F_n(x)| \, dx \\
\leq M \int_{K} \log(1 + |f(x + iy)|) \, dx + M\varepsilon_n m(K) \overset{(\text{a})}{\leq} M C + M\varepsilon_n m(K) \to M, \quad \text{as} \quad n \to \infty.
\]

In the proof of (ii) we used the inequality \(|a + b| \leq |a| + |b|\), monotonicity of the function \(\log x\) and the inequality \(\log(1 + a + b) \leq \log(1 + a) + b\), for \(a, b > 0\).

**Theorem.** Let \(\varphi_0\) be a locally integrable function on \(\mathbb{R}\) and \(T_{\varphi_0}\) be the distribution in \(D'\) generated by \(\varphi_0\). Let there exists a sequence of polynomials \(P_n(z)\), \(z \in \Pi^+\) such that the following conditions are satisfied:

(i) The sequence of distributions, generated by the boundary values \(P_n^*(x)\) of \(P_n(z)\) converges to \(T_{\varphi_0}\) in \(D'\) as \(n \to \infty\).

(ii) \(\lim_{n \to \infty} \int_{-\infty}^{+\infty} \log(1 + |P_n(x + iy)|)|\varphi(x)| \, dx < \infty\), for all \(x + iy \in \Pi^+, \varphi \in D\).

Then there exists a function \(f \in H(\Pi^+)\), such that

\[
\int_{K} \log(1 + |f(x + iy)|) \, dx < C < \infty, \quad \forall(x + iy) \in \Pi^+
\]

for every compact subset \(K\) of \(\mathbb{R}\) and

\[
\lim_{y \to 0^+} \int_{-\infty}^{+\infty} f(x + iy) \varphi(x) \, dx = \langle T_{\varphi_0}, \varphi \rangle, \quad \varphi \in D.
\]

**Proof.** Let the conditions of the Theorem be satisfied. In [6] it is proven that the condition (i), i.e.

\[
\lim_{n \to \infty} \int_{R} P_n^*(x) \varphi(x) \, dx = \int_{R} \varphi_0(x) \varphi(x) \, dx, \quad \varphi \in D,
\]
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implies
there exists a function $f \in H(\Pi^+)$, such that the sequence of polynomials
$
\{P_n(z)\}$ converges to $f(z)$ uniformly on compact subsets of $\Pi^+$ as $n \to \infty.$

(2)

First we will prove that this analytic function $f$ also satisfies

$$
\int_K \log(1 + |f(x + iy)|) \, dx < C < \infty, \quad \forall (x + iy) \in \Pi^+
$$

for every compact subset $K$ of $R$.

In order to do that, we will use the second condition (ii), i.e.,

$$
\lim_{n \to \infty} \int_K \log(1 + |P_n(x + iy)|) |\varphi(x)| \, dx < C < \infty, \quad \forall (x + iy) \in \Pi^+, \ \varphi \in D.
$$

(3)

Let $K$ be a compact subset of $R$. Then there exists $\varphi(x) \in C^\infty(R)$, $\varphi(x) = 1$,
$v x \in K$. Substituting $\varphi(x)$, chosen in this way, in (3), we get

$$
\lim_{n \to \infty} \int_K \log(1 + |P_n(x + iy)|) \, dx < C < \infty, \quad \forall (x + iy) \in \Pi^+.
$$

(4)

Now,

$$
\int_K \log(1 + |f(x + iy)|) \, dx = \int_K \lim_{n \to \infty} \log(1 + |P_n(x + iy)|) \, dx
\leq \lim_{n \to \infty} \int_K \log(1 + |P_n(x + iy)|) \, dx \tag{4}
$$

i.e. $\int_K \log(1 + |f(x + iy)|) \, dx < C < \infty$, for every compact subset $K$ of $R$ and for

every $x + iy \in \Pi^+$.

It remains to prove that

$$
\lim_{y \to 0^+} \int_{-\infty}^{+\infty} f(x + iy) \varphi(x) \, dx = \langle T_{\varphi^*}, \varphi \rangle, \quad \varphi \in D.
$$

(5)

Let $\varphi \in D$ and $\text{supp}(\varphi) = K \subset R$. Then

$$
\lim_{y \to 0^+} \int_R f(x + iy) \varphi(x) \, dx \overset{(2)}{=} \lim_{y \to 0^+} \int_K \lim_{n \to \infty} P_n(x + iy) \varphi(x) \, dx \overset{\text{u.c.}}{=}
$$

$$
= \lim_{y \to 0^+} \lim_{n \to \infty} \int_K P_n(x + iy) \varphi(x) \, dx
= \lim_{n \to \infty} \lim_{y \to 0^+} \int_K P_n(x + iy) \varphi(x) \, dx
= \lim_{n \to \infty} \int_K P_n^*(x) \varphi(x) \, dx
= \int_R \varphi_0(x) \varphi(x) \, dx = \langle T_{\varphi^*}, \varphi \rangle, \quad \forall \varphi \in D.
$$

In the proof above, we used that

$$
\lim_{y \to 0^+} \lim_{n \to \infty} \int_K P_n(x + iy) \varphi(x) \, dx = \lim_{n \to \infty} \lim_{y \to 0^+} \int_K P_n(x + iy) \varphi(x) \, dx.
$$

(6)

We will show that (6) holds.

Let us consider the sequence $\{g_n(y)\}$, where

$$
g_n(y) = \int_K P_n(x + iy) \varphi(x) \, dx, \quad x + iy \in K_1,
$$


$K_1$ is any compact set in $\Pi^+$ whose elements $z \in K_1$ satisfy $\text{Re} z \in K$. Since 
\[
\lim_{n \to \infty} g_n(y) = \lim_{n \to \infty} \int_K P_n(x + iy)\varphi(x) \, dx = \int_K f(x + iy)\varphi(x) \, dx = g(y),
\] 
i.e., the sequence $\{g_n(y)\}$ converges to $g(y)$, as $n \to \infty$. We will prove that the
convergence is uniform.

\[
0 \leq \sup_y |g_n(y) - g(y)| = \sup_y \left| \int_K P_n(x + iy)\varphi(x) \, dx - \int_K f(x + iy)\varphi(x) \, dx \right|
\]
\[
= \sup_y \left| \int_K [P_n(x + iy) - f(x + iy)]\varphi(x) \, dx \right|
\]
\[
\leq \sup_y \int_K |P_n(x + iy) - f(x + iy)||\varphi(x)| \, dx
\]
\[
\leq M \sup_y \int_K |P_n(x + iy) - f(x + iy)| \, dx.
\]

Since 
\[
\lim_{n \to \infty} \int_K |P_n(x + iy) - f(x + iy)| \, dx = 0,
\]
we get that $\lim_{n \to \infty} \sup_y |g_n(y) - g(y)| = 0$.

So we have proved that $\{g_n(y)\}$ converges to $g(y)$ uniformly on $K_1$, as $n \to \infty$, 
which implies (6). This concludes the proof of (5) and of Theorem 2.

Comment. This work is a continuation of [6], where two similar theorems were proved in the spaces $H^p$, $1 \leq p < \infty$.

Similar theorems can be given in the Smirnov space.

REFERENCES


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