r-FUZZY STRONGLY PREOPEN SETS
IN FUZZY TOPOLOGICAL SPACES

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Abstract. We introduce r-fuzzy strongly preopen and r-fuzzy strongly preclosed sets in fuzzy topological spaces in the sense of the definition of Sostak [8] and investigate some of their properties. Fuzzy strongly precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings between fuzzy topological spaces are defined. Their properties and the relationship between these mappings and other mappings introduced previously are investigated.

1. Introduction and preliminaries

A. P. Sostak [8] introduced a fuzzy topology as an extension of Chang’s fuzzy topology [2]. It has been developed in many directions [3,4,7]. B. Krsteska [1] introduced fuzzy strongly preopen and fuzzy strongly preclosed sets in the Chang’s fuzzy topology.

In this paper we define r-fuzzy strongly preopen and r-fuzzy strongly preclosed sets in a fuzzy topological space in view of the definition of Sostak [8] and investigate some of their properties. We show that fuzzy strong precontinuity implies fuzzy precontinuity, but the converse is not true. We obtain some properties of fuzzy strongly precontinuous mappings.

Throughout this paper, let $X$ be a non-empty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\check{\alpha}(x) = \alpha$, for all $x \in X$. All other notations and definitions are standard in the fuzzy set theory.

Definition 1.1. [8] A function $\tau : I^X \to I$ is called a fuzzy topology on $X$ if it satisfies the following conditions:

1. $\tau(\check{0}) = \tau(\check{1}) = 1$,
2. $\tau(\mu_1 \land \mu_2) \geq \tau(\mu_1) \land \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,
3. $\tau\left(\bigvee_{i \in I} \mu_i\right) \geq \bigwedge_{i \in I} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in I} \subseteq I^X$.

The pair $(X, \tau)$ is called a fuzzy topological space (fts, for short).

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Remark 1.2. Let \((X, \tau)\) be an fts. Then, for each \(\alpha \in I\), \(\tau_\alpha = \{ \mu \in I^X : \tau(\mu) \geq \alpha \}\) is a Chang’s fuzzy topology on \(X\).

Theorem 1.1. [7] Let \((X, \tau)\) be an fts. For each \(r \in I_0\), \(\lambda \in I^X\), define an operator \(C_r : I^X \times I_0 \to I^X\) as follows

\[
C_r(\lambda, r) = \bigwedge \{ \mu \in I^X : \lambda \leq \mu, \tau(\mu) \geq r \}.
\]

For each \(\lambda, \mu \in I^X\) and \(r, s \in I_0\) it satisfies the following conditions:
1. \(C_r(0, r) = 0\).
2. \(\lambda \leq C_r(\lambda, r)\).
3. \(C_r(\lambda, r) \lor C_r(\mu, r) = C_r(\lambda \lor \mu, r)\).
4. \(C_r(\lambda, r) \leq C_r(\lambda, s)\) if \(r \leq s\).
5. \(C_r(\lambda, r, r) = C_r(\lambda, r)\).

Theorem 1.2. [6] Let \((X, \tau)\) be an fts. For each \(r \in I_0\), \(\lambda \in I^X\), define an operator \(I_r : I^X \times I_0 \to I^X\) as follows

\[
I_r(\lambda, r) = \bigvee \{ \mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r \}.
\]

For each \(\lambda, \mu \in I^X\) and \(r, s \in I_0\) it satisfies the following conditions:
1. \(I_r(1, \lambda, r) = 1 - C_r(\lambda, r)\) and \(C_r(\lambda - r, r) = 1 - I_r(\lambda, r)\).
2. \(I_r(1, r) = 1\).
3. \(I_r(\lambda, r) \leq \lambda\).
4. \(I_r(\lambda, r) \land I_r(\mu, r) = I_r(\lambda \land \mu, r)\).
5. \(I_r(\lambda, r) \geq I_r(\lambda, s)\) if \(r \leq s\).
6. \(I_r(1, \lambda, r, r) = I_r(1, \lambda, r)\).

Definition 1.2. [6] Let \((X, \tau)\) be an fts. For \(\lambda \in I^X\) and \(r \in I_0\):
1. \(\lambda\) is called an \(r\)-fuzzy semi-open (\(r\)-fso, for short) set if there exists \(\nu \in I^X\) with \(\tau(\nu) \geq r\) such that \(\nu \leq \lambda \leq C_r(\nu, r)\). Equivalently, \(\lambda \leq C_r(I_r(\lambda, r, r))\).
2. \(\lambda\) is called an \(r\)-fuzzy semi-closed (\(r\)-fsc, for short) set if there exists \(\nu \in I^X\) with \(\tau(\nu) \geq r\) such that \(I_r(\lambda, \nu) \leq \lambda \leq \nu\). Equivalently, \(I_r(C_r(\lambda, r, r)) \leq \lambda\).

Definition 1.3. [6] Let \((X, \tau)\) and \((Y, \tau^*)\) be fts’s. A mapping \(f : X \to Y\) is said to be:
1. fuzzy continuous iff \(\tau^*(\mu) \leq \tau(f^{-1}(\mu))\) for each \(\mu \in I^X\).
2. fuzzy open iff \(\tau^*(f(\mu)) \geq \tau(\mu)\) for each \(\mu \in I^X\).
3. fuzzy closed iff \(\tau^*(1 - f(\mu)) \geq \tau(1 - \mu)\) for each \(\mu \in I^X\).
\textbf{Definition 1.4.} [6] Let \((X, \tau)\) and \((Y, \tau^+)\) be fts's. A mapping \(f: X \rightarrow Y\) is said to be:

1. fuzzy semi-continuous iff \(f^{-1}(\mu)\) is \(r\)-fso for each \(\mu \in I^Y, \ r \in I_0\) with \(\tau^+(\mu) \geq r\).
2. fuzzy open iff \(f(\mu)\) is \(r\)-fso for each \(\mu \in I^X, \ r \in I_0\) with \(\tau(\mu) \geq r\).
3. fuzzy semi-closed iff \(f(\mu)\) is \(r\)-fsc for each \(\mu \in I^X, \ r \in I_0\) with \(\tau(1-\mu) \geq r\).

\textbf{2. \(r\)-fuzzy strongly preopen and \(r\)-fuzzy strongly preclosed sets}

\textbf{Definition 2.1.} [6] Let \((X, \tau)\) be an fts. For \(\lambda \in I^X\) and \(r \in I_0\):

1. \(\lambda\) is called \(r\)-fuzzy preopen (\(r\)-fpo, for short) iff
   
   \[\lambda \leq I_r(C_r(\lambda, r), r).\]

2. \(\lambda\) is called \(r\)-fuzzy preclosed (\(r\)-fpc, for short) iff
   
   \[C_r(I_r(\lambda, r), r) \leq \lambda.\]

3. \(\lambda\) is called \(r\)-fuzzy strongly semi-open (\(r\)-fso, for short) iff
   
   \[\lambda \leq I_r(C_r(I_r(\lambda, r), r), r).\]

4. \(\lambda\) is called \(r\)-fuzzy strongly semi-closed (\(r\)-fsc, for short) iff
   
   \[C_r(I_r(C_r(\lambda, r), r), r) \leq \lambda.\]

\textbf{Definition 2.2.} [6] Let \((X, \tau)\) be an fts. For \(\lambda \in I^X\) and \(r \in I_0\):

1. the \(r\)-fuzzy preinterior of \(\lambda\), denoted by \(PI_r(\lambda, r)\), is defined by
   
   \[PI_r(\lambda, r) = \bigvee \{ \nu \in I^X : \nu \leq \lambda, \ \nu \text{ is } r\text{-fpo} \} ,\]

2. the \(r\)-fuzzy preclosure of \(\lambda\), denoted by \(PC_r(\lambda, r)\), is defined by
   
   \[PC_r(\lambda, r) = \bigwedge \{ \nu \in I^X : \nu \geq \lambda, \ \nu \text{ is } r\text{-fpc} \} ,\]

3. the \(r\)-fuzzy strongly semi-interior of \(\lambda\), denoted by \(SSI_r(\lambda, r)\), is defined by
   
   \[SSI_r(\lambda, r) = \bigvee \{ \nu \in I^X : \nu \leq \lambda, \ \nu \text{ is } r\text{-fso} \} ,\]

4. the \(r\)-fuzzy strongly semi-closure of \(\lambda\), denoted by \(SSC_r(\lambda, r)\), is defined by
   
   \[SSC_r(\lambda, r) = \bigwedge \{ \nu \in I^X : \nu \geq \lambda, \ \nu \text{ is } r\text{-fsc} \} .\]
Theorem 2.1. Let \((X, \tau)\) be an fts, \(\lambda \in I^X\) and \(r \in I_0\). Then

1. \(\lambda \lor C_r(I_r(\lambda, r), r) \leq PC_r(\lambda, r)\),
2. \(PI_r(\lambda, r) \leq \lambda \land C_r(I_r(\lambda, r), r)\),
3. \(I_r(PC_r(\lambda, r), r) \leq I_r(C_r(\lambda, r), r)\),
4. \(I_r(C_r(I_r(\lambda, r), r), r) \leq I_r(PC_r(\lambda, r), r)\),
5. \(PC_r(\lambda - \lambda, r) = I_r - PL_r(\lambda, r)\) and \(PL_r(\lambda - \lambda, r) = I_r - PC_r(\lambda, r)\).

Proof. (1) Since \(PC_r(\lambda, r)\) is an \(r\)-fpc set, we have
\[
C_r(I_r(\lambda, r), r) \leq C_r(I_r(PC_r(\lambda, r), r), r) \leq PC_r(\lambda, r).
\]
Thus, \(\lambda \lor C_r(I_r(\lambda, r), r) \leq PC_r(\lambda, r)\).

(2) It can be shown as (1).

(3) It follows from the relation \(PC_r(\lambda, r) \leq C_r(\lambda, r)\).

(4) From (1) we have
\[
I_r(PC_r(\lambda, r), r) \geq I_r(\lambda \lor C_r(I_r(\lambda, r), r), r) \geq I_r(C_r(I_r(\lambda, r), r), r).
\]

(5) Straightforward. ■

Definition 2.3. Let \((X, \tau)\) be an fts. For \(\lambda \in I^X\) and \(r \in I_0\):

1. \(\lambda\) is called \(r\)-fuzzy strongly preopen (\(r\)-fspo, for short) iff
\[
\lambda \leq I_r(\lambda, r),
\]
2. \(\lambda\) is called \(r\)-fuzzy strongly preclosed (\(r\)-fspc, for short) iff
\[
C_r(\lambda, r) \leq \lambda.
\]
3. The \(r\)-fuzzy strong preinterior of \(\lambda\), denoted by \(SPL_r(\lambda, r)\), is defined by
\[
SPL_r(\lambda, r) = \bigvee\{\nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-fspo}\}.
\]
4. The \(r\)-fuzzy strong preclosure of \(\lambda\), denoted by \(SPC_r(\lambda, r)\), is defined by
\[
SPC_r(\lambda, r) = \bigwedge\{\nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-fspc}\}.
\]

Theorem 2.2. Let \((X, \tau)\) be an fts, \(\lambda \in I^X\) and \(r \in I_0\). Then

1. if \(\tau(\lambda) \geq r\), then \(\lambda\) is an \(r\)-fspo set;
2. if \(\lambda\) is \(r\)-fsso, then \(\lambda\) is \(r\)-fspo;
3. if \(l\) is \(r\)-fspo, then \(\lambda\) is \(r\)-fsso.

Proof. It follows from Theorem 2.1. ■
The following examples show that the converses in the above theorem are not true in general.

**Example 2.1.** Let $X = \{a, b, c\}$. Define fuzzy sets $\lambda_1, \lambda_2, \mu \in I^X$ as follows:

\[
\begin{align*}
\lambda_1(a) &= 0.3, & \lambda_1(b) &= 0.2, & \lambda_1(c) &= 0.7, \\
\lambda_2(a) &= 0.8, & \lambda_2(b) &= 0.8, & \lambda_2(c) &= 0.4, \\
\mu(a) &= 0.8, & \mu(b) &= 0.7, & \mu(c) &= 0.6.
\end{align*}
\]

Define a fuzzy topology $\tau: I^X \to I$ as follows:

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \lambda = \lambda_1, \\
\frac{1}{3}, & \text{if } \lambda = \lambda_2, \\
\frac{2}{3}, & \text{if } \lambda = \lambda_1 \lor \lambda_2, \\
\frac{3}{4}, & \text{if } \lambda = \lambda_1 \land \lambda_2, \\
0, & \text{otherwise}.
\end{cases}
\]

For the fts $(X, \tau)$ with $0 < r \leq \frac{1}{3}$, $\mu$ is an $r$-fspo set from

\[
\mu \leq I_r(\text{PC}_r(\mu, r), r) = \overline{1}.
\]

For $0 < r \leq \frac{1}{4}$, $\mu$ is neither $r$-fspo nor $\tau(\mu) \geq r$ from

\[
\mu \not\leq I_r(\text{C}_r(I_r(\mu, r), r), r) = \lambda_1 \land \lambda_2.
\]

**Example 2.2.** Let $X$ be a non-empty set. We define fuzzy topology $\tau: I^X \to I$ as follows:

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{3}, & \text{if } \lambda = \overline{0.2}, \\
\frac{1}{2}, & \text{if } \lambda = \overline{0.4}, \\
\frac{1}{4}, & \text{if } \lambda = \overline{0.5}, \\
\frac{3}{4}, & \text{if } \lambda = \overline{0.6}, \\
0, & \text{otherwise}.
\end{cases}
\]

For $0 < r \leq \frac{1}{3}$, $\overline{0.7}$ is an $r$-fpo set but it is not $r$-fspö, from the following:

\[
\begin{align*}
\overline{0.7} &\leq I_r(\text{C}_r(\overline{0.7}, r), r) = \overline{1}, \\
\overline{0.7} &> I_r(\text{PC}_r(\overline{0.7}, r), r) = I_r(\overline{0.7}, r) = \overline{0.4}.
\end{align*}
\]

For $\frac{1}{3} < r \leq \frac{2}{3}$, $\overline{0.3}$ is $r$-fso but it is not $r$-fspö, from the following:

\[
\begin{align*}
\overline{0.3} &\leq C_r(I_r(\overline{0.3}, r), r) = \overline{0.4}, \\
\overline{0.3} &> I_r(\text{PC}_r(\overline{0.3}, r), r) = I_r(\overline{0.3}, r) = \overline{0.2}.
\end{align*}
\]
Remark 2.1. From Example 2.1, for $0 < r \leq \frac{1}{3}$, $\mu$ is an r-fspo set but not an r-fso set because

$$\mu \not\subseteq C_r(\mu, r) = \overline{1} - \lambda_1 \wedge \lambda_2.$$ 

Also, in Example 2.2, $\overline{0.3}$ is an r-fso but not r-fspo set for $\frac{1}{3} < r \leq \frac{5}{8}$. From the above discussion there is no relation between the concepts of r-fso sets and r-fspo sets.

Theorem 2.3. Let $(X, \tau)$ be an fts and $r \in I_0$.

(1) Any union of r-fspo sets is an r-fspo set.

(2) Any intersection of r-fspo sets is an r-fspo set.

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r-fspo sets. For each $\alpha \in \Gamma$, $\lambda_\alpha \subseteq I_r(\text{PC}_r(\lambda_\alpha, r), r)$. Hence we have

$$\bigvee_{\alpha \in \Gamma} \lambda_\alpha \subseteq \bigvee_{\alpha \in \Gamma} I_r(\text{PC}_r(\lambda_\alpha, r), r) \subseteq I_r\left(\text{PC}_r\left(\bigvee_{\alpha \in \Gamma} \lambda_\alpha, r\right), r\right).$$

So, $\bigvee_{\alpha \in \Gamma} \lambda_\alpha$ is an r-fspo set.

(2) It is easily proved in the same manner.

Remark 2.2. The intersection of two r-fspo sets need not be an r-fspo set. And the union of two r-fspo sets need not be an r-fspo set. We will show it in the next example.

Example 2.3. Consider the fts $(X, \tau)$ form Example 2.1. The fuzzy set $\rho$ defined as:

$$\rho(a) = 0.4, \quad \rho(b) = 0.2, \quad \rho(c) = 0.8,$$

is a $\frac{1}{3}$-fspo set, but $\lambda_2 \wedge \rho$ is not a $\frac{1}{3}$-fspo set. Also, $(\overline{1} - \lambda_2) \vee (\overline{1} - \rho)$ is not a $\frac{1}{3}$-fspo set in $(X, \tau)$.

Theorem 2.4. Let $(X, \tau)$ be an fts. For $\lambda \in I^X$ and $r \in I_0$ the following statements hold:

(1) $C_r(\lambda, r)$ is r-fspo.

(2) $\lambda$ is r-fspo iff $\lambda = \text{SPI}_r(\lambda, r)$.

(3) $\lambda$ is r-fspo iff $\lambda = \text{SPC}_r(\lambda, r)$.

(4) $I_r(\lambda, r) \subseteq \text{SPI}_r(\lambda, r) \subseteq \text{PI}_r(\lambda, r) \subseteq \lambda \subseteq \text{PC}_r(\lambda, r) \subseteq \text{SPC}_r(\lambda, r) \subseteq C_r(\lambda, r)$.

(5) $\text{SPI}_r(\overline{1} - \lambda, r) = \overline{1} - \text{SPC}_r(\lambda, r)$ and $\text{SPC}_r(\overline{1} - \lambda, r) = \overline{1} - \text{SPI}_r(\lambda, r)$.

(6) $C_r(\text{SPI}_r(\lambda, r), r) = \text{SPI}_r(C_r(\lambda, r), r) = C_r(\lambda, r)$.

Proof. (1), (2), (3) and (4) follow from the definitions.

(5) For $\lambda \in I^X$, $r \in I_0$ we have the following:

$$\overline{1} - \text{SPI}_r(\lambda, r) = \overline{1} - \bigvee \{\nu : \nu \leq \lambda, \nu \text{ is r-fspo}\}$$

$$= \bigwedge \{\overline{1} - \nu : \overline{1} - \lambda \leq \overline{1} - \nu, \overline{1} - \nu \text{ is r-fspo}\}$$

$$= \text{SPC}_r(\overline{1} - \lambda, r).$$

(6) From (1) and (3), \(SPC_r(C, (\lambda, r), r) = C_r(\lambda, r)\). We only show that
\[C_r(SPC_r(\lambda, r), r) = C_r(\lambda, r)\].
Since \(\lambda \leq SPC_r(\lambda, r)\), \(C_r(\lambda, r) \leq C_r(SPC_r(\lambda, r), r)\). Suppose that
\[C_r(\lambda, r) \neq C_r(SPC_r(\lambda, r), r)\].
There exist \(x \in X\) and \(r \in I_0\) such that
\[C_r(\lambda, r)(x) < C_r(SPC_r(\lambda, r), r)(x)\].
By the definition of \(C_r\), there exists \(\rho \in I^X\) with \(\lambda \leq \rho\) and \(\tau(1 - \rho) \geq r\) such that
\[C_r(SPC_r(\lambda, r), r)(x) > \rho(x) \geq C_r(\lambda, r)(x)\].
On the other hand, since \(\rho = C_r(\rho, r), \lambda \leq \rho\) implies
\[SPC_r(\lambda, r) \leq SPC_r(\rho, r) = SPC_r(C_r(\rho, r), r) = C_r(\rho, r) = \rho\].
Thus,
\[C_r(SPC_r(\lambda, r), r) \leq \rho\].
It is a contradiction. Hence, \(C_r(SPC_r(\lambda, r), r) \leq C_r(\lambda, r)\).

3. Fuzzy strong precontinuity

**Definition 3.1.** Let \((X, \tau)\) and \((Y, \tau^*)\) be fts's and \(f : X \to Y\) be a mapping. Then, \(f\) is called fuzzy strongly precontinuous (resp. fuzzy strongly semi-continuous, fuzzy precontinuous) if \(f^{-1}(\mu)\) is an \(r\)-fspo (resp. \(r\)-fss, \(r\)-fpo) set in \(X\) for each \(\mu \in I^Y\) and \(r \in I_0\) with \(\tau^*(\mu) \geq r\).

The implications contained in the following diagram are true.

\[
\begin{array}{ccc}
\text{fuzzy continuity} & \Downarrow & \text{fuzzy strong semi-continuity} \\
\Downarrow & & \Downarrow \\
\text{fuzzy strong precontinuity} & \Downarrow & \text{fuzzy precontinuity}
\end{array}
\]

The following examples show that the reverse may not be true in general.

**Example 3.1.** Let \(X = \{a, b\}\). Define \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X\) as follows:
\[
\begin{align*}
\lambda_1(a) &= 0.2, & \lambda_1(b) &= 0.4, & \lambda_2(a) &= 0.6, & \lambda_2(b) &= 0.5; \\
\lambda_3(a) &= 0.3, & \lambda_3(b) &= 0.4, & \lambda_4(a) &= 0.6, & \lambda_4(b) &= 0.7.
\end{align*}
\]
Let \(\tau, \tau^* : I^X \to I\) be fuzzy topologies on \(X\) defined as:
\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0 \text{ or } 1, \\
\frac{1}{2}, & \text{if } \lambda = \lambda_1 \text{ or } 0.4, \\
\frac{3}{4}, & \text{if } \lambda = \lambda_2 \text{ or } 0.7, \\
1, & \text{otherwise}.
\end{cases}
\]
\[
\tau^*(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0 \text{ or } 1, \\
\frac{1}{2}, & \text{if } \lambda = \lambda_3, \\
\frac{1}{2}, & \text{if } \lambda = \lambda_4, \\
0, & \text{otherwise}.
\end{cases}
\]
Then the identity mapping \( id_X : (X, \tau) \rightarrow (X, \tau^*) \) is fuzzy precontinuous but not fuzzy strongly precontinuous because: for \( 0 < r \leq \frac{1}{2}, \lambda_3 \) is r-fpo and \( \lambda_4 \) is r-fpo but not r-fspo because
\[
\begin{align*}
\lambda_3 &\leq I_r(C_r(\lambda_3, r), r) = I(\tau)(I - \lambda_2, r) = 0, \\
\lambda_4 &\leq I_r(C_r(\lambda_4, r), r) = I, \\
\lambda_4 &> I_r(PC_r(\lambda_4, r), r) = I_r(\lambda_4, r) = \lambda_2.
\end{align*}
\]

**Example 3.2.** We consider Example 2.1 and put
\[
\tau^*(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0 \text{ or } I, \\
\frac{1}{2}, & \text{if } \lambda = \mu, \\
0, & \text{otherwise}.
\end{cases}
\]
For fts \((X, \tau^*)\) the identity mapping \( id_X : (X, \tau) \rightarrow (X, \tau^*) \) is fuzzy strongly precontinuous but it is neither fuzzy continuous nor fuzzy strongly semi-continuous.

**Theorem 3.1.** Let \((X, \tau)\) and \((Y, \tau^*)\) be fts’s and \( f : X \rightarrow Y \). The following statements are equivalent:

1. The mapping \( f \) is fuzzy strongly precontinuous.
2. \( f^{-1}(\mu) \) is r-fspo in \( X \) for each \( \mu \in I_Y, r \in I_0 \) with \( \tau^*(I - \mu) \geq r \).
3. \( f(SPC_r(\lambda, r)) \subseteq C_\tau(f(\lambda), r) \), \( \forall \lambda \in I^X, r \in I_0 \).
4. \( SPC_r(f^{-1}(\mu), r) \subseteq f^{-1}(C_\tau(\mu, r)) \), \( \forall \mu \in I^Y, r \in I_0 \).
5. \( f^{-1}(I_r(\lambda, r)) \subseteq SPL_r(f^{-1}(\mu), r) \), \( \forall \mu \in I^Y, r \in I_0 \).

**Proof.** (1) \( \iff \) (2). Clear.

(1) \( \implies \) (3). For all \( \lambda \in I^X, r \in I_0 \), since \( \tau^*(I - C_\tau(f(\lambda), r)) \geq r \) from the definition of \( C_\tau \), and Definition 1.1 (O3),
\[
f^{-1}(I - C_\tau(f(\lambda), r)) = I - f^{-1}(C_\tau(\lambda, r))
\]
is r-fspo. By Definition 2.3(2), \( f^{-1}(C_\tau(\lambda, r)) \) is an r-fspo set in \( X \). Since
\[
\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(C_\tau(f(\lambda), r)),
\]
we have \( SPC_r(\lambda, r) \subseteq f^{-1}(C_\tau(\lambda, r)) \). Hence
\[
f(f(SPC_r(\lambda, r))) \leq f(f^{-1}(C_\tau(f(\lambda), r))) \subseteq C_\tau(f(\lambda), r).
\]

(3) \( \implies \) (4). For \( \mu \in I^Y, r \in I_0 \), let \( \lambda = f^{-1}(\mu) \). By (3)
\[
f(f(SPC_r(f^{-1}(\mu), r))) \subseteq C_\tau(f(f^{-1}(\mu), r)) \subseteq C_\tau(\mu, r).
\]

It implies \( SPC_r(f^{-1}(\mu), r) \subseteq f^{-1}(C_\tau(\mu, r)) \).
(4) $\Rightarrow$ (5). For $\mu \in I^Y, r \in I_0$, by (4) we have

$$f^{-1}(C_\ast(1 - \mu, r)) \supseteq SPC_r(f^{-1}(1 - \mu), r) = SPC_r(1 - f^{-1}(\mu), r).$$

Then by Theorem 2.4.(5) we have

$$f^{-1}(C_\ast(1 - \mu, r)) \supseteq 1 - SPI_r(f^{-1}(\mu), r),$$

$$1 - f^{-1}(1 - I_\ast(\mu, r)) \subseteq SPI_r(f^{-1}(\mu), r),$$

$$f^{-1}(I_\ast(\mu, r)) \subseteq SPI_r(f^{-1}(\mu), r).$$

(5) $\Rightarrow$ (1). For each $\mu \in I^Y, r \in I_0$ with $\tau^*(\mu) \geq r$, since $I_\ast(\mu, r) = \mu$,

$$f^{-1}(\mu) = f^{-1}(I_\ast(\mu, r)) \subseteq SPI_r(f^{-1}(\mu), r).$$

Hence, by definition of $SPI_r(f^{-1}(\mu), r), f^{-1}(\mu) = SPI_r(f^{-1}(\mu), r)$. By Theorem 2.4.(2), $f^{-1}(\mu)$ is $r$-fspo. Therefore $f$ is a fuzzy strongly precontinuous mapping. ■

**Theorem 3.2.** Let $(X, \tau)$ and $(Y, \tau^*)$ be fts’s and $f: X \rightarrow Y$. The following statements are equivalent:

1. The mapping $f$ is fuzzy strongly precontinuous.
2. $C_\ast(PL_r(f^{-1}(\mu), r), r) \subseteq f^{-1}(C_\ast(\mu, r)), \forall \mu \in I^Y, r \in I_0.$
3. $f(C_\ast(PL_r(\lambda, r), r)) \subseteq C_\ast(f(\lambda), r), \forall \lambda \in I^X, r \in I_0.$

**Proof.** The proof is standard and therefore omitted. ■

**Theorem 3.3.** Let $(X, \tau)$ and $(Y, \tau^*)$ be fts’s and $f: X \rightarrow Y$. The following statements are equivalent:

1. The mapping $f$ is fuzzy precontinuous.
2. $f^{-1}(\mu)$ is $r$-fspe in $X$ for each $\mu \in I^Y, r \in I_0$ with $\tau^*(1 - \mu) \geq r$.
3. $f(PC_\ast(\lambda, r)) \subseteq C_\ast(f(\lambda), r), \forall \lambda \in I^X, r \in I_0.$
4. $PC_\ast(f^{-1}(\mu), r) \subseteq f^{-1}(C_\ast(\mu, r)), \forall \mu \in I^Y, r \in I_0.$
5. $f^{-1}(I_\ast(\mu, r)) \subseteq PL_r(f^{-1}(\mu), r), \forall \mu \in I^Y, r \in I_0.$

**Proof.** Similar to the proof of Theorem 3.1. ■

**Theorem 3.4.** Let $(X, \tau)$ and $(Y, \tau^*)$ be fts’s and $f: X \rightarrow Y$ be a bijective mapping. The following statements are equivalent:

1. The mapping $f$ is fuzzy strongly precontinuous.
2. $I_\ast(f(\lambda), r) \subseteq f(SPL_r(\lambda, r)), \forall \lambda \in I^X, r \in I_0.$

**Proof.** (1) $\Rightarrow$ (2). Let $f$ be a fuzzy strongly precontinuous mapping and $\lambda \in I^X, r \in I_0$. Then, $f^{-1}(I_\ast(f(\lambda), r))$ is an $r$-fspe set in $X$. By Theorem 3.1.(5), and the fact that $f$ is injective, we have

$$f^{-1}(I_\ast(f(\lambda), r)) \subseteq SPI_r(f^{-1}(f(\lambda)), r) = SPI_r(\lambda, r).$$
Since $f$ is surjective, we have
\[
I_\ast(f(\lambda), r) = f(f^{-1}(I_\ast(f(\lambda), r))) \leq f(SPL(\lambda, r)).
\]

(2) $\implies$ (1). Let $\mu \in I^r$, $r \in I_0$ with $\tau^\ast(\mu) \geq r$. Then $I_\ast(\mu, r) = \mu$ and by (2) we have
\[
\mu = I_\ast(\mu, r) \leq f(SPL(f^{-1}(\mu), r)).
\]
Since $f$ is injective we have
\[
f^{-1}(\mu) \leq f^{-1}f(SPL(f^{-1}(\mu), r)) = SPL(f^{-1}(\mu), r).
\]
Then, by the definition of $SPL(f^{-1}(\mu), r)$ we have $f^{-1}(\mu) = SPL(f^{-1}(\mu), r)$. By Theorem 2.4, (2) we have $f^{-1}(\mu)$ is an $r$-fspo set in $X$. Thus, $f$ is a fuzzy strongly precontinuous mapping.

4. Fuzzy strongly preopen and fuzzy strongly preclosed mappings

**Definition 4.1.** Let $(X, \tau)$ and $(Y, \tau^\ast)$ be fts's and $f: X \to Y$ be a mapping. Then $f$ is called:

(1) fuzzy strongly preopen (resp. fuzzy strongly semi-open, fuzzy preopen) if $f(\lambda)$ is an $r$-fspo (resp. $r$-fss, $r$-fpo) set in $Y$ for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\lambda) \geq r$.

(2) fuzzy strongly preclosed (resp. fuzzy strongly semi-closed, fuzzy preclosed) if $f(\lambda)$ is an $r$-fpsc (resp. $r$-fssc, $r$-fpc) set in $Y$ for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\tilde{\lambda} - \lambda) \geq r$.

The implications contained in the following diagram are true.

\[
\begin{array}{ccc}
\text{fuzzy open (closed)} & \Downarrow & \Downarrow \\
\text{fuzzy strongly semi-open (semi-closed)} & \Downarrow & \Downarrow \\
\text{fuzzy strongly preopen (preclosed)} & \Downarrow & \\
\text{fuzzy preopen (preclosed)} & & \\
\end{array}
\]

The following examples show that the reverse may not be true.

**Example 4.1.** In Example 3.1, the identity mapping $id_X : (X, \tau^\ast) \to (X, \tau)$ is a fuzzy preopen mapping but not a fuzzy strongly preopen mapping.

**Example 4.2.** In Example 3.2, the identity mapping $id_X : (X, \tau^\ast) \to (X, \tau)$ is a fuzzy strongly preopen mapping but it is neither fuzzy open nor fuzzy strongly semi-open mapping.
Theorem 4.1. Let $f : (X, \tau) \to (Y, \tau^*)$ be a mapping. Then the following statements are equivalent:

1. $f$ is a fuzzy strongly preopen mapping.
2. $f(I_x(\lambda, r)) \subseteq SPI_{\star}(f(\lambda), \tau)$ for each $\lambda \in I^X$, $r \in I_0$.
3. $I_x(f^{-1}(\mu), r) \subseteq f^{-1}(SPI_{\star}(\mu), r)$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) $\implies$ (2). For all $\lambda \in I^X$, $r \in I_0$, since $\tau(I_x(\lambda, r)) \supseteq r$, $f(I_x(\lambda, r))$ is $r$-$\tau^*$-fspo. From Theorem 2.4.(2),

$$f(I_x(\lambda, r)) = SPI_{\star}(f(I_x(\lambda, r), r)) \subseteq SPI_{\star}(f(\lambda), r).$$

(2) $\implies$ (1). For all $\lambda \in I^X$, $r \in I_0$, with $\tau(\lambda) \supseteq r$ we have $I_x(\lambda, r) = \lambda$. From (2), we have

$$f(\lambda) = f(I_x(\lambda, r)) \subseteq SPI_{\star}(f(\lambda), r) \subseteq f(\lambda).$$

Then, $f(\lambda) = SPI_{\star}(f(\lambda), r)$, and by Theorem 2.4.(2) we have that $f(\lambda)$ is an $r$-fspo set in $Y$. Therefore $f$ is a fuzzy strongly preopen mapping.

(2) $\implies$ (3). For all $\mu \in I^Y$, $r \in I_0$, by (2) we have

$$f(I_x(f^{-1}(\mu), r)) \subseteq SPI_{\star}(f f^{-1}(\mu), r) \subseteq SPI_{\star}(\mu, r).$$

It implies that $I_x(f^{-1}(\mu), r) \subseteq f^{-1}(I_x(f^{-1}(\mu), r)) \subseteq f^{-1}(SPI_{\star}(\mu, r)).$

(3) $\implies$ (2). For all $\lambda \in I^X$, $r \in I_0$, by (3) we have

$$I_x(\lambda, r) \subseteq I_x(f^{-1}(f(\lambda)), r) \subseteq f^{-1}(SPI_{\star}(\mu, r)).$$

It implies that $f(I_x(\lambda, r)) \subseteq f(f^{-1}(SPI_{\star}(f(\lambda), r))) \subseteq SPI_{\star}(f(\lambda), r)$.

Theorem 4.2. Let $f : (X, \tau) \to (Y, \tau^*)$ be a mapping. Then the following statements are equivalent:

1. $f$ is a fuzzy strongly preclosed mapping.
2. $SPC_{\star}(f(\lambda), r) \subseteq f(C_x(\lambda, r))$ for each $\lambda \in I^X$, $r \in I_0$.

Proof. Similar to the proof of Theorem 4.1. •

Theorem 4.3. Let $f : (X, \tau) \to (Y, \tau^*)$ be a bijective mapping. Then the following statements are equivalent:

1. $f$ is a fuzzy strongly preclosed mapping.
2. $f^{-1}(SPC_{\star}(\mu, r)) \subseteq C_x(f^{-1}(\mu), r)$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) $\implies$ (2). For all $\mu \in I^Y$, $r \in I_0$, since $\tau(1 - C_x(f^{-1}(\mu), r)) \supseteq r$, from (1) we have $f(C_x(f^{-1}(\mu), r))$ is an $r$-fspc set in $Y$. Then by Theorem 2.4.(3) we have

$$f(C_x(f^{-1}(\mu), r)) = SPC_{\star}(f(C_x(f^{-1}(\mu), r)), r) \supseteq SPC_{\star}(f f^{-1}(\mu), r).$$

By the surjectivity of $f$ we have

$$f(C_x(f^{-1}(\mu), r)) \supseteq SPC_{\star}(\mu, r).$$
Also, by injectivity of \( f \) we have
\[
f^{-1}(SPC, (\mu, r)) \leq f^{-1}(f(C, f^{-1}(\mu), r))) = C, (f^{-1}(\mu), r).
\]

(2) \( \implies \) (1). For all \( \lambda \in I^X \), \( r \in I_0 \) with \( \tau(\overline{1} - \lambda) \geq r \), from (2) we have
\[
f^{-1}(SPC, (\lambda, r)) \leq C, (f^{-1}(\lambda), r).
\]

By the injectivity of \( f \) we have
\[
f^{-1}(SPC, (\lambda, r)) \leq C, (\lambda, r) = \lambda
\]
and by surjectivity of \( f \) we have
\[
SPC, (\lambda, r) \leq f(\lambda) \leq SPC, (\lambda, r).
\]

Then \( f(\lambda) = SPC, (f(\lambda), r) \). Hence, \( f(\lambda) \) is an \( r \)-fspc set in \( Y \). So, \( f \) is a fuzzy strongly preclosed mapping.

**Theorem 4.4.** Let \( f: (X, \tau) \to (Y, \tau^*) \) be a mapping. Then \( f \) is fuzzy strongly preopen iff for each \( \nu \in I^Y \) and each \( \lambda \in I^X \), \( r \in I_0 \) with \( \tau(\overline{1} - \lambda) \geq r \), when \( f^{-1}(\nu) \leq \lambda \), there exists an \( r \)-fspc set \( \mu \) in \( Y \) such that \( \nu \leq \mu \) and \( f^{-1}(\mu) \leq \lambda \).

**Proof.** Suppose that \( f \) is a fuzzy strongly preopen mapping, \( \nu \in I^Y \) and \( \lambda \in I^X \), \( r \in I_0 \) with \( \tau(\overline{1} - \lambda) \geq r \) such that \( f^{-1}(\nu) \leq \lambda \). Then, \( f(\overline{1} - \lambda) \leq f(\overline{1} - \nu) \leq \overline{1} - \nu \). Since \( \tau(\overline{1} - \lambda) \geq r \) and \( f \) is a fuzzy strongly preopen mapping, then \( f(\overline{1} - \lambda) \) is an \( r \)-fspc set in \( Y \). So,

\[
f(\overline{1} - \lambda) = SPI, (f(\overline{1} - \lambda), r) \leq SPI, (\overline{1} - \nu, r),
\]
\[
(\overline{1} - \lambda) \leq f^{-1}(f(\overline{1} - \lambda)) \leq f^{-1}(SPI, (\overline{1} - \nu, r)),
\]
\[
(\overline{1} - \nu) \leq f^{-1}(SPI, (\overline{1} - \nu, r)) = f^{-1}(\overline{1} - SPI, (\overline{1} - \nu, r)) \leq \lambda
\]
\[
f^{-1}(SPC, (\nu, r)) \leq \lambda.
\]

Let \( \mu = SPC, (\nu, r) \). Then, \( \mu \) is an \( r \)-fspc set in \( Y \), \( \nu \leq SPC, (\nu, r) = \mu \) and \( f^{-1}(\mu) \leq \lambda \).

Conversely, for all \( \eta \in I^X \), \( r \in I_0 \) with \( \tau(\eta) \geq r \), take \( \lambda = \overline{1} - \eta \in I^X \) and \( \nu = \overline{1} - f(\eta) \in I^Y \). Then, \( \tau(\overline{1} - \lambda) \geq r \) and since \( \eta \leq f^{-1}f(\eta) \) we have

\[
\overline{1} - f^{-1}(f(\eta)) = f^{-1}(\overline{1} - f(\eta)) \leq \overline{1} - \eta,
\]
which implies that \( f^{-1}(\nu) \leq \lambda \). Then there exists an \( r \)-fspc set in \( Y \) such that \( \nu = \overline{1} - f(\eta) \leq \mu \) and \( f^{-1}(\mu) \leq \overline{1} - \eta \). Then, \( \overline{1} - \mu = f(\eta) \). But \( \mu \) is an \( r \)-fspc set in \( Y \) and so \( \overline{1} - \mu = f(\eta) \) is an \( r \)-fspc set in \( Y \). Hence, \( f \) is a fuzzy strongly preopen mapping.
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