DIRECT AND INVERSE THEOREMS FOR SZÁSZ-LUPAS TYPE OPERATORS IN SIMULTANEOUS APPROXIMATION

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Abstract. In this paper we give the direct and inverse theorems for Szász-Lupas operators and study the simultaneous approximation for a new modification of the Szász operators with the weight function of Lupas operators.

1. Introduction

Let $f$ be a function defined on the interval $[0, \infty)$ with real values. For $f \in [0, \infty)$ and $n \in \mathbb{N}$, the Szász operator $S_n(f, x)$ is defined as follows:

$$S_n(f, x) = \sum_{k=0}^{\infty} s_{n,k}(x) f(k/n), \text{ where } s_{n,k}(x) = e^{-nx} (nx)^k / k!.$$

The Szász-type operator $L_n(f, x)$ is defined by

$$L_n(f, x) = \sum_{k=0}^{\infty} s_{n,k}(x) \phi_{n,k}(f),$$

where

$$\phi_{n,k}(f) = \begin{cases} f(0), & \text{for } k = 0 \\ n \int_{0}^{\infty} s_{n,k}(t) f(t) dt, & \text{for } k = 1, 2, \ldots \end{cases}$$

In [10], Mazhar and Totik introduced the Szász-type operator and showed some approximation theorems. Lupas proposed a family of linear positive operators mapping $C[0, \infty)$ into $C[0, \infty)$, the class of all bounded and continuous functions on $[0, \infty)$ namely,

$$(B_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f(k/n), \text{ where } p_{n,k}(x) = \binom{n+k-1}{k} x^k (1 + x)^{n+k}.$$

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Motivated by the integration of Bernstein polynomials of Derriennic [4], Sahai and Prasad [11] modified the operators \( B_n \) for function integrable on \([0, \infty)\) as
\[
(M_n f)(x) = (n - 1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) \, dt.
\]

Now we consider another modification of operators with the weight function of Lupas operators, which are defined as
\[
(V_n f)(x) = (n - 1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t) \, dt. \quad (1.1)
\]
The norm \( \|f\|_{C_\alpha} \) on the space \( C_\alpha[0, \infty) = \{ f \in C[0, \infty) : |f(t)| \leq K t^\alpha \text{ for some } \alpha > 0 \text{ and } K > 0 \} \) is defined by
\[
\|f\|_{C_\alpha} = \sup_{0 \leq t < \infty} |f(t)| t^{-\alpha}.
\]

To improve the saturation order \( O(n^{-1}) \) for the operator (1.1), we use the technique of linear combination as described below:
\[
V_n (f, k, x) = \sum_{j=0}^{k} C(j, k) V_{d_j, n}(f, x),
\]
where
\[
C(j, x) = \prod_{i=0, i \neq j}^{k} \frac{d_j}{d_j - d_i} \text{ for } k \neq 0 \text{ and } C(0, 0) = 1
\]
and \( d_0, d_1, d_2, \ldots, d_k \) are \((k+1)\) arbitrary, fixed and distinct positive integers. For our convenience we shall write the operator (1.1) as
\[
V_n(f, x) = \int_0^{\infty} W(n, x, t) f(t) \, dt,
\]
where
\[
W(n, x, t) = (n - 1) \sum_{k=0}^{\infty} s_{n,k}(x) p_{n,k}(t).
\]

The function \( f \) is said to belong to the generalized Zygmund class \( \text{Liz}(\alpha, k, a, b) \) if there exists a constant \( M \) such that
\[
\omega_{2k}(f, \eta, a, b) \leq M \eta^{\alpha k}, \eta > 0,
\]
where \( \omega_{2k}(f, \eta, a, b) \) denotes the modulus of continuity of \( 2k \)-th order of \( f(x) \) on the interval \([a, b] \). The class \( \text{Liz}(\alpha, a, b) \) is more commonly denoted by \( \text{Lip}^{*}(\alpha, a, b) \).

Let \( f \in C_\alpha[0, \infty) \) and \( 0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty \). Then for \( m \in N \) the Steklov mean \( f_{\eta, m} \) of the \( m \)-th order corresponding to \( f \), for sufficiently small values of \( \eta > 0 \) is defined by
\[
f_{\eta, m}(x) = \eta^{-m} \left( \int_{-\eta/2}^{\eta/2} \left\{ f(x) + (-1)^{m-1} \sum_{i=1}^{m} \Delta_{\eta}^{m} f(x) \right\} dx_i \right)^{m} \prod_{i=1}^{m} dx_i, \quad (1.2)
\]
where \( x \in [a_1, b_1] \) and \( \Delta_{\eta}^{m} f(x) \) is the \( m \)-th order forward difference with step length \( \eta \).
The direct results in ordinary and simultaneous approximation for such type of modified Szász-Mirakyan operators were studied by many researchers see e.g. [2], [5], [6] and [12].

2. Auxiliary results

In this section, we shall give some basic results, which will be useful in proving the main results.

**Lemma 2.1.** [9] For $m \in \mathbb{N} \cup \{0\}$, let the $m$-th order moment for the Szász operator be defined by

$$
\mu_{n,m}(x) = \sum_{k=0}^{\infty} s_{n,k}(x) \left( \frac{k}{n} - x \right)^m.
$$

Then we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$ and

$$
n \mu_{n,m+1}(x) = x \left( \mu_{n,m}^\prime(x) + m \mu_{n,m-1}(x) \right), \quad \text{for } n \in \mathbb{N}.
$$

Consequently,

(i) $\mu_{n,m}(x)$ is a polynomial in $x$ of degree $\lfloor m/2 \rfloor$;

(ii) for every $x \in [0, \infty)$, $\mu_{n,m}(x) = O\left(n^{-\lfloor(m+1)/2\rfloor}\right)$, where $\lfloor \beta \rfloor$ denotes the integral part of $\beta$.

**Lemma 2.2.** Let the $m$-th moment for the Szász operator be defined by

$$
\mu_{n,m}(x) = (n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_{0}^{\infty} p_{n,k}(t)(t-x)^m dt.
$$

Then

(i) $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{(1+2x)}{(n-2)}$, $n > 2$;

(ii) $(n-m-2)\mu_{n,m+1}(x) = x \left[ \mu_{n,m}^\prime(x) + (m+2)x \mu_{n,m-1}(x) \right] + (m+1) \times \left(1+2x\right)\mu_{n,m}(x)$

(iii) $\mu_{n,m}(x) = O\left(n^{-\lfloor(m+1)/2\rfloor}\right)$ for all $x \in [0, \infty)$.

**Proof.** By the definition of $\mu_{n,m}(x)$, we can easily obtain (i). Now the proof of (ii) goes as follows:

$$
x \mu_{n,m}^\prime(x) = (n-1) \sum_{k=0}^{\infty} x s_{n,k}^\prime(x) \int_{0}^{\infty} p_{n,k}(t)(t-x)^m dt - m x \mu_{n,m-1}(x).
$$

Using relations $t(1+t)p_{n,k}^\prime(t) = (k-nt)p_{n,k}(t)$ and $x s_{n,k}^\prime(x) = (k-nt)s_{n,k}(x)$, we get

$$
x \left[ \mu_{n,m}^\prime(x) + m \mu_{n,m-1}(x) \right]
= (n-1) \sum_{k=0}^{\infty} (k-nt) s_{n,k}(x) \int_{0}^{\infty} p_{n,k}(t)(t-x)^m dt
$$
Again using Leibnitz’s theorem in (1.1)

\[
\left( V_n^{(r)}f \right)(x) = (n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty f_n \left( x \right) f(t) dt
\]

and

\[
\left( V_n^{(r)}f \right)(x) = (n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty f_n \left( x \right) f(t) dt
\]

Again using Leibnitz’s theorem

\[
p_{n,r-k+1} \left( t \right) = \frac{(n-1)!}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} p_{n,i}(t)
\]

integrating by parts \( r \) times, we get the required result. 

**Lemma 2.4.** For the function \( f_{\eta,m}(x) \) defined in (1.2), there hold:

(i) \( f_{\eta,m} \in C[a_1, b_1] \);

(ii) \( \|f_{\eta,m}^{(r)}\|_{C[a_2, b_2]} \leq M_r \eta^{-r} \omega_r(f, \eta, a_1, b_1), \quad r = 1, 2, \ldots, m \);

(iii) \( \|f - f_{\eta,m}\|_{C[a_2, b_2]} \leq M_{m+1} \omega_m(f, \eta, a_1, b_1) \);

(iv) \( \|f_{\eta,m}\|_{C[a_2, b_2]} \leq M_{m+2} \|f\|_{C[a_2, b_2]} \leq M' \|f\|_{C[a_1, b_1]} \)

where \( M' \) are certain constants independent of \( f \) and \( \eta \).

For the proof of the above properties of the function \( f_{\eta,m}(x) \) we refer to [12, page 167].
Theorems for Szász-Lupas type operators

Lemma 2.5. [8, 9] There exist polynomials \( q_{i,j,r}(x) \) independent of \( n \) and \( k \) such that
\[
x^r \frac{d^r}{dx^r} [e^{-nx}(nx)^k] = \sum_{i+j \leq r} n^i |k - nx|^j q_{i,j,r}(x) [e^{-nx}(nx)^k]
\]

Lemma 2.6. Let \( f \in C_\alpha[0, \infty) \). If \( f^{(2k+2)} \) exists at a point \( x \in (0, \infty) \), then
\[
\lim_{n \to \infty} n^{k+1} \{ V_n(f, k, x) - f(x) \} = \sum_{p=0}^{2k+2} Q(p, k, x) f^{(p)}(x),
\]
where \( Q(p, k, x) \) is a certain polynomial in \( x \) of degree \( p \).

The proof of Lemma 2.6 follows along the lines of [7].

Lemma 2.7. Let \( \delta \) and \( \gamma \) be any two positive numbers and \( [a, b] \subset [0, \infty) \). Then, for any \( m > 0 \) there exists a constant \( M_m \) such that
\[
\left\| \int_{|t-x| \geq \delta} V_n(f(x)) t^\gamma dt \right\|_{C[a,b]} \leq M_m n^{-m}.
\]

The proof of this result follows easily by using Schwarz inequality and Lemma 2.7 from [1].

3. Main results

Theorem 3.1. (Direct Theorem) Let \( f \in C_\alpha[0, \infty) \). Then, for sufficiently large \( n \), there exists a constant \( M \) independent of \( f \) and \( n \) such that
\[
\left\| V_n(f, k, .) - f \right\|_{C[a_2,b_2]} \leq \max \left\{ C_2 \omega^{2k+2}(f; n^{-1/2}, a_1, b_1) C_2 n^{-(k+1)} \left\| f \right\|_{C_\alpha} \right\},
\]
where \( C_1 = C_1(k) \) and \( C_2 = C_2(k, f) \).

Proof. By linearity property
\[
\left\| V_n(f, k, .) - f \right\|_{C[a_2,b_2]} \leq \left\| V_n \left( (f - f^{2k+2, \eta}), k, . \right) \right\|_{C[a_2,b_2]}
+ \left\| V_n \left( f^{2k+2, \eta}, k, . \right) - f^{2k+2, \eta} \right\|_{C[a_2,b_2]} + \left\| f - f^{2k+2, \eta} \right\|_{C[a_2,b_2]}
= A_1 + A_2 + A_3, \quad \text{say.}
\]

By property (iii) of Steklov mean, we get
\[
A_3 \leq C_1 \omega^{2k+2}(f, \eta, a_1, b_1).
\]

Next, by Lemma 2.6, we get
\[
A_2 \leq C_2 n^{-(k+1)} \left\| f^{2k+2, \eta} \right\|_{C[a_1, b_1]}.
\]
Using the interpolation property \([5]\) and properties of Steklov mean,

\[
A_2 \leq C_3 n^{-(k+1)} \left\{ \| f \|_{C_n} + \eta^{-2(k+2)} \omega_{2(k+2)}(f, \eta) \right\}.
\]

To estimate \(A_1\), we choose \(a_2, b_2\) such that

\[
0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < \infty.
\]

Also let \(\psi(t)\) be the characteristic function of the interval \([a_2, b_2]\), then

\[
A_1 \leq \left\| V_n \left( (\psi(t) (f(t) - f_{2k+2, \eta}(t)), k_{\cdot} \right) \right\|_{C[a_2, b_2]}
+ \left\| V_n \left( (1 - \psi(t) (f(t) - f_{2k+2, \eta}(t)), k_{\cdot} \right) \right\|_{C[a_2, b_2]}
\]

\[
= A_4 + A_5,
\]

say.

We note that in order to estimate \(A_4\) and \(A_5\), it is sufficient to consider their expressions without the linear combination. It is clear that by Lemma 2.3, we obtain

\[
V_n \left( (\psi(t) (f(t) - f_{2k+2, \eta}(t)), x) \right)
= (n - 1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty p_{n,k}(t) \psi(t) (f(t) - f_{2k+2, \eta}(t)) \, dt.
\]

Hence,

\[
\left\| V_n \left( (\psi(t) (f(t) - f_{2k+2, \eta}(t)), \cdot) \right) \right\|_{C[a_1, b_1]} \leq C_4 \| f - f_{2k+2, \eta} \|_{C[a_2, b_2]}.
\]

Now for \(x \in [a_3, b_3]\) and \(t \in [0, \infty) / [a_2, b_2]\) we can choose an \(\eta_1\) satisfying \(|t - x| \geq \eta_1\). Therefore by Lemma 2.5 and Schwarz inequality, we have

\[
I \equiv \left| V_n \left( (1 - \psi(t)) (f(t) - f_{2k+2, \eta}(t)), x \right) \right| \leq \left| (n - 1) \sum_{i+j \leq r \atop i, j \geq 0} n^i \phi_{i,j,r}(x) \sum_{k=0}^{\infty} s_{n,k}(x) |k - nx|^j \right| \times
\]

\[
\int_0^\infty p_{n,k}(t) (1 - \psi(t)) |f(t) - f_{2k+2, \eta}(t)| \, dt.
\]

\[
\leq C_5 \| f \|_{C_n} \left( n - 1 \right) \sum_{i+j \leq r \atop i, j \geq 0} n^i \sum_{k=0}^{\infty} s_{n,k}(x) |k - nx|^j \int_{|t - x| \geq \eta_1} p_{n,k}(t) \, dt.
\]

\[
\leq C_5 \eta_1^{-2s} \left\{ (n - 1) \sum_{i+j \leq r \atop i, j \geq 0} n^i \sum_{k=0}^{\infty} s_{n,k}(x) |k - nx|^j \right| \times
\]

\[
\left( \int_0^\infty p_{n,k}(t) \, dt \right)^{1/2} \left( \int_0^\infty p_{n,k}(t) (t - x)^{4s} \, dt \right)^{1/2}
\]

\[
\leq C_5 \eta_1^{-2s} \| f \|_{C_n} \sum_{i+j \leq r \atop i, j \geq 0} n^i \left\{ \sum_{k=0}^{\infty} s_{n,k}(x) (k - nx)^{2j} \right\}^{1/2} \times
\]

\[
\left( n - 1 \right) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty p_{n,k}(t) (t - x)^{4s} \, dt \right)^{1/2}.
\]
Hence, by Lemma 2.1 and Lemma 2.2, we have
\[ I \leq C_6 \|f\|_{C^0_n} \sum n^{(i+\frac{1}{2})} \leq C_6 n^{-q} \|f\|_{C^0_n} \]
where \( q = (s - m/2) \). Now choose \( s > 0 \) such that \( q \geq k + 1 \). Then
\[ I \leq C_6 n^{-(k+1)} \|f\|_{C^0_n}. \]

Therefore by property (iii) of Steklov mean, we get
\[ A_1 \leq C_7 \|f - f_{2k+2}\|_{C[a_2, b_2]} + C_6 n^{-(k+1)} \|f\|_{C^0_n} \]
\[ \leq C_8 n^{-2k+2}(f, \eta, a_1, b_1) + C_6 n^{-(k+1)} \|f\|_{C^0_n} \]

Hence with \( \eta = n^{-1/2} \), the theorem follows. \( \blacksquare \)

**Theorem 3.2.** (Inverse Theorem) If \( 0 < \alpha < 2 \) and \( f \in C_\alpha [0, \infty) \) then in the following statements (i) \( \Rightarrow \) (ii):

(i) \( \|V_n(f, k, x) - f(x)\|_{C[a_1, b_1]} = O \left(n^{-\alpha(k+1)/2}\right) \), where \( f \in C_\alpha [a, b] \).

(ii) \( f \in Liz(\alpha, k + 1, a_2, b_2) \).

**Proof.** Let us choose points \( a', a'' \), \( b', b'' \) in such a way that \( a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1 \). Also suppose \( g \in C_\alpha^{\infty} \) with \( \text{supp}(g) \subseteq [a'', b''] \) and \( g(x) = 1 \)
for \( x \in [a_2, b_2] \). It is sufficient to show that
\[ \|V_n(fg, k, \cdot) - fg\|_{C[a', b']} = O \left(n^{-\alpha(k+1)/2}\right) \Rightarrow (ii). \tag{3.1} \]

Using \( F \) in place of \( fg \) for all the values of \( r > 0 \), we get
\[ \|\Delta_r^{2k+2}F\|_{C[a', b']} \leq \|\Delta_r^{2k+2}(F - V_n(F, k, \cdot))\|_{C[a'', b'']} + \|\Delta_r^{2k+2}V_n(F, k, \cdot)\|_{C[a'', b'']} \tag{3.2} \]

By the definition of \( \Delta_r^{2k+2} \),
\[ \|\Delta_r^{2k+2}V_n(F, k, \cdot)\|_{C[a'', b'']} \]
\[ = \left\| \int_0^\infty \cdots \int_0^r V_n \left(F, k, x + \sum_{i=1}^{2k+2} x_i\right) dx_1 \cdots dx_{2k+2} \right\|_{C[a'', b'']} \]
\[ \leq \|\Delta_r^{2k+2}V_n\|_{C[a'', b'']}^{(2k+2)}(F, k, \cdot) \]
\[ \leq r^{2k+2} \left( \|V_n\|_{C[a'', b'']}^{(2k+2)}(F, k, \cdot) \right) \]
\[ + \left\| \Delta_r^{2k+2}V_n\right\|_{C[a'', b'']}^{(2k+2)}(F, k, \cdot) \] \tag{3.3}

where \( F, k, 2k+2 \) is the Steklov mean of \( (2k + 2) \)-th order corresponding to \( F \). By Lemma 3 from [1], we get
\[ \int_0^\infty \left| \frac{\partial^{2k+2}}{\partial x_{2k+2}} W_n(t, x) \right| dt \]
\[ \leq \sum_{\substack{i+j \geq 2k+2 \atop i, j \geq 0}} (n - 1) \sum_{k=0}^{n-1} n! |k - nx|^j \left| q_{i+j, 2k+2}(x) \right| s_{n,k}(x) \int_0^\infty p_{n,k}(t) dt. \]
Since $\int_0^\infty p_{n,k}(t)\,dt = \frac{1}{n^k}$, by Lemma 2.1,
\[
\sum_{k=0}^\infty s_{n,k}(x)(k-nx)^{2j} = n^{2j} \sum_{k=0}^\infty s_{n,k}(x) \left(\frac{k}{n} - x\right)^{2j} = O(n^j) \tag{3.4}
\]

Using Schwarz inequality and Lemma 2.1, we obtain
\[
\left\|V_n^{(2k+2)}(F-F_{n,2k+2,k,\cdot})\right\|_{C[a^p,b^p+(2k+2)r]} \leq K_1 n^{k+1} \left\|F-F_{n,2k+2}\right\|_{C[a^p,b^p]} \tag{3.5}
\]

By Lemma 2 from [1], we get
\[
\int_0^\infty \left[\frac{\partial^k}{\partial x^k} W_n(t,x)\right] (t-x)^i\,dt = 0, \quad \text{for} \quad k > i. \tag{3.6}
\]

By Taylor’s expansion, we obtain
\[
F_{n,2k+2}(t) = \sum_{i=0}^{2k+1} \frac{F_{n,2k+2}^{(i)}(x)}{i!} (t-x)^i + F_{n,2k+2}^{(2k+2)}(\xi) \frac{(t-x)^{2k+2}}{(2k+2)!}, \tag{3.7}
\]

where $t < \xi < x$. By (3.6) and (3.7), we get
\[
\left\|\frac{\partial^{2k+2}}{\partial x^{2k+2}} V_n(F_{n,2k+2,k,\cdot})\right\|_{C[a^p,b^p+(2k+2)r]} \leq \sum_{j=0}^k \frac{|C(j,k)|}{(2k+2)!} \left\|F_{n,2k+2}^{(2k+2)}\right\|_{C[a^p,b^p]} \int_0^\infty \left[\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_{n,t}(t,x)\right] (t-x)^{2k+2} \,dt \right\|_{C[a^p,b^p]}. \tag{3.8}
\]

Again applying Schwarz inequality for integration and summation and Lemma 3 from [1], we obtain
\[
I \equiv \int_0^\infty \left[\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(t,x)\right] (t,x)^{2k+2} \,dt \leq (n-1) \sum_{i,j \geq 0} \sum_{k=0}^{2k+2} n^i s_{n,k}(x) (k-nx)^j \left|\frac{q_{i,j,2k+2}(x)}{x^{2k+2}}\right| \int_0^\infty p_{n,k}(t)(t-x)^{2k+2} \,dt \leq \sum_{i,j \geq 0} \sum_{k=0}^{2k+2} n^i s_{n,k}(x) (k-nx)^j \left\{\sum_{k=0}^{2k+2} s_{n,k}(x) (k-nx)^j\right\}^{1/2} \times \left\{(n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty p_{n,k}(t)(t-x)^4 \,dt\right\}^{1/2}. \tag{3.9}
\]

Using Lemma 2 from [1],
\[
(n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^\infty p_{n,k}(t)(t-x)^4 \,dt = T_{n,4k+4}(x) = O \left(n^{-(2k+2)}\right). \tag{3.9}
\]

Using (3.4) and (3.9) in (3.8), we obtain
\[
I \leq \sum_{i,j \geq 0} \sum_{k=0}^{2k+2} n^i \left|\frac{q_{i,j,2k+2}(x)}{x^{2k+2}}\right| O(n^{j/2}) O \left(n^{-(k+1)}\right) = O(1). \tag{3.9}
\]
Hence
\[
\|W_n^{(2k+2)}(F_{\eta, 2k+2}, k, \cdot)\|_{C[a'', b''+2(2k+2)r]} \leq K_2 \|F_{\eta, 2k+2}\|_{C[a'', b'']},
\] (3.10)

On combining (3.2), (3.3), (3.5) and (3.10) it follows
\[
\|\Delta_r^{2k+2}F\|_{C[a'', b'']} \leq \|\Delta_r^{2k+2}(F - V_n(F, k, \cdot))\|_{C[a'', b'']} + K_3r^{2k+2}\left(n^{k+1}\|F - F_{\eta, 2k+2}\|_{C[a'', b'']} + \|F_{\eta, 2k+2}\|_{C[a'', b'']}\right).
\]

Since for small values of \(r\) the above relation holds, it follows from the properties of \(F_{\eta, 2k+2}\) and (3.1) that
\[
\omega_{2k+2}(F, h, [a'', b'']) = O(h^{\alpha(k+1)}).
\] (3.11)

Since \(F(x) = f(x)\) in \([a_2, b_2]\), from (3.11) we have
\[
\omega_{2k+2}(f, h, [a_2, b_2]) = O(h^{\alpha(k+1)}), \text{ i.e., } f \in L_{{\alpha, k+1}, a_2, b_2}.
\]

Let us assume (i). Putting \(\tau = \alpha(k+1)\), we first consider the case \(0 < \tau \leq 1\).

For \(x \in [a', b']\), we get
\[
V_n(f, k, x) - f(x)g(x) = g(x)V_n((f(t) - f(x)), k, x) + \sum_{j=0}^{k} C(j, k) \int_{a_1}^{b_1} W_{d_j, n}(t, x) f(x)(g(t) - g(x)) dt + O\left(n^{-k+1}\right)
\]
\[
= I_1 + I_2 + O\left(n^{-k+1}\right),
\] (3.12)

where the \(O\)-term holds uniformly for \(x \in [a', b']\). Now by assumption
\[
\|V_n(f, k, \cdot) - f\|_{C[a_1, b_1]} = O\left(n^{-\frac{\tau}{2}}\right),
\]
we have
\[
\|I_1\|_{C[a', b']} \leq \|g\|_{C[a', b']}\|V_n(f, k, \cdot) - f\|_{C[a', b']} \leq K_{5} n^{-\frac{\tau}{2}}.
\] (3.13)

By the Mean Value Theorem, we get
\[
I_2 = \sum_{j=0}^{k} C(j, k) \int_{a_2}^{b_2} W_{d_j, n}(t, x) f(t) \{g'(\xi)(t - x)\} dt.
\]

Once again applying Cauchy-Schwarz inequality and Lemma 2 from [1], we get
\[
\|I_2\|_{C[a', b']} \leq \|f\|_{C[a_1, b_1]}\|g'\|_{C[a', b']} \left(\sum_{j=0}^{k} C(j, k)\right) \times \max_{0 \leq j \leq k} \int_{0}^{\infty} W_{d_j, n}(t, x)(t - x)^2 dt \|_{C[a', b']}^{1/2} = O\left(n^{-\frac{\tau}{2}}\right).
\] (3.14)

Combining (3.12–3.14), we obtain
\[
\|V_n(f, g, k, \cdot) - f g\|_{C[a', b']} = O\left(n^{-\frac{\tau}{2}}\right), \text{ for } 0 < \tau \leq 1.
\]
Now to prove the implication for \(0 < \tau < 2k + 2\), it is sufficient to assume it for \(\tau \in (m - 1, m)\) and prove if for \(\tau \in (m - 1, m + 1)\), \((m = 1, 2, 3, \ldots, 2k + 1)\). Since the result holds for \(\tau \in (m - 1, m)\), we choose two points \(x_1, y_1\) in such a way that \(a_1 < x_1 < a' < b' < y_1 < b_1\). Then in view of assumption \((i) \Rightarrow (ii)\) for the interval \((m - 1, m)\) and equivalence of \((ii)\) it follows that \(f^{(m-1)}\) exists and belongs to the class \(\text{Lip}(1 - \delta, x_1, y_1)\) for any \(\delta > 0\).

Let \(g \in C_0^\infty\) be such that \(g(x) = 1\) on \([a'', b']\) and \(\text{supp}\ g \subset [a'', b']\). Then with \(\chi_2(t)\) denoting the characteristic function of the interval \([x_1, y_1]\), we have
\[
\|V_n(f, g, k,) - f\|_{C[a', b']^I} \leq \|V_n(g(x)f(t) - f(x)), k,.)\|_{C[a', b']^I} + \|V_n(f(t)(g(t) - g(x)))\chi(t), k,.)\|_{C[a', b']^I} + O\left(n^{-\gamma(k+1)}\right).
\]

Now
\[
\|V_n(g(x)(f(t) - f(x)), k,.)\|_{C[a', b']^I} \leq \|g\|_{C[a'', b']}\|V_n(f, k,.) - f\|_{C[a_1, b_1]} = O\left(n^{-\gamma/2}\right).
\]

Applying Taylor’s expansion of \(f\), we have
\[
I_3 \equiv \|V_n(f(t)g(t) - g(x))\chi(t), k,.)\|_{C[a', b']^I} =
\]
\[
\|\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!} (t-x)^i \frac{f^{(m-1)}(x) - f^{(m-1)}(x)}{(m-1)!} (g(t)-g(x))\chi(t), k,.)\|_{C[a', b']^I}
\]
where \(\xi\) lies between \(t\) and \(x\). Since \(f^{(m-1)} \in \text{Lip}(1 - \delta, x_1, y_1)\),
\[
|f^{(m-1)}(\xi) - f^{(m-1)}(x)| \leq K_6|\xi - x|^{1-\delta} \leq K_6|t - x|^{1-\delta},
\]
where \(K_6\) is the \(\text{Lip}(1 - \delta, x_1, y_1)\) constant for \(f^{(m-1)}\), we have
\[
I_3 \leq \|V_n\left(\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!} (t-x)^i (g(t)-g(x))\chi(t), k,.)\right)\|_{C[a', b']^I}
\]
\[
+ \frac{K_6}{(m-1)!}\|g\|_{C[a'', b']}\left(\sum_{j=0}^{k} |C(j, k)|\right)\|V_n(t - x)^{m+1-\delta}\chi(t,.)\|_{C[a', b']^I}
\]
\[
= I_4 + I_5 \quad \text{say.}
\]

By Taylor’s expansion of \(g\) and Lemma 2.6, we have
\[
I_4 = O\left(n^{-\gamma(k+1)}\right).
\]

Also, by Hölder’s expansion of \(g\) and Lemma 2 from \([1]\), we have
\[
I_5 \leq \frac{K_6}{(m-1)!}\|g\|_{C[a'', b']}\left(\sum_{j=0}^{k} |C(j, k)|\right) \times
\]
\[
\times \max_{0 \leq j \leq k} \left\|\int_{x_1}^{y_1} W_{d_j, n}(t - x)|t - x|^{m+1-\delta} dt\right\|_{C[a', b']^I}
\]
\[
\leq K_7 \max_{0 \leq j \leq k} \|\int_{x_1}^{y_1} W_{d_j, n}(t - x)(t - x)^{2(m+1)} dt\|_{C[a', b']^I}
\]
\[
= O\left(n^{-\gamma(k+1)/2}\right) = O\left(n^{\gamma/2}\right),
\]
\[ (3.19) \]
by choosing $\delta$ such that $0 < \delta < m + 1 - \delta$. Combining the estimates (3.15–3.19), we get

$$\| V_n(fg, k, \cdot) - fg \|_{C^{[a', b']}} = O \left( n^{7/2} \right).$$

This completes the proof of the Theorem 3.2. \( \blacksquare \)

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