ON PSEUDO-BCI IDEALS OF PSEUDO-BCI ALGEBRAS

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Abstract. The notions of pseudo-atoms, pseudo-BCI ideals and pseudo-BCI homomorphisms in pseudo-BCI algebras are introduced. Characterizations of a pseudo-BCI ideal are displayed, and conditions for a subset to be a pseudo-BCI ideal are given. The concept of a ω-medial pseudo-BCI algebra is also introduced, and its characterization is provided. We show that every pseudo-BCI homomorphic image and preimage of a pseudo-BCI ideal is also a pseudo-BCI ideal.

1. Introduction

G. Georgescu and A. Iorgulescu [1] introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebras. In [2], Y. B. Jun, one of the present authors, gave a characterization of pseudo-BCK algebra, and provided conditions for a pseudo-BCK algebra to be ∧-semi-lattice ordered (resp. ∩-semi-lattice ordered). Y. B. Jun et al. [4] introduced the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then they investigated some of their properties. In [2], W. A. Dudek and Y. B. Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties. In this paper, we introduce the concepts of pseudo-atoms, pseudo-BCI ideals and pseudo-BCI homomorphisms in pseudo-BCI algebras. We display characterizations of a pseudo-BCI ideal, and provide conditions for a subset to be a pseudo-BCI ideal. We also introduced the notion of a ω-medial pseudo-BCI algebra, and give its characterization. We prove that every pseudo-BCI homomorphic image and preimage of a pseudo-BCI ideal is also a pseudo-BCI ideal.

2. Preliminaries

Recall that a BCI-algebra is an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following axioms: for every \(x, y, z \in X\),

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\[ ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0, \]
\[ (x \ast (x \ast y)) \ast y = 0, \]
\[ x \ast x = 0, \]
\[ x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y. \]

For any BCI-algebra \( X \), the relation \( \leq \) defined by \( x \leq y \) if and only if \( x \ast y = 0 \) is a partial order on \( X \). A nonempty subset \( I \) of a BCI-algebra \( X \) is called a BCI-ideal of \( X \) if it satisfies
\[ 0 \in I, \]
\[ \forall x, y \in X, x \ast y \in I, y \in I \Rightarrow x \in I. \]

3. Properties of Pseudo-BCI algebras

**Definition 3.1.** A pseudo-BCI algebra is a structure \( X = (X, \preceq, \ast, \circ, 0) \), where \( \preceq \) is a binary relation on a set \( X \), \( \ast \) and \( \circ \) are binary operations on \( X \) and \( 0 \) is an element of \( X \), verifying the axioms: for all \( x, y, z \in X, \)

(a1) \( (x \ast y) \circ (x \ast z) \leq z \ast y, (x \circ y) \ast (x \circ z) \leq z \circ y, \)
(a2) \( x \ast (x \circ y) \leq y, x \circ (x \ast y) \leq y, \)
(a3) \( x \preceq x, \)
(a4) \( x \leq y, y \leq x \Rightarrow x = y, \)
(a5) \( x \leq y \iff x \ast y = 0 \iff x \circ y = 0. \)

Note that every pseudo-BCI algebra satisfying \( x \ast y = x \circ y \) for all \( x, y \in X \) is a BCI-algebra. Every pseudo-BCK algebra is a pseudo-BCI algebra.

**Proposition 3.2.** [2] In a pseudo-BCI algebra \( X \) the following holds:

(p1) \( x \leq 0 \Rightarrow x = 0. \)
(p2) \( x \leq y \Rightarrow z \ast y \leq z \ast x, z \circ y \leq z \circ x. \)
(p3) \( x \preceq y, y \preceq z \Rightarrow x \preceq z. \)
(p4) \( (x \ast y) \circ z = (x \circ z) \ast y. \)
(p5) \( x \ast y \preceq z \iff x \circ z \preceq y. \)
(p6) \( (x \ast y) \ast (z \ast y) \leq x \ast z, (x \circ y) \circ (z \circ y) \leq x \circ z. \)
(p7) \( x \leq y \Rightarrow x \ast z \leq y \ast z, x \circ z \leq y \circ z. \)
(p8) \( x \ast 0 = x = x \circ 0. \)
(p9) \( x \ast (x \circ y) = x \ast y \text{ and } x \circ (x \ast y) = x \circ y. \)

**Example 3.3.** Let \( X = [0, \infty] \) and let \( \leq \) be the usual order on \( X \). Define binary operations \( \ast \) and \( \circ \) on \( X \) by

\[
\begin{align*}
x \ast y & : \begin{cases} 0 & \text{if } x \leq y, \\
\frac{\pi}{2} \arctan(\ln(\frac{x}{y})) & \text{if } y < x,
\end{cases} \\
x \circ y & : \begin{cases} 0 & \text{if } x \leq y, \\
x e^{-\tan(\frac{\pi}{2})} & \text{if } y < x,
\end{cases}
\end{align*}
\]
for all \(x, y \in X\). Then \(X := (X, \leq, \ast, \circ, 0)\) is a pseudo-\(BCK\) algebra, and so a pseudo-\(BCI\) algebra.

**Proposition 3.4** In a pseudo-\(BCI\) algebra \(X\), the following holds for all \(x, y \in X\):

(i) \(0 \ast (x \circ y) \preceq y \circ x\).

(ii) \(0 \circ (x \ast y) \preceq y \ast x\).

(iii) \(0 \ast (x \ast y) = (0 \circ x) \circ (0 \ast y)\).

(iv) \(0 \circ (x \circ y) = (0 \ast x) \ast (0 \circ y)\).

**Proof.** (i) and (ii). We have \(0 \ast (x \circ y) = ((x \ast y) \circ y) \circ (x \ast y) \preceq y \circ x\) and \(0 \circ (x \ast y) = (x \ast x) \circ (x \ast y) \preceq y \ast x\) by (a1) and (a3).

(iii) and (iv). Using (a3) and (p4), we obtain

\[
(0 \circ x) \circ (0 \ast y) = ((((x \ast y) \circ (x \ast y)) \circ x) \circ (0 \ast y) = ((x \ast y) \circ (x \ast y)) \circ (0 \ast y) = ((0 \ast y) \ast (x \ast y)) \circ (0 \ast y) = (0 \ast y) \circ (0 \ast y) = (x \ast y) = 0 \ast (x \ast y)
\]

and

\[
(0 \ast x) \ast (0 \circ y) = (((x \circ y) \ast (x \circ y)) \ast x) \ast (0 \circ y) = (((x \circ y) \ast x) \ast (x \circ y)) \ast (0 \circ y) = ((0 \circ y) \circ (x \circ y)) \ast (0 \circ y) = ((0 \circ y) \ast (0 \circ y)) \circ (x \circ y) = 0 \circ (x \circ y).
\]

**Definition 3.5.** An element \(w\) of a pseudo-\(BCI\) algebra \(X\) is called a pseudo-atom if for every \(x \in X\), \(x \preceq w\) implies \(x = w\).

Obviously, 0 is a pseudo-atom of \(X\).

**Proposition 3.6.** Let \(X\) be a pseudo-\(BCI\) algebra. If an element \(w\) of \(X\) satisfies the identity \(y \ast (y \circ (w \ast x)) = w \ast x\) for all \(x, y \in X\), then \(w\) is a pseudo-atom of \(X\).

**Proof.** Let \(y \in X\) be such that \(y \preceq w\). Then

\[w = w \ast 0 = y \ast (y \circ (w \ast 0)) = y \ast (y \circ w) = y \ast 0 = y.\]

Hence \(w\) is a pseudo-atom of \(X\).

**Proposition 3.7.** Let \(X\) be a pseudo-\(BCI\) algebra and let \(w\) be a pseudo-atom of \(X\). Then the following are true.
(i) \( w = x \circ (x \ast w), \forall x \in X. \)
(ii) \( (x \ast y) \circ (x \ast w) = w \ast y, \forall x, y \in X. \)
(iii) \( w \ast (x \circ y) \preceq y \circ (x \ast w), \forall x, y \in X. \)
(iv) \( (w \circ x) \ast (y \circ z) \preceq (z \circ (y \ast w)) \circ x, \forall x, y, z \in X. \)
(v) \( 0 \circ (y \ast w) = w \ast y, \forall y \in X. \)

**Proof.**

(i) Since \( x \circ (x \ast w) \preceq w \) by (a2), it follows that \( w = x \circ (x \ast w). \)

(ii) For every \( x, y \in X \), we have
\[
(x \ast y) \circ (x \ast w) = (x \circ (x \ast w)) \ast y = w \ast y
\]
by (p4) and (i).

(iii) Using (i), (a2), (p4) and (p7), we have
\[
w \ast (x \circ y) = (x \circ (x \ast w)) \ast y = (x \circ (x \circ y)) \circ (x \ast w) \preceq y \circ (x \ast w).
\]

(iv) Using (p4), (p7) and (iii), we get
\[
(w \circ x) \ast (y \circ z) = (w \ast (y \circ z)) \circ x \preceq (z \circ (y \ast w)) \circ x.
\]

(v) For every \( y \in X \), we obtain
\[
w \ast y = (w \circ 0) \ast (y \circ 0) \quad \text{by (p8)}
\]
\[
\preceq (0 \circ (y \ast w)) \circ 0 \quad \text{by (iv)}
\]
\[
= 0 \circ (y \ast w) \quad \text{by (p8)}
\]
\[
\preceq w \ast y, \quad \text{by Proposition 3.4(ii)}
\]
and so \( 0 \circ (y \ast w) = w \ast y. \)

**Definition 3.8.** A pseudo-BCI algebra \( X \) is said to be \( \circ \)-medial if it satisfies the following identity:

(M1) \( (x \ast y) \circ (z \ast u) = (x \ast z) \circ (y \ast u), \forall x, y, z, u \in X. \)

**Proposition 3.9.** A pseudo-BCI algebra \( X \) is \( \circ \)-medial if and only if it satisfies:

(M2) \( x \circ (y \ast z) = (x \ast y) \circ (0 \ast z), \forall x, y, z \in X. \)

**Proof.** Assume that \( X \) is \( \circ \)-medial. Putting \( z = 0 \) and \( u = z \) in (M1) and using (p8), we have
\[
(x \ast y) \circ (0 \ast z) = (x \circ 0) \ast (y \ast z) = x \circ (y \ast z).
\]
Suppose that \( X \) satisfies the condition (M2). Then
\[
(x \ast y) \circ (z \ast u) = (x \circ (z \ast u)) \ast y \quad \text{by (p4)}
\]
\[
= ((x \ast z) \circ (0 \ast u)) \ast y \quad \text{by (M2)}
\]
\[
= ((x \ast z) \ast y) \circ (0 \ast u) \quad \text{by (p4)}
\]
\[
= (x \ast z) \circ (y \ast u). \quad \text{by (M2)}
\]
Therefore \( X \) is \( \circ \)-medial.
**Proposition 3.10.** Every \(\circ\)-medial pseudo-\(BCI\) algebra \(X\) satisfies the following identities.

(i) \(x \circ y = 0 \circ (y \circ x)\).
(ii) \(0 \circ (0 \circ x) = x\).
(iii) \(x \circ (x \circ y) = y\).

**Proof.** (i) For any \(x, y \in X\), we have
\[
x \circ y = (x \circ y) \circ 0 = (x \circ y) \circ (x \circ x) = (x \circ x) \circ (y \circ x) = 0 \circ (y \circ x).
\]

(ii) If we put \(y = 0\) in (i), then we have (ii).

(iii) Using (ii), (a3) and (p8), we get
\[
x \circ (x \circ y) = (x \circ 0) \circ (y \circ x) = (x \circ x) \circ (0 \circ y) = 0 \circ (0 \circ y) = y.
\]

4. Pseudo-\(BCI\) ideals

Let \(X\) be a pseudo-\(BCI\) -algebra. For any nonempty subset \(J\) of \(X\) and any element \(y\) of \(X\), we denote
\[
*(y, J) := \{x \in X \mid x \circ y \in J\} \quad \text{and} \quad \circ (y, J) := \{x \in X \mid x \circ y \in J\}.
\]

Note that \(*(y, J) \cap \circ (y, J) = \{x \in X \mid x \circ y \in J, x \circ y \in J\}\).

**Definition 4.1.** A nonempty subset \(J\) of a pseudo-\(BCI\) algebra \(X\) is called a pseudo-\(BCI\) ideal of \(X\) if it satisfies

(I1) \(0 \in J\),

(I2) \(\forall y \in J, *(y, J) \subseteq J \quad \text{and} \quad \circ (y, J) \subseteq J\).

Note that if \(X\) is a pseudo-\(BCI\) algebra satisfying \(x \circ y = x \circ y\) for all \(x, y \in X\), then the notion of a pseudo-\(BCI\) ideal and a \(BCI\)-ideal coincide.

**Proposition 4.2.** Let \(J\) be a pseudo-\(BCI\) ideal of a pseudo-\(BCI\) algebra \(X\). If \(x \in J\) and \(y \preceq x\), then \(y \in J\).

**Proof** is straightforward. \(\blacksquare\)

**Theorem 4.3.** For any element \(a\) of a pseudo-\(BCI\) algebra \(X\), the initial section \(\downarrow a := \{x \in X \mid x \preceq a\}\) is a pseudo-\(BCI\) ideal of \(X\) if and only if the following implications hold:

(i) \(\forall x, y, z \in X, x \circ y \preceq z, y \preceq z \Rightarrow x \preceq z\),

(ii) \(\forall x, y, z \in X, x \circ y \preceq z, y \preceq z \Rightarrow x \preceq z\).

**Proof.** Assume that for each \(a \in X\), \(\downarrow a\) is a pseudo-\(BCI\) ideal of \(X\). Let \(x, y, z \in X\) be such that \(x \circ y \preceq z, x \circ y \preceq z, \text{and} \ y \preceq z\). Then \(x \circ y \in \downarrow z, x \circ y \in \downarrow z, \text{and} \ y \in \downarrow z\), that is, \(y \in \downarrow z, x \in *(y, \downarrow z)\) and \(x \in \circ (y, \downarrow z)\). Since \(\downarrow z\) is a pseudo-\(BCI\) ideal of \(X\), it follows from (I2) that \(x \in \downarrow z\), i.e., \(x \preceq z\). Conversely, consider \(\downarrow z\) for any \(z \in X\). Obviously \(0 \in \downarrow z\). For every \(y \in \downarrow z\), let \(a \in *(y, \downarrow z)\) and \(b \in \circ (y, \downarrow z)\).
Then \( a \ast y \in \downarrow z \) and \( b \circ y \in \downarrow z \), i.e., \( a \ast y \leq z \) and \( b \circ y \leq z \). Since \( y \in \downarrow z \), it follows from the hypothesis that \( a \leq z \) and \( b \leq z \), i.e., \( a \in \downarrow z \) and \( b \in \downarrow z \). This shows that \( *(y, \downarrow z) \subseteq \downarrow z \) and \( \circ(y, \downarrow z) \subseteq \downarrow z \). Hence \( \downarrow z \) is a pseudo-\( BCI \) ideal of \( \mathfrak{X} \) for every \( z \in \mathcal{X} \).

**Theorem 4.4.** If \( J \) is a pseudo-\( BCI \) ideal of a pseudo-\( BCI \) algebra \( \mathfrak{X} \), then

(i) \( \forall x, y, z \in X, \ x, y, z \in J, \ x \ast y \leq z \Rightarrow z \in J \),
(ii) \( \forall a, b, c \in X, \ a, b \in J, \ c \circ b \leq a \Rightarrow c \in J \).

**Proof.** Suppose that \( J \) is a pseudo-ideal of \( \mathfrak{X} \) and let \( x, y, z \in X \) be such that \( x, y \in J \) and \( z \leq y \). Then \( (z \ast y) \circ x = 0 \in J \), and so \( z \ast y \in \circ(x, J) \subseteq J \). It follows that \( z \in *(y, J) \subseteq J \) so that (i) is valid. Now let \( a, b, c \in X \) be such that \( a, b \in J \) and \( c \circ b \leq a \). Then \( (c \circ b) \ast a = 0 \in J \), and thus \( c \circ b \in *(a, J) \subseteq J \). Hence \( c \in \circ(b, J) \subseteq J \), which shows (ii).

A pseudo-\( BCI \) subalgebra of a pseudo-\( BCI \) algebra \( \mathfrak{X} \) is a subset \( S \) of \( \mathfrak{X} \) which satisfies \( x \ast y \in S \) and \( x \circ y \in S \) for all \( x, y \in S \). We provide conditions for a pseudo-\( BCI \) subalgebra to be a pseudo-\( BCI \) ideal.

**Theorem 4.5.** Let \( J \) be a pseudo-\( BCI \) subalgebra of a pseudo-\( BCI \) algebra \( \mathfrak{X} \). Then \( J \) is a pseudo-\( BCI \) ideal of \( \mathfrak{X} \) if and only if

\[
\forall x, y \in X, \ x \in J, \ y \in X - J \Rightarrow y \ast x \in X - J \quad \text{and} \quad y \circ x \in X - J.
\]

**Proof.** Assume that \( J \) is a pseudo-\( BCI \) ideal of \( \mathfrak{X} \) and let \( x, y \in X \) be such that \( x \in J \) and \( y \in X - J \). If \( y \ast x \notin X - J \), then \( y \ast x \in J \), i.e., \( y \in *(x, J) \subseteq J \) which is a contradiction. Hence \( y \ast x \in X - J \). Now if \( y \circ x \notin X - J \), then \( y \circ x \in J \) and so \( y \in \circ(x, J) \subseteq J \). This is a contradiction, and therefore \( y \circ x \in X - J \). Conversely, assume that

\[
\forall x, y \in X, \ x \in J, \ y \in X - J \Rightarrow y \ast x \in X - J \quad \text{and} \quad y \circ x \in X - J.
\]

Since \( J \) is a pseudo-\( BCI \) subalgebra, therefore \( 0 \in J \). For every \( x \in J \), let \( y \in \ast(x, J) \). Then \( y \ast x \in J \). If \( y \notin J \), then \( y \ast x \in X - J \) by assumption. This is a contradiction, and so \( y \in J \) which shows that \( \ast(x, J) \subseteq J \). Now let \( z \in \circ(x, J) \). Then \( z \circ x \in J \). It follows from the hypothesis that \( z \in J \) so that \( \circ(x, J) \subseteq J \). Consequently, \( J \) is a pseudo-\( BCI \) ideal of \( \mathfrak{X} \).

Using [2, Theorem 3.5], we know that every pseudo-\( BCI \) algebra \( \mathfrak{X} \) contains a maximal pseudo-\( BCK \) algebra \( K(\mathfrak{X}) := \{ x \in X \mid 0 \preceq x \} \).

**Proposition 4.6.** Let \( \mathfrak{X} \) be a pseudo-\( BCI \) algebra. If \( x \in K(\mathfrak{X}) \) and \( y \in X - K(\mathfrak{X}) \), then \( x \ast y \in X - K(\mathfrak{X}) \) and \( x \circ y \in X - K(\mathfrak{X}) \).

**Proof.** If \( x \ast y \in K(\mathfrak{X}) \), then \( x \circ (x \ast y) \in K(\mathfrak{X}) \) because \( K(\mathfrak{X}) \) is a pseudo-\( BCI \) subalgebra of \( \mathfrak{X} \). Hence \( 0 \preceq x \circ (x \ast y) \preceq y \), and so \( y \in K(\mathfrak{X}) \). This is a contradiction. Now if \( x \circ y \in K(\mathfrak{X}) \), then \( x \ast (x \circ y) \in K(\mathfrak{X}) \) and so \( 0 \preceq x \ast (x \circ y) \preceq y \) by (a2). Therefore \( y \in K(\mathfrak{X}) \), a contradiction.

**Theorem 4.7.** Let \( \mathfrak{X} \) be a pseudo-\( BCI \) algebra. Then the maximal pseudo-\( BCK \) algebra \( K(\mathfrak{X}) \) is a pseudo-\( BCI \) ideal of \( \mathfrak{X} \).
Proof. Let } \textit{a}, y \in X \textit{ be such that } x \in K(\mathfrak{X}) \textit{ and } y \in X - K(\mathfrak{X}) \textit{. Using (a1) and (p8), we have}
\[(y \circ x) \circ y = (y \circ x) \circ (y \circ 0) \preceq 0 \circ x = 0\]
and
\[(y \circ x) \circ y = (y \circ x) \circ (y \circ 0) \preceq 0 \circ x = 0\]
since } x \in K(\mathfrak{X}) \textit{. It follows from (p1)} that } (y \circ x) \circ y = 0 \textit{ and } (y \circ x) \circ y = 0 \textit{ so that } y \circ x \preceq y \textit{ and } y \circ x \preceq y \textit{. If } y \circ x \in K(\mathfrak{X}) \textit{, then } 0 \preceq y \circ x \preceq y \textit{, and so } y \in K(\mathfrak{X}) \textit{ which is a contradiction. Now if } y \circ x \in K(\mathfrak{X}) \textit{, then } 0 \preceq y \circ x \preceq y \textit{ which implies that } y \in K(\mathfrak{X}) \textit{, a contradiction. Hence } y \circ x \in X - K(\mathfrak{X}) \textit{ and } y \circ x \in X - K(\mathfrak{X}) \textit{. By means of Theorem 4.5, we know that } K(\mathfrak{X}) \textit{ is a pseudo-BCI ideal of } \mathfrak{X}. \]

**Theorem 4.8.** Let } J \textit{ be a pseudo-BCI ideal of a pseudo-BCI algebra } \mathfrak{X}. \textit{Then the following are equivalent.}

(i) } J \textit{ contains the maximal pseudo-BCK algebra } K(\mathfrak{X}).

(ii) } \forall x, y \in X, x \preceq y, x \in J \Rightarrow y \in J. \]

**Proof.** The sufficiency is straightforward. Assume that } K(\mathfrak{X}) \subseteq J \textit{. Let } x, y \in X \textit{ be such that } x \preceq y \textit{ and } x \in J \textit{. Then } x \circ y = 0 \textit{, and so}
\[0 = 0 \circ 0 = 0 \circ (x \circ y) = (x \circ x) \circ (x \circ y) \preceq y \circ x.\]
Thus } y \circ x \in K(\mathfrak{X}) \subseteq J \textit{, which implies that } y \in \ast(x, J) \subseteq J. \]

**Definition 4.9.** Let } \mathfrak{X} \textit{ and } \mathfrak{Y} \textit{ be pseudo-BCI algebras. A mapping } f : \mathfrak{X} \to \mathfrak{Y} \textit{ is called a pseudo-BCI homomorphism if } f(x \ast y) = f(x) \ast f(y) \textit{ and } f(x \circ y) = f(x) \circ f(y) \textit{ for all } x, y \in \mathfrak{X}. \]

Note that if } f : \mathfrak{X} \to \mathfrak{Y} \textit{ is a pseudo-BCI homomorphism, then } f(0_\mathfrak{X}) = 0_\mathfrak{Y} \textit{ where } 0_\mathfrak{X} \textit{ and } 0_\mathfrak{Y} \textit{ are zero elements of } \mathfrak{X} \textit{ and } \mathfrak{Y} \textit{, respectively.}

**Theorem 4.10.** Let } f : \mathfrak{X} \to \mathfrak{Y} \textit{ be a pseudo-BCI homomorphism of pseudo-BCI algebras } \mathfrak{X} \textit{ and } \mathfrak{Y}. \textit{(i) If } J \textit{ is a pseudo-BCI ideal of } \mathfrak{Y}, \textit{ then } f^{-1}(J) \textit{ is a pseudo-BCI ideal of } \mathfrak{X}. \textit{(ii) If } f \textit{ is surjective and } I \textit{ is a pseudo-BCI ideal of } \mathfrak{X}, \textit{ then } f(I) \textit{ is a pseudo-BCI ideal of } \mathfrak{Y}. \]

**Proof.** (i) Assume that } J \textit{ is a pseudo-BCI ideal of } \mathfrak{Y}. \textit{Obviously } 0_\mathfrak{X} \in f^{-1}(J) \textit{. For every } y \in f^{-1}(J), \textit{ let}
\[a \in \ast(y, f^{-1}(J)) \textit{ and } b \in \circ(y, f^{-1}(J)).\]
Then } a \ast y \in f^{-1}(J) \textit{ and } b \circ y \in f^{-1}(J). \textit{ It follows that } f(a \ast y) = f(a \ast y) \in J \textit{ and } f(b) \circ f(y) = f(b \circ y) \in J \textit{ so that } f(a) \in \ast(f(y), J) \subseteq J \textit{ and } f(b) \in \circ(f(y), J) \subseteq J \textit{ because } J \textit{ is a pseudo-BCI ideal of } \mathfrak{X} \textit{ and } f(y) \textit{ is in } J. \textit{ Hence } a \in f^{-1}(J) \textit{ and } b \in f^{-1}(J) \textit{, which shows that } f(y, f^{-1}(J)) \subseteq f^{-1}(J) \textit{ and } \circ(y, f^{-1}(J)) \subseteq f^{-1}(J) \textit{. Hence } f^{-1}(J) \textit{ is a pseudo-BCI ideal of } \mathfrak{X}. \]

(ii) Assume that } f \textit{ is surjective and let } I \textit{ be a pseudo-BCI ideal of } \mathfrak{X}. \textit{Obviously, } 0_\mathfrak{Y} \in f(I) \textit{. For every } y \in f(I), \textit{ let } a, b \in Y \textit{ be such that } a \in \ast(y, f(I)) \textit{ and } b \in \circ(y, f(I)). \textit{ Then } a \ast y \in f(I) \textit{ and } b \circ y \in f(I). \textit{ It follows that there exist
$x_\ast, x_\circ \in I$ such that $f(x_\ast) = a \ast y$ and $f(x_\circ) = b \circ y$. Since $y \in f(I)$, there exists $x_y \in I$ such that $f(x_y) = y$. Also since $f$ is surjective, there exist $x_a, x_b \in X$ such that $f(x_a) = a$ and $f(x_b) = b$. Hence

$$f(x_\ast \ast x_y) = f(x_\ast) \ast f(x_y) = a \ast y \in f(I)$$

and

$$f(x_\circ \circ x_y) = f(x_\circ) \circ f(x_y) = b \circ y \in f(I),$$

which imply that $x_\ast \ast x_y \in I$ and $x_\circ \circ x_y \in I$. Since $I$ is a pseudo-BCI ideal of $X$, we get $x_\ast \in (x_y, I) \subseteq I$ and $x_\circ \in \circ(x_y, I) \subseteq I$, and thus $a = f(x_\ast) \in f(I)$ and $b = f(x_\circ) \in f(I)$. This shows that $*$ and $\circ$ of $f(I)$ are $f$-homomorphisms of $f(I)$ with $f(I)$ as the pseudo-BCI ideal of $Y$. Therefore $f(I)$ is a pseudo-BCI ideal of $Y$. \qed

**Corollary 4.11.** Let $f: X \rightarrow Y$ be a pseudo-BCI homomorphism of pseudo-BCI algebras $X$ and $Y$. Then the kernel

$$\text{Ker}(f) := \{ x \in X \mid f(x) = 0_Y \}$$

of $f$ is a pseudo-BCI ideal of $X$.

**Proof** is straightforward. \qed

**REFERENCES**


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