SPACES RELATED TO $\gamma$-SETS

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Abstract. We characterize Ramsey theoretically two classes of spaces which are related to $\gamma$-sets.

1. Introduction

The notation and terminology are mainly as in \cite{2}. $X$ will denote an infinite Hausdorff topological space.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose members are families of subsets of an infinite set $X$. Then (see \cite{7}, \cite{4}):

$S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:
For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:
For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite (not necessarily non-empty) sets such that for each $n \in \mathbb{N}$, $B_n \subset A_n$ and $\bigcup_{n \in \mathbb{N}} B_n$ is an element of $\mathcal{B}$.

The symbol $G_1(\mathcal{A}, \mathcal{B})$ \cite{7} denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $\langle A_1, b_1; \cdots ; A_n, b_n; \cdots \rangle$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

If ONE does not have a winning strategy in the game $G_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $S_1(\mathcal{A}, \mathcal{B})$ is true, but the converse need not be always true. In many cases the game characterizes the corresponding selection principle.

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For positive integers \( n \) and \( m \) the symbol \( A \to (B)^n_m \) denotes the statement:

For each \( A \in A \) and for each function \( f : [A]^n \to \{1, \ldots, m\} \) there are a set \( B \subset A, B \in B \), and an \( i \in \{1, \ldots, m\} \) such that for each \( Y \in [B]^n \), \( f(Y) = i \).

Here \( [A]^n \) denotes the set of \( n \)-element subsets of \( A \). We call \( f \) a “coloring” and say that “\( B \) is homogeneous of color \( i \) for \( f \)”.

This symbol is called the ordinary partition symbol [7]. Several selection principles of the form \( S_1(A, B) \) have been characterized by the ordinary partition relation (see [7], [4], [6]).

An open cover \( U \) of a space \( X \) is an \( \omega \)-cover (resp. \( k \)-cover) if \( X \) does not belong to \( U \) and every finite (resp. compact) subset of \( X \) is contained in a member of \( U \). Because we deal with \( k \)-covers, we assume that spaces we consider are (infinite) non-compact. An open cover \( U \) of \( X \) is called a \( \gamma \)-cover \([3]\) if it is infinite and each \( x \in X \) belongs to all but finitely many elements of \( U \). Notice that it is equivalent to the assertion: Each finite subset of \( X \) belongs to all but finitely many members of \( U \). An open cover of a space \( X \) is called a \( \gamma_k \)-cover of \( X \) if each compact subset of \( X \) is contained in all but finitely many elements of \( U \) and \( X \) is not a member of the cover ([5]).

We suppose that all covers are countable. Recall that spaces in which every open \( k \)-cover contains a countable \( k \)-subcover are called \( k \)-Lindelöf.

For a topological space \( X \) we denote:

1. \( \Omega \) – the family of \( \omega \)-covers of \( X \);
2. \( \mathcal{K} \) – the family of \( k \)-covers of \( X \);
3. \( \Gamma \) – the family of \( \gamma \)-covers of \( X \);
4. \( \Gamma_k \) – the family of \( \gamma_k \)-covers of \( X \).

Let us observe that we have

\[ \Gamma_k \subset \Gamma \subset \Omega, \quad \Gamma_k \subset \mathcal{K} \subset \Omega. \]

In [3], Gerlits and Nagy introduced the following notion: a space \( X \) is a \( \gamma \)-space (or a \( \gamma \)-set) if each \( \omega \)-cover \( U \) of \( X \) contains a countable family \( \{U_n : n \in \mathbb{N}\} \) which is a \( \gamma \)-cover of \( X \). They have also proved that the \( \gamma \)-set property of a space \( X \) is equivalent to the statement that \( X \) satisfies the selection property \( S_1(\Omega, \Gamma) \). It was shown in [4] that the \( \gamma \)-set property is equivalent also to the selection hypothesis \( S_{fin}(\Omega, \Gamma) \).

In [7], it was proved:

**Theorem 1.** For a space \( X \) the following statements are equivalent:

(a) \( X \) is a \( \gamma \)-set;

(b) \( \text{ONE has no winning strategy in the game } G_1(\Omega, \Gamma) \text{ on } X \);

(c) For all \( n, m \in \mathbb{N} \), \( X \) satisfies \( \Omega \to (\Gamma)^n_m \).

We shall prove here that similar results are true for two recently introduced classes of spaces which are similar to \( \gamma \)-sets. In fact, we give Ramsey theoretical characterizations of those classes of spaces.
2. $k$-$\gamma$-sets

In [1], the class of $k$-$\gamma$-sets was introduced as the class of $S_1(\mathcal{K}, \Gamma)$-sets and the following result regarding that class of spaces was shown.

**Theorem 2.** For a space $X$ the following are equivalent:
1. $X$ is a $k$-$\gamma$-set;
2. $X$ satisfies $S_{fin}(\mathcal{K}, \Gamma)$;
3. $\text{ONE}$ has no winning strategy in the game $G_1(\mathcal{K}, \Gamma)$ on $X$.

We show here that this class of spaces also can be described Ramsey-theoretically.

**Theorem 3.** For a $k$-Lindel"of space $X$ the following are equivalent:
1. $X$ is a $k$-$\gamma$-set;
2. For positive integers $n$ and $m$, $X$ satisfies $\mathcal{K} \rightarrow (\Gamma)^n_m$.

**Proof.** We consider the case $n = m = 2$, because the general case can be easily obtained from it by standard induction arguments.

$1) \Rightarrow 2$: Let $\mathcal{U} = \{U_1, U_2, \cdots \}$ be a countable $k$-cover of $X$ and let $f : [\mathcal{U}]^2 \rightarrow \{1, 2\}$ be a coloring. For $j \in \{1, 2\}$ let $\mathcal{H}_j = \{V \in \mathcal{U} : f(\{U_1, V\}) = j\}$. Then at least one of the sets $\mathcal{H}_1$ and $\mathcal{H}_2$ is a $k$-cover of $X$. Denote such a set by $\mathcal{U}_1$ and the corresponding $j$ by $i_1$. In a similar way define inductively sets $\mathcal{U}_n$ of $k$-covers of $X$ and elements $i_n$ from $\{1, 2\}$ such that

$$\mathcal{U}_n = \{V \in \mathcal{U}_{n-1} : f(\{U_n, V\}) = i_n\}.$$  

Apply now the $S_1(\mathcal{K}, \Gamma)$ property of $X$ to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ to choose for each $n \in \mathbb{N}$ a $V_n \in \mathcal{U}_n$ such that $\mathcal{V} = \{V_n : n \in \mathbb{N}\} \in \Gamma$. Consider now the sets $\mathcal{V}_1 := \{V_m \in \mathcal{V} : i_m = 1\}$ and $\mathcal{V}_2 := \{V_m \in \mathcal{V} : i_m = 2\}$. At least one of them is infinite and so is a $\gamma$-cover of $X$, as each infinite subset of a $\gamma$-cover is also a $\gamma$-cover. So, one may suppose that there is an $i \in \{1, 2\}$ satisfying: for each $U_m \in \mathcal{V}$, $i_m = i$. We have that $f(\{A, B\}) = i$ for each $\{A, B\} \in [\mathcal{V}]^2$.

$2) \Rightarrow 1$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of countable $k$-covers of $X$ and let us suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Define now $\mathcal{V}$ to be the family of all nonempty sets of the form $U_{1,n} \cap U_{n,m}$, $n, m \in \mathbb{N}$. Clearly, $\mathcal{V}$ is a $k$-cover of $X$. Let $f : [\mathcal{V}]^2 \rightarrow \{1, 2\}$ be define by

$$f(U_{1,n_1} \cap U_{n_1,m_1}, U_{1,n_2} \cap U_{n_2,m_2}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}$$

Apply $\mathcal{K} \rightarrow (\Gamma)^2_2$ to find a $\gamma$-cover $\mathcal{W} \subset \mathcal{V}$ and an $i \in \{1, 2\}$ such that whenever $U$ and $V$ are from $\mathcal{W}$, then $f(\{U, V\}) = i$. Consider two possibilities:

(i) $i = 1$: Then there is some $n \in \mathbb{N}$ such that for each $W \in \mathcal{W}$ we have $W \subset U_{1,n}$. However, this implies that $\mathcal{W}$ is not a $\gamma$-cover of $X$ and this contradiction shows that this case is impossible.
(ii) $i = 2$: Whenever $W \in \mathcal{W}$ is of the form $U_{1,n} \cap U_{n,m}$ choose (one element) $H_n = U_{n,m} \in \mathcal{U}_n$. Otherwise, let $H_n = \emptyset$. Then the set $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ is a $\gamma$-cover of $X$ (because $\mathcal{W}$ refines $\mathcal{H}$), and the sequence $(H_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $X$ satisfies $S_{\text{fin}}(\mathcal{K}, \Gamma)$. By Theorem 2 it is equivalent to $S_1(\mathcal{K}, \Gamma)$, i.e. (1) holds.

3. $\gamma_k'$-sets

The following class of spaces was introduced in [5]. A space $X$ is said to be a $\gamma_k'$-set if it satisfies the selection hypothesis $S_1(\mathcal{K}, \Gamma_k)$.

A characterization of $\gamma_k'$-sets from [5] is given in the next theorem.

**Theorem 4.** For a space $X$ the following are equivalent:

1. $X$ is a $\gamma_k'$-set;
2. $X$ satisfies $S_{\text{fin}}(\mathcal{K}, \Gamma_k)$;
3. ONE does not have a winning strategy in the game $G_1(\mathcal{K}, \Gamma_k)$ played on $X$.

We give now a Ramsey-theoretical characterization of $\gamma_k'$-sets.

**Theorem 5.** For a $k$-Lindelöf space $X$ the following are equivalent:

1. $X$ is a $\gamma_k'$-set;
2. For all $n, m \in \mathbb{N}$, $X$ satisfies $\mathcal{K} \rightarrow (\Gamma_k)_m^n$.

**Proof.** We again consider only the case $n = m = 2$.

$(1) \Rightarrow (2)$: We shall use that $(1)$ is equivalent to the fact that ONE has no winning strategy in the game $G_1(\mathcal{K}, \Gamma_k)$ played on $X$ (Theorem 4).

Suppose $\mathcal{U} = \{U_1, U_2, \cdots\}$ is a $k$-cover of $X$ and let $f : [\mathcal{U}]^2 \rightarrow \{1, 2\}$ be a coloring. Let us define a strategy $\sigma$ for ONE in the game $G_1(\mathcal{K}, \Gamma_k)$.

In the first round ONE plays $\sigma(\emptyset) = \mathcal{U}$. Then choose $i_n \in \{1, 2\}$, $n \in \mathbb{N}$, such that $\sigma(U_n) = \{V \in \mathcal{U} : \sigma(\{U_n, V\}) = i_n\}$ is a $k$-cover of $X$ (see the proof of Theorem 3). Let us write $\sigma(U_n) = \{U_{n,m} : m \in \mathbb{N}\}$. Suppose for each finite sequence $(n_1, \ldots, n_p)$ of natural numbers we have defined sets $U_{n_1, \ldots, n_p}$ and $i_{n_1, \ldots, n_{p-1}} \in \{1, 2\}$ satisfying the condition $\{U_{n, \ldots, n_p, m} : m \in \mathbb{N}\}$ is a $k$-cover of $X$ which is equal to the set

$$\{V \in \sigma(U_{n_1, U_{n_2, \ldots, U_{n_p, \ldots, n_1}}) : f(\{U_{n_1, n_2, \ldots, n_p, V\}) = i_{n_1, n_2, \ldots, n_p}\}.$$

In this way one defines a strategy $\sigma$ for ONE in $G_1(\mathcal{K}, \Gamma_k)$. As ONE has no winning strategy, there is a play (for TWO)

$$U_{n_1, U_{n_2, \ldots, U_{n_1, n_2, \ldots, n_m, \ldots}}},$$

which defeats this strategy. The set $\{U_{n_1, \ldots, U_{n_1, n_2, \ldots, n_m, \ldots}\}$ is a $\gamma_k$-cover of $X$. Besides, if $p < q$, then

$$f(\{U_{n_1, n_2, \ldots, n_p, U_{n_1, n_2, \ldots, n_q}\}) = i_{n_1, n_2, \ldots, n_p},$$
We may choose \( i \in \{1, 2\} \) such that for infinitely many \( m \) we have \( i_{n_1, n_2, \ldots, n_m} = i \).

Then define

\[
\mathcal{V} = \{ U_{n_1, n_2, \ldots, n_m} : i_{n_1, n_2, \ldots, n_m} = i \} \subset \mathcal{U}.
\]

This set is a \( \gamma_k \)-cover of \( X \) (because an infinite subset of a \( \gamma_k \)-cover is also a \( \gamma_k \)-cover) and, by construction, is homogeneous for \( f \) of color \( i \).

\[2 \Rightarrow 1\): Let \((\mathcal{U}_n : n \in \mathbb{N})\) be a sequence of countable \( k \)-covers of \( X \) and suppose that for each \( n \), \( \mathcal{U}_n = \{ U_{n;m} : m \in \mathbb{N} \} \). Consider now the set \( \mathcal{V} \) of all nonempty sets of the form \( U_{1;m} \cap U_{m;k}, n, k \in \mathbb{N} \). Clearly, \( \mathcal{V} \) is a \( k \)-cover of \( X \).

Define \( f : [\mathcal{V}]^2 \to \{1, 2\} \) by

\[
f(U_{1;n_1} \cap U_{n_1;k}, U_{1;n_2} \cap U_{n_2;m}) = \begin{cases} 1, & \text{if } n_1 = n_2, \\ 2, & \text{otherwise.} \end{cases}
\]

Since \( K \to (\Gamma_k)^2 \) holds there are \( j \in \{1, 2\} \) and a homogeneous for \( f \) of color \( j \) collection \( \mathcal{W} \subset \mathcal{V} \) such that \( \mathcal{W} \in \Gamma_k \).

Consider two possibilities:

(i) \( j = 1 \): Then there is some \( n \) such that for each \( W \in \mathcal{W} \) we have \( W \subset U_{1,n} \).

However, this means that \( W \) is not a \( \gamma_k \)-cover of \( X \) and we have a contradiction which shows that this case is impossible.

(ii) \( j = 2 \): For each \( W \in \mathcal{W} \) choose, when it is possible, \( U_{n,k_n} \) to be the second term in the chosen representation of \( W \); otherwise let \( U_{n,k_n} = \emptyset \). Let \( \mathcal{V}' \) be the set of all \( U_{n,k_n} \)'s chosen in this way. Then \( \mathcal{V}' \) is a \( \gamma_k \)-cover of \( X \) witnessing for \((\mathcal{U}_n : n \in \mathbb{N})\) that \( X \) satisfies \( S_{\text{fin}}(K, \Gamma_k) \). Apply now Theorem 4

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