METRIZABLE GROUPS AND STRICT o-BOUNDEDNESS

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Abstract. We show that for metrizable topological groups being a strictly o-bounded group is equivalent to being a Hurewicz group. In [5] Hernandez, Robbie and Tkachenko ask if there are strictly o-bounded groups G and H for which G×H is not strictly o-bounded. We show that for metrizable strictly o-bounded groups the answer is no. In the same paper the authors also ask if the product of an o-bounded group with a strictly o-bounded group is again an o-bounded group. We show that if the strictly o-bounded group is metrizable, then the answer is yes.

1. Definitions and notation

Let H and G be topological spaces with G a subspace of H. We shall use the notations:

- \( \mathcal{O}_H \): The collection of open covers of H.
- \( \mathcal{O}_{HG} \): The collection of covers of G by sets open in H.

An open cover \( \mathcal{U} \) of a topological space H is said to be

- an \( \omega \)-cover if it is infinite, and each infinite subset of it is still an open cover of the space. The symbol \( \omega \) denotes the collection of \( \omega \)-covers of H.
- groupable if there is a partition \( \mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n \), where each \( \mathcal{U}_n \) is finite, and for each \( x \in H \) the set \( \{ n : x \notin \bigcup \mathcal{U}_n \} \) is finite [9]. The symbol \( \mathcal{O}_{gp} \) denotes the collection of groupable open covers of the space.
- \( \gamma \)-cover if it is infinite, and each infinite subset of it is still an open cover of the space. The symbol \( \gamma \) denotes the collection of \( \gamma \)-covers of the space.
- large if each element of the space is contained in infinitely many elements of the cover. The symbol \( \Lambda \) denotes the collection of large covers of the space.

Now let \((G, *)\) be a topological group with identity element \( e \). We will assume that \( G \) is not compact. For a neighborhood \( U \) of \( e \), and for a finite subset \( F \) of \( G \) the set \( F \ast U \) is a neighborhood of the finite set \( F \). Thus, the set \( \{ F \ast U : F \subset G \text{ finite} \} \) is an \( \omega \)-cover of \( G \), which is denoted by the symbol \( \Omega(U) \). The set

\[
\Omega_{nbd} = \{ \Omega(U) : U \text{ a neighborhood of } e \}
\]

is the set of all such \( \omega \)-covers of \( G \).
The set \( O(U) = \{ x \ast U : x \in G \} \) is an open cover of \( G \). The symbol
\[
O_{\text{nbhd}} = \{ O(U) : U \text{ a neighborhood of } e \}
\]
denotes the collection of all such open covers of \( G \). Now we describe the selection principles relevant to this topic. Let \( S \) be an infinite set, and let \( A \) and \( B \) be collections of families of subsets of \( S \).

The symbol \( S_1(A, B) \) denotes the statement that there is for each sequence \((O_n : n \in \mathbb{N})\) of elements of \( A \) a sequence \((T_n : n \in \mathbb{N})\) such that for each \( n \) \( T_n \in O_n \), and \( \{ T_n : n \in \mathbb{N} \} \in B \). The earliest example of this sort of selection principle was introduced in [14] by Rothberger, and is \( S_1(\mathcal{O}, \mathcal{O}) \) in our notation.

The symbol \( S_{\text{fin}}(A, B) \) is defined as follows: For each sequence \((O_n : n \in \mathbb{N})\) from \( A \) there is a sequence \((T_n : n \in \mathbb{N})\) of finite sets such that for each \( n \) \( T_n \subset O_n \), and \( \bigcup_{n \in \mathbb{N}} T_n \in B \). The earliest example of this selection principle was introduced by Hurewicz in [7], and in our notation is \( S_{\text{fin}}(\mathcal{O}, \mathcal{O}) \). Because of its equivalence in metric spaces with a basis property of Menger, \( S_{\text{fin}}(\mathcal{O}, \mathcal{O}) \) is also called the Menger property. In [7] Hurewicz introduced also a second selection principle, now called the Hurewicz property, and characterized in [9] as the statement \( S_{\text{fin}}(\Lambda, \mathcal{O}^{\text{op}}) \).

Each selection principle has a corresponding game. The game that we will use in this paper is the game corresponding to \( S_1(A, B) \). The game is denoted \( G_1(A, B) \) and is played as follows: In the \( n \)-th inning ONE chooses an \( O_n \in A \), and TWO responds with a \( T_n \in O_n \). They play an inning for each natural number \( n \). A play
\[
O_1, T_1, \ldots, O_n, T_n, \ldots
\]
is won by TWO if \( \{ T_n : n \in \mathbb{N} \} \) is in \( B \). Otherwise, ONE wins.

### 2. Hurewicz-bounded groups

The notion of Hurewicz-bounded group was introduced by Kočinac in 1998 in one of his unpublished notes. A topological group \((G, \ast)\) is said to be a Hurewicz-bounded group if it satisfies the selection principle \( S_1(\Omega_{\text{nbhd}}(G), \mathcal{O}_G^{\text{op}}) \). Let \((G, \ast)\) be a subgroup of the group \((H, \ast)\). Then \( G \) is Hurewicz-bounded \( H \) if the selection principle \( S_1(\Omega_{\text{nbhd}}(H), \mathcal{O}_H^{\text{op}}) \) holds.

**Theorem 1.** For a subgroup \((G, \ast)\) of an infinite topological group \((H, \ast)\) the following are equivalent:

1. \( S_1(\Omega_{\text{nbhd}}(H), \mathcal{O}_H^{\text{op}}) \).
2. \( S_1(\Omega_{\text{nbhd}}(H), \Gamma_H) \).

**Proof.** We need to prove only that 1 \( \Rightarrow \) 2. For each \( n \) let \( U_n \) be an element of \( \Omega_{\text{nbhd}}(H) \), and choose for each \( n \) an open neighborhood \( U_n \) of \( e \) such that \( U_n = \Omega(U_n) \). For each \( n \) put \( V_n = \bigcap_{j \leq n} U_j \). For each \( n \) put \( V_n = \Omega(V_n) \). Apply \( S_1(\Omega_{\text{nbhd}}, \mathcal{O}^{\text{op}}) \) to \( \{ V_n : n \in \mathbb{N} \} \). For each \( n \) choose \( W_n \in V_n \) such that \( \{ W_n : n \in \mathbb{N} \} \) is a groupable open cover of \( G \). Choose a sequence \( m_1 < m_2 < \cdots < m_n < \cdots \) such that for each \( x \in G \), for all but finitely many \( n \), \( x \in \bigcup_{m_n \leq j < m_{n+1}} W_j \). For
each $n$ also choose a finite set $F_n \subset H$ with $W_n = F_n \ast V_n$. Now define, for each $k$, the finite set $G_k$ by

$$G_k = \begin{cases} 
\bigcup_{i \leq m_1} F_i, & \text{if } k \leq m_1 \\
\bigcup_{m_n < i \leq m_{n+1}} F_i, & \text{if } m_n < k \leq m_{n+1}
\end{cases}$$

For each $n$ put $A_n = G_n \ast U_n$, an element of $\Omega(U_n)$. We claim that $\{A_n : n \in \mathbb{N}\}$ is a $\gamma$-cover of $G$. For consider $g \in G$. Choose $M \in \mathbb{N}$ so that for all $n \geq M$ we have $g \in \bigcup_{m_n < i \leq m_{n+1}} W_i$. But for $m_n < i \leq m_{n+1}$ we have $W_i = F_i \ast V_i \subset A_k = G_k \ast U_k$ for $m_n < k \leq m_{n+1}$. Thus, for all $k > m_M$ we have $g \in A_k$. It follows that $\{A_k : k \in \mathbb{N}\}$ is a $\gamma$-cover of $G$. ■

**Theorem 2.** Let $(G, \ast)$ be a subgroup of the topological group $(H, \ast)$. Then the following are equivalent:

1. $S_1(\Omega_{nbd}(H), C^{gp}_{HG})$.
2. $S_1(\Omega_{nbd}(G), C^{gp}_{G})$.

**Proof.** The implication $(2) \Rightarrow (1)$ is evidently true. We must show that $(1) \Rightarrow (2)$. Thus, let $(\Omega(U_n) : n \in \mathbb{N})$ be a sequence in $\Omega_{nbd}(G)$. Since each $U_n$ is a neighborhood (in $G$) of the group identity we may choose for each $n$ a neighborhood $T_n$ in $H$ of the identity, such that $U_n = T_n \cap G$. Next, choose for each $n$ a neighborhood $S_n$ in $H$ of the identity, such that $S_n^{-1} S_n \subseteq T_n$.

Apply $(2)$ to the sequence $(\Omega(S_n) : n \in \mathbb{N})$ we get for each $n$ a finite set $F_n \subset H$ such that for each element $x$ of $G$ there is an $N$ such that for all $n \geq N$ we have $x \in F_n \ast S_n$. For each $n$, and for each $f \in F_n$, choose a $g_f \in G$ as follows:

$$g_f \begin{cases} 
\in G \cap f \ast S_n, & \text{if nonempty} \\
= e, & \text{the group identity, otherwise}
\end{cases}$$

Then put $G_n = \{g_f : f \in F_n\}$, a finite subset of $G$. For each $n$ we have $G_n \ast U_n \in \Omega(U_n) \in \Omega_{nbd}(G)$. We must show that for each $x \in G$ there is an $N$ such that for each $n \geq N$, $x \in G_n \ast U_n$.

Let $x \in G$ be given. Choose $N$ so that for all $n \geq N$ we have $x \in F_n \ast S_n$. Fix $n \geq N$ and choose $f_x \in F_n$ so that $x \in f_x \ast S_n$. Then evidently $G \cap f_x \ast S_n$ is nonempty, and so $g_{f_x} \in G$ is defined as an element of this intersection. Since $g_{f_x} \in f_x \ast S_n$, we have $f_x \in g_{f_x} \ast S_n^{-1}$, and so $x \in g_{f_x} \ast S_n^{-1} S_n \subseteq g_{f_x} \ast T_n$. Now $g_{f_x}^{-1} \ast x \in G \cap T_n = U_n$, and so we have $x \in g_{f_x} \ast U_n \subset G_n \ast U_n$. This completes the proof. ■

This result implies the following:

**Corollary 3.** If $(H, \ast)$ has property $S_1(\Omega_{nbd}(H), \Gamma_H)$, then for each infinite subgroup $G$ of $H$, $S_1(\Omega_{nbd}(G), \Gamma_G)$ holds.
3. A characterization of strict $\sigma$-boundedness for metrizable groups

According to Hausdorff [4, 25] a metric space $(X, d)$ is totally bounded if there is for each $\delta > 0$ a partition of $X$ into finitely many sets, each of diameter less than $\delta$. The metric space is $\sigma$-totally bounded if it is a union of countably many sets, each totally bounded.

Measure-like properties of metrizable spaces can be equivalent to selection properties. Similarly, for metrizable groups measure-like properties with respect to left-invariant metrics can be equivalent to selection properties. Let $A$ be a collection of sets, each a set of subsets of $H$. As in [1] we say: Metric space $(H, d)$ has $A$-measure zero if for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers, there is a sequence $(F_n : n \in \mathbb{N})$ where:

1. For each $n$, $F_n$ is a finite set of subsets of $H$,
2. Each element of $F_n$ has diameter less than $\epsilon_n$, and
3. $\bigcup_{n \in \mathbb{N}} F_n \in A$.

The following theorem of Kakutani will be used below:

**Theorem 4.** [6] Let $(U_k : k < \infty)$ be a sequence of subsets of the topological group $(H, \ast)$ where $\{U_k : k < \infty\}$ is a neighborhood basis of the identity element $e$ and each $U_k$ is symmetric$^1$, and for each $k$ also $U_{k+1}^2 \subseteq U_k$. Then there is a left-invariant metric $d$ on $H$ such that

1. $d$ is uniformly continuous in the left uniform structure on $H \times H$.
2. If $y^{-1} \ast x \in U_k$ then $d(x, y) \leq (\frac{1}{2})^{k-2}$.
3. If $d(x, y) < (\frac{1}{2})^k$ then $y^{-1} \ast x \in U_k$.

The equivalence of the first two statements of the following theorem apparently has been independently obtained also by H. Michalewski [13].

**Theorem 5.** For a subgroup $(G, \ast)$ of a metrizable group $(H, \ast)$ the following are equivalent:

1. TWO has a winning strategy in the game $G_1(\Omega_{\text{nbd}}(H), \mathcal{O}_{HG})$.
2. $(G, \ast)$ is $\sigma$-totally bounded in all left-invariant metrics generating the topology of $H$.
3. $S_1(\Omega_{\text{nbd}}(H), \Gamma_{HG})$ holds.
4. $H$ has the $\mathcal{O}_{HG}$-measure zero property in all left invariant metrics on $H$.

**Proof.** 1 $\Rightarrow$ 2: Since $(H, \ast)$ is a metrizable group, it is first countable. Let $d$ be a left-invariant metric of $H$ and let $(U_n : n \in \mathbb{N})$ be a neighborhood basis of the identity element $e$ of $H$ such that for each $n$, $U_n \supset U_{n+1}$ and $\text{diam}_d(U_n) < \frac{1}{2^n}$. Let $\sigma$ be TWO’s winning strategy. Define:

$$G_\emptyset = \bigcap_{n \in \mathbb{N}} \sigma(\Omega(U_n)).$$

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$^1$ $U_k$ is symmetric if $U_k = U_k^{-1}$.
Then, for \( n_1, \ldots, n_k < \infty \) given, define
\[
G_{n_1, \ldots, n_k} = \bigcap_{n \in \mathbb{N}} \sigma(\Omega(U_{n_1}), \ldots, \Omega(U_{n_k}), \Omega(U_n)).
\]

First, we show that
\[
G \subseteq \bigcup_{\tau \in \omega^\omega} G_{\tau}.
\] (1)

For suppose on the contrary that (1) is false. Choose an \( x \in G \setminus \bigcup_{\tau \in \omega^\omega} G_{\tau} \). Since \( x \) is not in \( G_0 \), choose \( n_1 \) so that \( x \notin \sigma(\Omega(U_{n_1})) \). Since \( x \) is not in \( G_{n_1} \), choose \( n_2 \) so that \( x \notin \sigma(\Omega(U_{n_1}), \Omega(U_{n_2})) \), and so on. In this way we obtain a \( \sigma \)-play
\[
\Omega(U_{n_1}), \sigma(\Omega(U_{n_1})), \Omega(U_{n_2}), \sigma(\Omega(U_{n_1}), \Omega(U_{n_2})), \ldots
\]
which is lost by TWO, since \( x \) is never covered by TWO. This contradicts the fact that \( \sigma \) is a winning strategy for TWO.

Next, we observe that each \( G_{\tau} \) is totally bounded. Fix \( \tau = (n_1, \ldots, n_k) \). Let an \( \epsilon > 0 \) be given. Choose \( n \) so large that \( \frac{1}{2^n} < \epsilon \). Now \( G_{\tau} \) is a subset of \( \sigma(\Omega(U_{n_1}), \ldots, \Omega(U_k), \Omega(U_n)) \), which is of the form \( F \star U_n \) for some finite subset \( F \) of \( G \). But then \( \{ f \star U_n : f \in F \} \) is a finite family of sets, each of diameter less than \( \frac{1}{2^n} < \epsilon \), and covers \( G_{\tau} \).

This completes the proof that \( G \) is \( \sigma \)-totally bounded.

2 \( \Rightarrow \) 1: Fix a left-invariant metric \( d \) of \( H \) and assume that \( G \) is \( \sigma \)-totally bounded. Write \( G = \bigcup_{n \in \mathbb{N}} G_n \), where each \( G_n \) is totally bounded. We define a winning strategy \( \sigma \) for TWO as follows:

When player ONE plays \( \Omega(U_1) \), put \( \epsilon_1 = \text{diam}(U_1) \), and \( \delta_1 = \frac{\epsilon_1}{2} \). Since \( G_1 \) is totally bounded, choose a finite family \( F \) of open sets, each of diameter less than \( \delta_1 \), so that \( G_1 \subseteq \bigcup F \). For each \( F \in F \), choose a point \( x_F \in F \). Then \( F \subseteq x_F \star U_1 \) (by diameter considerations), and so, setting \( S_1 = \{ x_F : F \in F \} \), TWO responds with \( \sigma(\Omega(U_1)) = S_1 \star U_1 \in \Omega(U_1) \).

Suppose it is the \( n \)-th inning, and ONE has played \( \Omega(U_1), \ldots, \Omega(U_n) \) so far. Put \( \epsilon_n = \min \{ \text{diam}(U_j) : j \leq n \} \), and put \( \delta_n = \frac{\epsilon_n}{2^n} \). Since \( G_1 \cup \cdots \cup G_n \) is totally bounded, choose a finite set \( F \) of open subsets of \( G \), each of diameter less than \( \delta_n \), such that \( G_1 \cup \cdots \cup G_n \subseteq \bigcup F \). For each \( F \in F \) choose an \( x_F \in F \) and put \( S_n = \{ x_F : F \in F \} \). Then by diameter considerations, \( \bigcup F \subseteq S_n \star U_n \). Now TWO plays \( \sigma(\Omega(U_1), \ldots, \Omega(U_n)) = S_n \star U_n \in \Omega(U_n) \).

It is evident that \( \sigma \) is a winning strategy for TWO.

2 \( \Rightarrow \) 3: The winning strategy \( \sigma \) described for TWO above has the effect of choosing for a sequence \( \{ \Omega(U_n) : n \in \mathbb{N} \} \) a corresponding sequence \( \{ T_n : n \in \mathbb{N} \} \) such that for each \( n \) we have \( T_n \in \Omega(U_n) \), and \( G_1 \cup \cdots \cup G_n \subseteq T_n \). This witnesses \( S_1(\text{ndb}(H), \Gamma_{HG}) \).

3 \( \Rightarrow \) 2: Since \( H \) is metrizable it is first countable and has a left-invariant metric. Choose a left invariant metric \( d \) for \( H \) and a sequence \( \{ U_n : n \in \mathbb{N} \} \) of neighborhoods of the identity \( e \), which forms a neighborhood basis at \( e \), and such
that $\lim_{n \to \infty} \text{diam}_d(U_n) = 0$. Apply $S_1(\Omega_{\text{nbhd}}(H), \Gamma_{HG})$ to the sequence $(\Omega(U_n) : n \in \mathbb{N})$: For each $n$ we find a $T_n = S_n \ast U_n \in \Omega(U_n)$ where $S_n$ is a finite subset of $G$, such that $\{T_n : n \in \mathbb{N}\}$ is a $\gamma$-cover of $G$. For each $n \in \mathbb{N}$ define $G_n = \cap_{k \geq n} T_k$. Then $G \subseteq \bigcup_{n \in \mathbb{N}} G_n$. Each $G_n$ is totally bounded: For let an $n$ be given and let $\epsilon > 0$ be given. Choose an $N > n$ so large that $\text{diam}(U_N) < \epsilon$. Then $T_N$ is a union of finitely many sets of the form $x \ast U_N$, and $G_n \subseteq T_N$.

$3 \Rightarrow 4$: Let $d$ be a left-invariant metric on $H$ and let $(\epsilon_n : n \in \mathbb{N})$ be a sequence of positive real numbers. Choose for each $n$ a neighborhood $U_n$ of the identity element $e$ of $H$ such that $\text{diam}_d(U_n) < \epsilon_n$. Then apply $S_1(\Omega_{\text{nbhd}}(H), \mathcal{O}^{\text{pp}}_{HG})$ to the sequence $(\Omega(U_n) : n \in \mathbb{N})$. For each $n$ we find $F_n$ of $H$ such that each element of $G$ is in all but finitely many of the $F_n \ast U_n$’s. Since $d$ is left-invariant, each element of $V_n = \{x \ast U_n : x \in F_n\}$ has diameter less than $\epsilon_n$.

$4 \Rightarrow 3$: Let $(U_n : n \in \mathbb{N})$ be a sequence of neighborhoods of $e$, the identity element of $H$. Choose for each $n$ a symmetric neighborhood $V_n$ such that $V_{n+1} \subseteq V_n \subseteq U_n$. By Kakutani’s theorem choose a left-invariant metric $d$ as in Theorem 5 corresponding to the sequence $(V_n : n \in \mathbb{N})$. For each $n$ put $\epsilon_n = \frac{1}{2^n}$. We may assume that $G$ is not totally bounded in $d$. Choose $\delta > 0$ witnessing this. Choose $N$ large enough such that $\epsilon_N \geq \delta$, and apply 4 to $(\epsilon_n : n \geq N)$. For each $n \geq N$ choose a finite set $\mathcal{H}_n$ of subsets of $H$ such that each set in $\mathcal{H}_n$ has $d$-diameter less than $\epsilon$. For each $S \in \mathcal{H}_n$ choose $T_s \in \mathcal{O}_H(V_n)$ with $S \subseteq T_s$. Choose $W_s \in \mathcal{O}_H(U_n)$ with $S \subseteq W_s$. For each $n$, put $\mathcal{G} = \{W_s : S \in \mathcal{H}_n\}$, and put $G_n = \bigcup \mathcal{G} \in \Omega(U_n)$. Then, $\{G_n : n \geq N\}$ is a $\gamma$ cover of $G$. ■

Problem 4.1 of [5] asks if there are strictly $o$-bounded groups $G$ and $H$ for which $G \times H$ is not strictly $o$-bounded. We show that for metrizable strictly $o$-bounded groups the answer is no. Problem 4.2 asks if the product of an $o$-bounded group with a strictly $o$-bounded group is again an $o$-bounded group. We show that if the strictly $o$-bounded group is metrizable, then the answer is yes. Both of these answers are results of the following theorem.

**Theorem 6.** Let $(G, \ast)$ be a group satisfying $S_1(\Omega_{\text{nbhd}}, \Gamma)$. Let $A$ be one of $\mathcal{O}$, $\Omega$, $\Gamma$. If $(H, \ast)$ is a group with property $S_1(\Omega_{\text{nbhd}}, A)$, then $(G \times H, \ast)$ also has this property.

**Proof.** Consider a sequence $(\Omega(U_n) : n \in \mathbb{N})$ in $\Omega_{\text{nbhd}}(G \times H)$. For each $n$ choose neighborhoods $V_n$ of $e_G$ and $W_n$ of $e_H$ such that $V_n \times W_n \subseteq U_n$. Consider the sequences $(\Omega(V_n) : n \in \mathbb{N})$ and $(\Omega(W_n) : n \in \mathbb{N})$. For each $n$ choose a finite set $G_n \subseteq G$ and a finite set $H_n \subseteq H$ such that $(G_n \ast V_n : n \in \mathbb{N})$ is in $\Gamma$, and $(H_n \ast W_n : n \in \mathbb{N})$ is in $A$. For each $n$ put $F_n = G_n \times H_n$. We claim that $(F_n \ast U_n : n \in \mathbb{N})$ has the required properties:

1. $A = \mathcal{O}$: By the Remark following Theorem 1 of [2], we may assume that there is for each $y \in H$ infinitely many $n$ so that $y \in H_n \ast W_n$. Consider any $(x,y) \in G \times H$. Fix an $N$ so that for all $n \geq N$ we have $x \in G_n \ast V_n$. Then choose an $n > N$ so that also $y \in H_n \ast W_n$. Then $(x,y) \in F_n \ast U_n$. 


2. \( \mathcal{A} = \Omega \): Observe that for each finite subset \( K \) of \( G \) there is an \( N \) such that for all \( n \geq N \), \( K \subseteq G_n \cdot V_n \). Since for each finite subset \( L \) of \( H \) there are infinitely many \( n \) with \( L \subseteq H_n \cdot W_n \), apply the argument from before.

3. \( \mathcal{A} = \Gamma \): The argument is similar for this case. ■

**Corollary 7.** If \( (G_j, \ast_j), j \leq n \) are metrizable strictly \( o \)-bounded groups, then so is their product.

**Proof.** By Theorem 5, metrizable strictly \( o \)-bounded groups are characterized by \( S_1(\Omega_{\text{nbd}}, \Gamma) \). Apply Theorem 6. ■

**Corollary 8.** If \( (G, \ast) \) is an \( o \)-bounded group and \( (H, \ast) \) is a metrizable strictly \( o \)-bounded group, then \( G \times H \) is an \( o \)-bounded group.

**Proof.** Apply Theorems 5 and 6. ■

We characterize for metrizable groups when TWO has a winning strategy in the game \( G_1(\mathcal{O}_{\text{nbd}}(H), \mathcal{O}_{HG}) \):

**Theorem 9.** Let \( (H, \ast) \) be a metrizable group with a subgroup \( (G, \ast) \). The following are equivalent:

1. TWO has a winning strategy in \( G_1(\mathcal{O}_{\text{nbd}}(H), \mathcal{O}_{HG}) \).
2. \( G \) is a countable set.

**Proof.** We only need to prove 1 \( \Rightarrow \) 2: Let \( \sigma \) be a winning strategy for TWO. Since \( (H, \ast) \) is metrizable, let \( d \) be a left-invariant metric for it, and let \( (U_n : n \in \mathbb{N}) \) be a neighborhood basis for \( e \) in \( H \) such that \( \lim_{n \to \infty} \text{diam}_d(U_n) = 0 \). Define \( G_\emptyset = \bigcap_{n \in \mathbb{N}} \sigma(\mathcal{O}(U_n)) \). For \( (n_1, \cdots, n_k) \) a given finite sequence, define \( G_{(n_1, \cdots, n_k)} = \bigcap_{n \in \mathbb{N}} \sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k}), \mathcal{O}(U_n)) \). We claim that

\[
G \subseteq \bigcup_{\tau \in \mathbb{N}} G_\tau. \quad (2)
\]

For suppose on the contrary \( x \) is not in \( \bigcup_{\tau \in \mathbb{N}} G_\tau \). Choose an \( n_1 \) with \( x \notin \sigma(\mathcal{O}(U_{n_1})) \), and then choose \( n_2 \) so that \( x \notin \sigma(\mathcal{O}(U_{n_2})) \), and so on. In this way we obtain a sequence \( n_1, n_2, \cdots, n_k, \cdots \) so that for each \( k \), \( x \notin \sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k})) \). But then we obtain a \( \sigma \)-play lost by TWO, a contradiction.

Next, we observe that each \( G_\tau \) has at most one element: This is because \( d \) is left-invariant, so that \( \text{diam}(\sigma(\mathcal{O}(U_{n_1}), \cdots, \mathcal{O}(U_{n_k}))) = \text{diam}(U_n) \), and since these \( U_n \)’s form a neighborhood base for \( e \), the diameter of \( H_{n_1, \cdots, n_k} \) is 0. ■

**REFERENCES**


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