ON SEQUENCE-COVERING msss-MAPS

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Abstract. This paper gives characterizations of metric spaces under some sequence-covering msss-maps by means of certain kind of \( \sigma \)-locally countable networks.

1. Introduction and definitions

A study of some images of metric spaces under certain maps is an important task on general topology. The paper [1] introduced the concept of msss-maps, and established the relationships between spaces with \( \sigma \)-locally countable \( k \)-networks (bases) and metric spaces by means of msss-maps. In this paper, we study some spaces with \( \sigma \)-locally countable networks, and give characterizations of some sequence-covering msss-images of metric spaces.

In this paper all spaces are regular and \( T_1 \), all maps are continuous and onto. \( N \) denotes the set of all natural numbers, \( \omega \) denotes \( N \cup \{0\} \). For two family \( A \) and \( B \) of subsets of a space \( X \), Denote \( A \land B = \{A \cap B : A \in A \text{ and } B \in B\} \). For the usual product space \( \prod_{i \in N} X_i \), \( p_i \) denotes the projection from \( \prod_{i \in N} X_i \) onto \( X_i \) For a space \( X \) and each \( x_n \in X \), \( (x_n) \) denotes a point of the usual product space \( X^\omega \) whose \( n \)-th coordinate is \( x_n \).

Definition 1.1. [10] Let \( X \) be a space, and \( P \subset X \). Then,

1. A sequence \( \{x_n\} \) in \( X \) is eventually in \( P \) if \( \{x_n\} \) converges to \( x \), and there exists \( m \in N \) such that \( \{x\} \cup \{x_n : n \geq m\} \subset P \);
2. \( P \) is a sequential neighborhood of \( x \) in \( X \) if whenever a sequence \( \{x_n\} \) in \( X \) converges to \( x \), then \( \{x_n\} \) is eventually in \( P \);
3. \( P \) is sequentially open in \( X \) if \( P \) is a sequential neighborhood at each of its points;
4. \( X \) is a sequential space if any sequentially open subset of \( X \) is open in \( X \).

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**DEFINITION 1.2.** Let \( \mathcal{P} = \bigcup \{ \mathcal{P}_x : x \in X \} \) be a family of subsets of a space \( X \) satisfying for each \( x \in X \),

1. \( \mathcal{P}_x \) is a network of \( x \) in \( X \) i.e., \( x \in \cap \mathcal{P}_x \) and for \( x \in U \) with \( U \) open in \( X \), \( P \subset U \) for some \( P \in \mathcal{P}_x \),

2. If \( U, V \in \mathcal{P}_x \), then \( W \subset U \cap V \) for some \( W \in \mathcal{P}_x \).

\( \mathcal{P} \) is a weak-base for \( X \) [8] if \( G \subset X \) such that for each \( x \in G \), there exists \( P \in \mathcal{P}_x \) satisfying \( P \subset G \), then \( G \) is open in \( X \). \( \mathcal{P} \) is an \( sn \)-network (i.e., sequential neighborhood network) for \( X \) if each element of \( \mathcal{P}_x \) is a sequential neighborhood of \( x \) in \( X \). \( \mathcal{P} \) is an \( so \)-network (i.e., sequential open network) for \( X \) if each element of \( \mathcal{P}_x \) is sequentially open in \( X \). The above \( \mathcal{P}_x \) respectively is a weak-base, an \( sn \)-network and an \( so \)-network of \( x \) in \( X \).

**DEFINITION 1.3.** [4] For a space \( X \), let \( \mathcal{P} \) be a family of subsets of \( X \), there exists \( \mathcal{P}_x \subset \mathcal{P}_x^\omega \) holding the following property: if \( (P_n) \in \mathcal{P}_x \), then \( \{P_n : n \in \mathbb{N} \} \) is a decrease network of \( x \) in \( X \). Denote \( \mathcal{P} \approx \bigcup \{ \mathcal{P}_x : x \in X \} \).

1. \( \mathcal{P} \) is an \( s \)-network (i.e., sequential network) for \( X \) if, whenever \( P \subset X \), and for each \( (P_n) \in \mathcal{P}_x \), \( P_m \subset P \) for some \( m \in M \), then \( P \) is sequentially open in \( X \);

2. \( \mathcal{P} \) is a sequential quasi-bases for \( X \) if, whenever \( P \subset X \), and for each \( (P_n) \in \mathcal{P}_x \), \( P_m \subset P \) for some \( m \in M \), then \( P \) is open in \( X \);

3. \( \mathcal{P} \) is a Fréchet quasi-bases for \( X \) if, whenever \( P \subset X \), and for each \( (P_n) \in \mathcal{P}_x \), \( P_m \subset P \) for some \( m \in M \), then \( P \) is a neighborhood of \( x \) in \( X \).

**DEFINITION 1.4.** Let \( \mathcal{P} \) be a family of subsets of a space \( X \).

1. \( \mathcal{P} \) is a \( cs \)-network [9] for \( X \) if whenever \( \{x_n\} \) is a sequence converging to a point \( x \in U \) with \( U \) open in \( X \), then there are \( P \in \mathcal{P} \) and \( m \in \mathbb{N} \) such that \( \{x_n : n \geq m\} \cup \{x\} \subset P \subset U \);

2. \( \mathcal{P} \) is a \( cs^1 \)-network for \( X \) if whenever \( \{x_n\} \) is a sequence converging to a point \( x \in U \) with \( U \) open in \( X \), then there are a subsequence \( \{x_{n_i}\} \) and \( P \in \mathcal{P} \) such that \( \{x_{n_i} : i \in \mathbb{N}\} \cup \{x\} \subset P \subset U \).

**DEFINITION 1.5.** Let \( f : X \to Y \) be a map.

1. \( f \) is a \( mssss \)-map [1] (i.e., metrizably stratified strong s-map) if \( X \) is a subspace of the product space \( \prod_{i \in \mathbb{N}} X_i \) of a family \( \{X_i : i \in \mathbb{N}\} \) of metric spaces and for each \( y \in Y \), there is a sequence \( \{V_i\} \) of open neighborhoods of \( y \) such that each \( p_i f^{-1}(V_i) \) is separable in \( X_i \);

2. \( f \) is a 1-sequence-covering map [2] if for each \( y \in Y \), there exists \( x \in f^{-1}(y) \) satisfying the following condition \((*)\): whenever \( y_n \to y \), then there exists \( x_n \in f^{-1}(y_n) \) such that \( x_n \to x \);

3. \( f \) is a 2-sequence-covering map [2] if for each \( y \in Y \) and each \( x \in f^{-1}(y) \) satisfying the above condition \((*)\);

4. \( f \) is a sequence-covering map [13] (resp. compact-covering map) if each convergent sequence (including its limit point) of \( Y \) (resp. each compact subset of \( Y \)) is the image of some compact subset of \( X \);
(5) $f$ is a strong sequence-covering map \cite{5} if each convergent sequence (including its limit point) in $Y$ is the image of some convergent sequence (including its limit point) in $X$;

(6) $f$ is a strong compact-covering map \cite{5} if it is both strong sequence-covering and compact-covering;

(7) $f$ is a sequentially quotient map \cite{7} if whenever $R \subseteq Y$ and $f^{-1}(R)$ is sequentially open in $X$, then $R$ is sequentially open in $Y$.

2. On 1-sequence-covering msss-images

**Theorem 2.1.** A space $X$ is a 1-sequence-covering msss-image of a metric space if and only if $X$ has a $\sigma$-locally countable $sn$-network.

**Proof.** Sufficiency. Suppose $P$ is a $\sigma$-locally countable $sn$-network for $X$. Let $P = \bigcup \{ P_i : i \in N \}$, where each $P_i = \{ P_{\alpha} : \alpha \in A_i \}$ is locally countable in $X$. We can assume that $P_i$ is closed under finite intersections and $X \subseteq P_1 \subseteq P_{i+1}$. For each $i \in N$, endow $A_i$ with discrete topology; then $A_i$ is a metric space. Let

$$ M = \left\{ (\alpha) \in \prod_{i \in N} A_i : \{ P_{\alpha} : i \in N \} \text{ is a network of some point } x_\alpha \text{ in } X \right\},$$

and endow $M$ with the subspace topology induced from the product topology of a family $\{ A_i : i \in N \}$ of metric spaces, then $M$ is a metric space. Since $X$ is Hausdorff, $x_\alpha$ is unique in $X$ for each $\alpha \in M$. We define $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. Since $P$ is a $\sigma$-locally countable $sn$-network for $X$, $f$ is onto. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x_\alpha$. Suppose $V$ is an open neighborhood of $x_\alpha$ in $X$, there exists $n \in N$ such that $x_\alpha \in P_{\alpha_n} \subseteq V$, set $W = \{ c \in M : \text{the } n-\text{the coordinate of } c \text{ is } \alpha_n \}$, then $W$ is an open neighborhood of $\alpha$ in $M$, and $f(W) \subseteq P_{\alpha_n} \subseteq V$. Hence $f$ is continuous. We will show that $f$ is a 1-sequence-covering msss-map.

(i) $f$ is an msss-map.

For each $x \in X$ and each $i \in N$, there exists an open neighborhood $V_i$ of $x$ in $X$ such that $\{ \alpha_i \in A_i : P_{\alpha_i} \cap V_i \neq \emptyset \}$ is countable. Put

$$ B_i = \{ \alpha_i \in A_i : P_{\alpha_i} \cap V_i \neq \emptyset \}, $$

then $p_i f^{-1}(V_i) \subseteq B_i$. Thus $p_i f^{-1}(V_i)$ is separable in $A_i$. Hence $f$ is a msss-map.

(ii) $f$ is a 1-sequence-covering map.

For each $x \in X$, by the definition of $P$, there exists $(\alpha_i) \in \prod_{i \in N} A_i$ such that $\{ P_{\alpha_i} : i \in N \} \subseteq P$ is an $sn$-network of $x$ in $X$. Denote $\beta = (\alpha_i)$, then $\beta \in f^{-1}(x)$. For each $n \in N$, let $R_n = \{ (\gamma_i) \in M : \text{if } i \leq n, \text{ then } \gamma_i = \alpha_i \}$. Then $\{ R_n : n \in N \}$ is a decreasing neighborhood base of $\beta$ in $M$. For each $n \in N$, it is easy to check that $f(R_n) = \bigcap_{i \leq n} P_{\alpha_i}$. Now suppose $x_j \to x$ in $X$. For each $n \in N$, since $f(B_n)$ is a sequential neighborhood of $x$ in $X$, there exists $i(n) \in N$ such that if $i \geq i(n)$, then $x_i \in f(R_n)$. Thus $f^{-1}(x_i) \cap R_n \neq \emptyset$. We may assume $1 < i(n) < i(n+1)$. For each $j \in N$, let

$$ \beta_j \in \begin{cases} 
    f^{-1}(x_j), & \text{if } j < i(1), \\
    f^{-1}(x_j) \cap R_n, & \text{if } i(n) \leq j < i(n+1), n \in N. 
\end{cases} $$
Then it is easy to show that sequence \{\beta_j\} converges to \beta in M. Hence f is a 1-sequence-covering map.

Necessity. Suppose \( f: M \rightarrow X \) is a 1-sequence-covering msss-map, where M is a metric space. Since f is a msss-map, then there exists a base \( \mathcal{B} \) for M such that \( \mathcal{P}^* = \{ f(B) : B \in \mathcal{B} \} \) is a \( \sigma \)-locally countable network for X by Lemma 1.2 of [1]. For each \( x \in X \), \( \beta \in f^{-1}(x) \) satisfies the condition (*) of Definition 1.5 (2). Put

\[
\mathcal{P}_x = \{ f(B) : \beta_x \in B \} \in \mathcal{B}, \quad \mathcal{P} = \bigcup \{ \mathcal{P}_x : x \in X \}.
\]

It is easy to show that each \( \mathcal{P}_x \) is an sn-network of \( x \) in X, and \( \mathcal{P} \) is an sn-network for X. Obviously, \( \mathcal{P} \subset \mathcal{P}^* \). Hence X has a \( \sigma \)-locally countable sn-network. \( \blacksquare \)

**Corollary 2.2.** A space X is a 1-sequence-covering and quotient msss-image of a metric space if and only if X has a \( \sigma \)-locally countable weak-base.

**Proof.** Sufficiency. Suppose X has a \( \sigma \)-locally countable weak-base, then X is a sequential space with a \( \sigma \)-locally countable sn-network by [3, Proposition 1.6.15, Corollary 1.6.18]. Thus X is a 1-sequence-covering msss-image of a metric space by Theorem 2.1. This 1-sequence-covering msss-map is quotient by Lemma 2.1 of [14].

Necessity. Suppose X is a 1-sequence-covering and quotient msss-image of a metric space. Then X is a sequential space with a \( \sigma \)-locally countable sn-network \( \mathcal{P} \). It is easy to prove that \( \mathcal{P} \) is a \( \sigma \)-locally countable weak-base for X. \( \blacksquare \)

3. On 2-sequence-covering msss-images

The following Theorem 3.1 can be proved by Lemma 3.1 of [2] according to the proof of Theorem 2.1.

**Theorem 3.1.** A space X is a 2-sequence-covering msss-image of a metric space if X has a \( \sigma \)-locally countable so-network.

**Corollary 3.2.** The following are equivalent for a space X:

1. X has a \( \sigma \)-locally countable base;
2. X is a 2-sequence-covering and quotient msss-image of a metric space;
3. X is an open msss-image of a space having a \( \sigma \)-locally countable base;
4. X is a countably-bi-quotient msss-image of a space having a \( \sigma \)-locally countable base.

**Proof.** (1) \( \implies \) (2) follows from Theorem 3.1 of [1] and Corollary 3.2 of [2].

(2) \( \implies \) (1). By Theorem 3.1, X is a sequential space with a \( \sigma \)-locally countable so-network \( \mathcal{P} \). It is easy to show that \( \mathcal{P} \) is a \( \sigma \)-locally countable base for X.

(1) \( \implies \) (3) follows from Theorem 3.1 of [1].

(3) \( \implies \) (4) is obvious.
(4) $\Rightarrow$ (1). Suppose $X$ is the image of $M$ under a countably-bi-quotient msss-map $f$, where $M$ is a space with a $\sigma$-locally countable base. Because $f$ is a msss-map, then there exists a base $B$ for $M$ such that $P = \{f(B) : B \in B\}$ is a $\sigma$-locally countable network for $X$ by Lemma 1.2 of [1]. Thus $P$ is a $\sigma$-locally countable $k$-network for $X$ by Lemma 2.5 of [1]. By Proposition 2.3.1 of [3], countably-bi-quotient maps preserve strong Fréchet property, thus $X$ is a strong Fréchet space with a $\sigma$-locally countable $k$-network. Hence $X$ has a $\sigma$-locally countable base by Theorem 3.9 of [6] and Proposition 3.2 of [13].

**Corollary 3.3.** A space with a $\sigma$-locally countable base is preserved by a countably-bi-quotient msss-map.

### 4. On sequence-covering msss-images

**Theorem 4.1.** The following are equivalent for a space $X$:

1. $X$ is a sequence-covering msss-image of a metric space;
2. $X$ is a sequentially quotient msss-image of a metric space;
3. $X$ has a $\sigma$-locally countable $s$-network.

**Proof.** (1) $\Rightarrow$ (2) follows from Proposition 2.1.17 of [3].

(2) $\Rightarrow$ (3). Suppose $f : M \rightarrow X$ is a sequentially quotient msss-map, where $M$ is a metric space. Since $f$ is a msss-map, there exists a base $B$ for $M$ such that $P = \{f(B) : B \in B\}$ is a $\sigma$-locally countable network for $X$ by Lemma 1.2 of [1]. Because $s$-networks are preserved by sequentially quotient maps by Lemma 3.1 of [4], $X$ has a $\sigma$-locally countable $s$-network $P$.

(3) $\Rightarrow$ (1). Suppose $P$ is a $\sigma$-locally countable $s$-network for $X$. then $P$ is a $\sigma$-locally countable cs*-network for $X$ by Theorem 2.4 of [4]. Hence $X$ is a sequence-covering msss-image of a metric space by Theorem 1 of [11].

The following corollaries can be proved by Theorem 4.1, Corollary 2.3 of [4], Lemma 3.1 of [4] and Proposition 2.1.16 of [3].

**Corollary 4.2.** A space $X$ has a $\sigma$-locally countable sequential quasi-base if and only if $X$ is a quotient msss-image of a metric space.

**Corollary 4.3.** A space $X$ has a $\sigma$-locally countable Fréchet quasi-base if and only if $X$ is a pseudo-open msss-image of a metric space.

### 5. On strong sequence-covering msss-images

**Theorem 5.1.** The following are equivalent for a space $X$:

1. $X$ is a strong sequence-covering msss-image of a metric space;
2. $X$ is a strong compact-covering msss-image of a metric space;
3. $X$ has a $\sigma$-locally-countable cs-network.

**Proof.** (1) $\Rightarrow$ (3) follows from Lemma 1.2 of [1] and the fact: cs-networks are preserved by strong sequence-covering maps.
(3) $\implies$ (2). Suppose $\mathcal{P}$ is a $\sigma$-locally-countable $cs$-network for $X$. Denote $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in N\}$, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is locally-countable in $X$. We can assume that each $\mathcal{P}_i$ is closed under finite intersections and $X \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. By the proof of Theorem 2.1, there exist a metric space $M$ and a msms-map $f : M \to X$. We will prove that $f$ is a strong compact-covering map. For each sequence $\{x_n\}$ converging to $x_0$, we can assume that all $x'_n$s are distinct, and that $x_n \neq x_0$ for each $n \in N$. Let $K = \{x_m : m \in \omega\}$. Suppose $V$ is an open neighborhood of $K$ in $X$. A subfamily $\mathcal{A}$ of $\mathcal{P}$ is said to have the following property, which is denoted by $F(K, V)$, if:

(a) $\mathcal{A}$ is finite,
(b) for each $P \in \mathcal{A}$, $\phi \neq P \cap K \subset P \subset V$,
(c) for each $z \in K$, there exists a unique $P_z \in \mathcal{A}$ such that $z \in P_z$,
(d) if $x_0 \in P \in \mathcal{A}$, then $K \setminus P$ is finite.

For each $i \in N$, put

$$\mathcal{P}_i(K) = \{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A} \text{ has the property } F(K, X)\};$$

then $|\mathcal{P}_i(K)| < \aleph_0$. Denote $\mathcal{P}'_i(K)$ by $\{\mathcal{P}_{ij} : j \in N\}$ (when $\mathcal{P}_i(K) = \{\mathcal{P}_{i1}, \ldots, \mathcal{P}_{is}\}$, denote $\mathcal{P}_{ij} = \mathcal{P}_{is}$ if $j > s$). For each $n \in N$, put

$$\mathcal{P}'_n = \bigwedge_{i,j \leq n} \mathcal{P}_{ij},$$

then $\mathcal{P}'_n \subset \mathcal{P}_n$ and $\mathcal{P}'_n$ also has the property $F(K, X)$.

For each $i \in N$ and each $m \in \omega$, there exists $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{P}'_i$. Let $b_m = (\alpha_{im}) \in \prod_{i \in N} A_i$. It is easy to prove that $\{P_{\alpha_{im}} : i \in N\}$ is a network of $x_m$ in $X$. Then $b_m \in M$ and $f(b_m) = x_m$ for each $m \in \omega$. For each $i \in N$, there exists $n(i) \in N$ such that $\alpha_{in} = \alpha_{i0}$ when $n \geq n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges $\alpha_{i0}$ in $A_i$. Thus, there is a sequence $\{b_n\}$ converging to $b_0$ in $X$. This shows that $f$ is sequence-covering.

Since $X$ has a $\sigma$-locally-countable $cs$-network, each compact subset $L$ of $X$ has a countable $cs$-network. So $L$ is metrizable. We can prove that $f$ is compact-covering by the proof of Theorem 2 in [5].

(2) $\implies$ (1) is obvious. $\blacksquare$

**Corollary 5.2.** The following are equivalent for a space $X$:

1. $X$ is a strong sequence-covering and quotient msms-image of a metric space;
2. $X$ is a strong compact-covering and msms-image of a metric space;
3. $X$ is a $k$-space with a $\sigma$-locally-countable $cs$-network.

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