SOME SUBSETS OF IDEAL TOPOLOGICAL SPACES

V. Jeyanthi, V. Renuka Devi and D. Sivaraj

Abstract. In ideal topological spaces, \( \star \)-dense in itself subsets are used to characterize ideals and mappings. In this note, properties of \( A_I \)-sets, \( I \)-locally closed sets and almost strong \( I \)-open sets are discussed. We characterize codense ideals by the collection of these sets. Also, we give a decomposition of continuous mappings and deduce some well-known results.

1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [19]. In this note, we discuss the properties of the \( \star \)-dense in itself sets, namely, \( A_I \)-sets, regular \( I \)-closed sets and almost strong \( I \)-open sets in ideal topological spaces.

An ideal \( I \) on a topological space \( (X, \tau) \) is a nonempty collection of subsets of \( X \) which satisfies: (i) \( A \in I \) and \( B \subset A \) implies \( B \in I \) and (ii) \( A \in I \) and \( B \in I \) implies \( A \cup B \in I \). Given a topological space \( (X, \tau) \) with an ideal \( I \) on \( X \) and if \( P(X) \) is the set of all subsets of \( X \), a set operator \( (\cdot)^*: P(X) \to P(X) \), called a local function [13] of \( A \) with respect to \( \tau \) and \( I \) is defined as follows: for \( A \subset X \), 
\[
A^*(I, \tau) = \{ x \in X \mid U \cap A \notin I \text{ for every } U \in \tau(x) \}
\]
where \( \tau(x) = \{ U \in \tau \mid x \in U \} \). We will make use of the basic facts about the local functions [9, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator \( cl^*(\cdot) \) for a topology \( \tau^*(I, \tau) \), called the \( \star \)-topology, finer than \( \tau \) is defined by \( cl^*(A) = A \cup A^*(I, \tau) \) [19]. When there is no chance for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \) and \( \tau^* \) or \( \tau^*(I) \) for \( \tau^*(I, \tau) \). If \( I \) is an ideal on \( X \), then \( (X, \tau, I) \) is called an ideal space. \( I \) is said to be codense [4] if \( \tau \cap I = \{ \emptyset \} \). \( N \) is the ideal of all nowhere dense subsets in \( (X, \tau) \).

By a space, we always mean a topological space \( (X, \tau) \) with no separation properties assumed. If \( A \subset X \), \( cl(A) \) and \( int(A) \) will, respectively, denote the closure and interior of \( A \) in \( (X, \tau) \) and \( cl^*(A) \) and \( int^*(A) \) will, respectively, denote the closure and interior of \( A \) in \( (X, \tau^*) \). An open subset \( A \) of a space \( (X, \tau) \) is

AMS Subject Classification: Primary: 54 A 05, 54 A 10; Secondary: 54 C 08, 54 C 10.

Keywords and phrases: Codense ideal, semiopen set, preopen set, \( I \)-locally closed set, \( f_I \)-set, regular \( I \)-closed set, \( A_I \)-set, semicontinuity, \( A_I \)-continuity, \( f_I \)-continuity.

75
said to be regular open if \( A = \text{int}(\text{cl}(A)) \). The complement of a regular open set is regular closed. The family of all regular open (resp. regular closed) set is denoted by \( RO(X, \tau) \) (resp. \( RC(X, \tau) \)). A subset \( A \) of a space \((X, \tau)\) is an \( \alpha \)-open \([16]\) (resp. semiopen \([14]\), preopen \([15]\), \( \beta \)-open or semipreopen \([1]\)) set if \( A \subset \text{int}(\text{cl}(\text{int}(A))) \) (resp. \( A \subset \text{cl}(\text{int}(A)), A \subset \text{int}(\text{cl}(A)), A \subset \text{cl}(\text{int}(\text{cl}(A))) \)). The complement of a semiopen (resp. preopen) set is semiclosed (resp. preclosed). The family of all \( \alpha \)-open (resp. semiopen, preopen) sets in \((X, \tau)\) is denoted by \( \tau^\alpha \) (resp. \( SO(X, \tau) \), \( PO(X, \tau) \)). The smallest preclosed set containing \( A \) is called the preclosure of \( A \) and is denoted by \( pcl(A) \). Also, \( pcl(A) = A \cup cl(int(A)) \) \([1, \text{Theorem 1.5(e)}]\). A subset \( A \) of a space \((X, \tau)\) is locally closed \([2]\) (resp. \( A \)-set \([18]\)) if \( A = U \cap V \) where \( U \) is open and \( V \) is closed (resp. regular closed). A subset \( A \) of an ideal space \((X, \tau, I)\) is said to be \( I \)-open \([10]\) if \( A \subset \text{int}(A^*) \). The largest \( I \)-open set contained in \( A \) is called the \( I \)-interior of \( A \) and is denoted by \( I\text{int}(A) \). The family of all \( I \)-open sets is denoted by \( IO(X, \tau) \). A subset \( A \) of an ideal space \((X, \tau, I)\) is \( \tau^* \)-closed \([8]\) (resp. \( \tau^* \)-dense in itself \([7]\), \( \tau^* \)-perfect \([7]\)) if \( A^* \subset A \) (resp. \( A \subset A^* \), \( A = A^* \)). Clearly, \( A \) is \( \tau^* \)-perfect if and only if \( A \) is \( \tau^* \)-closed and \( \tau^* \)-dense in itself. A subset \( A \) of an ideal space \((X, \tau, I)\) is \( I \)-locally closed \([3]\) if \( A = G \cap V \), where \( G \) is open and \( V \) is \( \tau^* \)-perfect. We will denote the collection of all \( I \)-locally closed sets in \((X, \tau, I)\) by \( ILC(X, \tau) \). Clearly, every \( \tau^* \)-perfect set is \( I \)-locally closed. A subset \( A \) of an ideal space \((X, \tau, I)\) is called an \( f_I \)-set \([12]\) (resp. regular \( I \)-closed \([11]\)) if \( A \subset (\text{int}(A))^* \) (resp. \( A = (\text{int}(A))^* \)). The family of all \( f_I \)-sets in a space \((X, \tau, I)\) will be denoted by \( f_I(X, \tau) \). A subset \( A \) of an ideal space \((X, \tau, I)\) is called \( \alpha - I \)-open \([6]\), semi-\( I \)-open \([6]\) if \( A \subset \text{int}(\text{cl}^*(A)) \) (resp. \( A \subset \text{int}(\text{cl}^*(\text{int}(A))), A \subset \text{cl}^*(\text{int}(A)) \)). The family of all \( \alpha - I \)-open (resp. semi-\( I \)-open) sets is denoted by \( P \alpha O(X, \tau) \) (resp. \( \alpha \tau O(X, \tau) \), \( S \alpha O(X, \tau) \)). Given a space \((X, \tau)\) and ideals \( I \) and \( \mathcal{I} \) on \( X \), the extension of \( I \) via \( \mathcal{I} \) \([10]\), denoted by \( I \star \mathcal{I} \), is the ideal given by \( I \star \mathcal{I} = \{ A \subset X \mid A^*(I) \in \mathcal{I} \} \). In particular, \( I \star \mathcal{N} = \{ A \subset X \mid \text{int}(A^*(I)) = \emptyset \} \) is an ideal containing both \( I \) and \( \mathcal{N} \) and \( I \star \mathcal{N} \) is usually denoted by \( \bar{I} \). The following lemmas will be useful in the sequel.

**Lemma 1.1.** \([9, \text{Theorem 6.1}]\) Let \((X, \tau, I)\) be an ideal space. Then the following are equivalent.

(a) \( I \) is codense.
(b) \( X = X^* \).
(c) \( G \subset G^* \) for every open set \( G \).
(d) \( G \subset G^* \) for every semiopen set \( G \) \([17, \text{Lemma 1(c)}]\).

**Lemma 1.2.** \([17, \text{Lemma 2}]\) Let \((X, \tau, I)\) be an ideal space. If \( A \) is \( \tau^* \)-dense in itself, then \( A^* = \text{cl}(A) = \text{cl}^*(A) \).

**Lemma 1.3.** \([17, \text{Theorem 3.1(b)}]\) Let \((X, \tau, I)\) be an ideal space. A subset \( A \) of \( X \) is \( I \)-locally closed if and only if \( A = G \cap A^* \) for some open set \( G \).

**Lemma 1.4.** Let \((X, \tau, I)\) be an ideal space and \( \Delta = \{ A \subset X \mid A \subset A^* \} \). Then \( \Delta \cap I = \{ \emptyset \} \).
Proof. Suppose \( A \in \Delta \cap \mathcal{I} \). Then \( A \in \mathcal{I} \) implies \( A^* = \emptyset \) and \( A \in \Delta \) implies that \( A \subseteq A^* \). Therefore, \( A = \emptyset \) which implies that \( \Delta \cap \mathcal{I} = \{\emptyset\} \).

2. \( \mathcal{A}_\mathcal{I} \)-sets.

A subset \( A \) of an ideal space \((X, \tau, \mathcal{I})\) is called an \( \mathcal{A}_\mathcal{I} \)-set [11] if \( A = U \cap V \) where \( U \in \tau \) and \( V \) is regular \( \mathcal{I} \)-closed. The family of all \( \mathcal{A}_\mathcal{I} \)-sets in a space \((X, \tau, \mathcal{I})\) will be denoted by \( \mathcal{A}_\mathcal{I}(X, \tau) \). The following Theorem 2.1 gives some properties of \( \mathcal{A}_\mathcal{I} \)-sets.

**Theorem 2.1.** (i) If \( A \) is an \( \mathcal{A}_\mathcal{I} \)-set of an ideal space \((X, \tau, \mathcal{I})\), then the following hold.

(a) \( A \) and \( \text{int}(A) \) are *-dense in itself.
(b) \( A^* = \text{cl}(A) = \text{cl}^*(A) \) and \( (\text{int}(A))^* = \text{cl}(\text{int}(A)) \).
(c) \( A \) is an \( f_\mathcal{I} \)-set.
(d) \( A^* = (\text{int}(A))^\ast = ((\text{int}(A))^\ast)^\ast = (A^*)^\ast \).
(e) \( A^* \) and \((\text{int}(A))^\ast \) are *-perfect and \( \mathcal{I} \)-locally closed.
(f) \( A^*(\mathcal{I}) = \text{cl}(\text{int}(A)) = A^*(\mathcal{I}) \) is regular closed.
(g) \( A^* = \text{pcl}(A) \).
(h) \( A^* \) is regular \( \mathcal{I} \)-closed.

(ii) In any ideal space \((X, \tau, \mathcal{I})\), \( \mathcal{A}_\mathcal{I}(X, \tau) \cap \mathcal{I} = \{\emptyset\} \).

**Proof.** (i) (a) If \( A \) is an \( \mathcal{A}_\mathcal{I} \)-set, then \( A = U \cap V \) where \( U \in \tau \) and \( V \) is regular \( \mathcal{I} \)-closed. Therefore, \( A = U \cap V = U \cap (\text{int}(V))^\ast \subseteq (U \cap \text{int}(V))^\ast = (\text{int}(U \cap V))^\ast = (\text{int}(A))^\ast \subseteq A^* \) which implies that \( \text{int}(A) \subseteq A \subseteq (\text{int}(A))^\ast \subseteq A^* \). Therefore, \( A \) and \( \text{int}(A) \) are *-dense in itself.

(b) By Lemma 1.2, we have \( A^* = \text{cl}(A) = \text{cl}^*(A) \) and \( (\text{int}(A))^\ast = \text{cl}(\text{int}(A)) \).

(c) From (a), \( A \subseteq (\text{int}(A))^\ast \) and so \( A \) is an \( f_\mathcal{I} \)-set.

(d) From (a), we have \( \text{int}(A) \subseteq A \subseteq (\text{int}(A))^\ast \subseteq A^* \) and so \( (\text{int}(A))^\ast \subseteq A^* \subseteq ((\text{int}(A))^\ast)^\ast \subseteq (\text{int}(A))^\ast \subseteq A^* \) and so \( A^* = (\text{int}(A))^\ast = ((\text{int}(A))^\ast)^\ast = (A^*)^\ast \).

(e) From (d), it follows that \( A^* \) and \((\text{int}(A))^\ast \) are *-perfect and hence are \( \mathcal{I} \)-locally closed.

(f) From (d), \( A^* = (\text{int}(A))^\ast \) and so by (b), \( A^* = \text{cl}(\text{int}(A)) \). Since \( A \) is *-dense in itself, \( A^* \subseteq \text{cl}(\text{int}(A^*)) \). Since \( \text{cl}(\text{int}(A^*)) = A^*(\mathcal{I}) \subseteq A^*(\mathcal{I}) \), we have \( A^*(\mathcal{I}) = \text{cl}(\text{int}(A)) = A^*(\mathcal{I}) \) and each is regular closed, since \( A^*(\mathcal{I}) \) is regular closed [10, Theorem 3.2].

(g) Since \( A^* = \text{cl}^*(A) = A \cup A^* = A \cup \text{cl}(\text{int}(A)) \) by (f), \( A^* = \text{pcl}(A) \).

(h) From (d), \( A^* = (\text{int}(A))^\ast \). Let \( B = (\text{int}(A))^\ast \). Then \( (\text{int}(B))^\ast = (\text{int}(A))^\ast = (A^*)^\ast \supseteq (\text{int}(A))^\ast = B \), since \( A \) is *-dense in itself. Therefore, \( B \subseteq (\text{int}(B))^\ast \). Also, \( \text{int}(B) \subseteq B \) implies that \( (\text{int}(B))^\ast \subseteq B^\ast = (\text{int}(A))^\ast \subseteq (\text{int}(A))^\ast = B \) and so \( (\text{int}(B))^\ast \subseteq B \). Therefore, \( B = (\text{int}(B))^\ast \) which implies that \( B \) is regular \( \mathcal{I} \)-closed. Therefore, \( A^* \) is regular \( \mathcal{I} \)-closed.

(ii) The proof follows from Lemma 1.4.
The following Theorems 2.2 and 2.4 give characterizations of codense ideals in terms of $A_\tau$-sets.

**Theorem 2.2.** Let $(X, \tau, I)$ be an ideal space. Then $I$ is codense if and only if $\tau \subset A_\tau(X, \tau)$.

*Proof.* If $I$ is codense, by Proposition 4(a) of [11], $\tau \subset A_\tau(X, \tau)$. Conversely, suppose the condition holds. By Theorem 2.1(ii), $A_\tau(X, \tau) \cap I = \{\emptyset\}$ and so $\tau \cap I = \emptyset$. Therefore, $I$ is codense. ■

**Corollary 2.3.** Let $(X, \tau, I)$ be an ideal space. Then the following are equivalent.

(a) $I$ is codense.

(b) $\tau = P_2O(X, \tau) \cap A_\tau(X, \tau)$.

(c) $\tau = \alpha_\tau O(X, \tau) \cap A_\tau(X, \tau)$.

(d) $\tau \subset A_\tau(X, \tau)$.

*Proof.* (a) implies (b) and (a) implies (c) follow from Proposition 6 of [11]. It is clear that (b) implies (d) and (c) implies (d). (d) implies (a) by Theorem 2.2. ■

Every $A_\tau$-set is an $A$-set [11, Proposition 5(b)] but not the converse [11, Example 5(3)]. Theorem 2.5 below shows that these two collection of sets are equal, if the ideal is codense and also it gives another characterization of codense ideals in terms of $A_\tau$-sets. Before that, we prove the following Theorem 2.4 which gives a characterization of codense ideals in terms of regular $I$-closed sets.

**Theorem 2.4.** Let $(X, \tau, I)$ be an ideal space. Then $I$ is codense if and only if $R_\tau C(X, \tau) = RC(X, \tau)$ where $R_\tau C(X, \tau)$ is the collection of all regular $I$-closed sets in $(X, \tau, I)$.

*Proof.* Suppose $I$ is codense. Then $A \in R_\tau C(X, \tau)$ if and only if $A = (int(A))^*$ if and only if $A = cl(int(A))$, by Lemma 1.1(c) and Lemma 1.2, if and only if $A \in RC(X, \tau)$. Conversely, suppose $R_\tau C(X, \tau) = RC(X, \tau)$. Since $X$ is regular closed, $X$ is regular $I$-closed and so $X = (int(X))^* = X^*$ which implies that $I$ is codense, by Lemma 1.1(b). ■

**Theorem 2.5.** Let $(X, \tau, I)$ be an ideal space. Then $I$ is codense if and only if $A_\tau(X, \tau) \subset A(X, \tau)$ where $A(X, \tau)$ is the collection of all $A$-sets in $(X, \tau)$.

*Proof.* Suppose $I$ is codense. $A_\tau(X, \tau) \subset A(X, \tau)$ by [11, Proposition 5(b)]. On the other hand, $A \in A(X, \tau)$ implies that $A = U \cap V$ where $U \in \tau$ and $V \in RC(X, \tau)$ and so $A = U \cap V$ where $U \in \tau$ and $V \in R_\tau C(X, \tau)$, by Theorem 2.4. So, $A \in A_\tau(X, \tau)$. Hence $A_\tau(X, \tau) = A(X, \tau)$. Conversely, suppose $A_\tau(X, \tau) = A(X, \tau)$. Since $X$ is an $A$-set, $X$ is an $A_\tau$-set and so $X \subset X^*$, by Theorem 2.1(a). Therefore $X = X^*$ which implies that $I$ is codense. ■

A function $f:(X, \tau, I) \rightarrow (Y, \sigma)$ is said to be $A_\tau-continuous$ [11] (resp. $A$-continuous [18]) if $f^{-1}(V)$ is an $A_\tau$-set (resp. $A$-set) in $X$ for every open set $V$ in $Y$. Every $A_\tau$-continuous function is $A$-continuous [11, Proposition 7(c)] but not
the converse [11, Example 6(3)]. The following Theorem 2.6 shows that the two concepts are equivalent, if the ideal $I$ is codense.

**Theorem 2.6.** Let $f: (X, \tau, I) \rightarrow (Y, \sigma)$ be a mapping and $I$ be codense. Then $f$ is $A_I$-continuous if and only if $f$ is $A$-continuous.

**Proof.** The proof follows from Theorem 2.5. □

Every $A_I$-set is $I$-locally closed [11, Proposition 5(a)] but not the converse [11, Example 5(2)]. The following Theorem 2.7 shows that every $A_I$-set is an $f_I$-set and characterizes $A_I$-set in terms of $f_I$-set and $I$-locally closed set. Example 2.8 below shows that $f_I$-sets need not be $A_I$-sets.

**Theorem 2.7.** Let $(X, \tau, I)$ be an ideal space. Then $A$ is an $A_I$-set if and only if $A$ is both an $f_I$-set and an $I$-locally closed set.

**Proof.** Suppose $A$ is an $A_I$-set. Then $A$ is $I$-locally closed by [11, Proposition 5(a)]. Also, $A = U \cap V$ where $U \in \tau$ and $V \in R_I C(X, \tau)$ and so $int(A) = int(U \cap V) = U \cap int(V)$. Now $A = U \cap V$ implies that $A = U \cap (int(V))^* \subset (U \cap int(V))^* = (int(A))^*$. Therefore, $A$ is an $f_I$-set. Conversely, suppose $A$ is both an $f_I$-set and an $I$-locally closed set. Since $A$ is an $f_I$-set, $A \subset (int(A))^*$ implies that $A^* \subset ((int(A))^*)^* \subset (int(A))^* \subset A^*$ and so $A^* = (int(A))^*$. As in the proof of Theorem 2.1(h), we can prove that $A^*$ is regular $I$-closed. $A$ is $I$-locally closed implies that $A = U \cap A^*$ for some $U \in \tau$, by Lemma 1.3. Since $A^*$ is regular $I$-closed, it follows that $A$ is an $A_I$-set. □

**Example 2.8.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{c\}\}$. If $A = \{a, c\}$, then $(int(A))^* = \{a, c, d\}$ and so $A$ is an $f_I$-set. Since $X$ is the only open containing $A$ and $A$ is not regular $I$-closed, $A$ is not an $A_I$-set.

A function $f: (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be $f_I$-continuous [12] (resp. $ILC$-continuous [3], semicontinuous [14], $LC$-continuous [5]) if $f^{-1}(V)$ is an $f_I$-set (resp. $I$-locally closed set, semipopen set, locally closed set) in $X$ for every open set $V$ in $Y$. Every $A_I$-continuous function is $ILC$-continuous [11, Proposition 7(b)] but not the converse [11, Example 6(2)]. The following Theorem 2.9 shows that the converse is true, if $f$ is $f_I$-continuous and hence we have a decomposition of $A_I$-continuous functions.

**Theorem 2.9.** A function $f: (X, \tau, I) \rightarrow (Y, \sigma)$ is $A_I$-continuous if and only if $f$ is both $f_I$-continuous and $ILC$-continuous.

**Proof.** The proof follows from Theorem 2.7. □

**Theorem 2.10.** Let $f: (X, \tau, I) \rightarrow (Y, \sigma)$ be a mapping and $I$ be codense. Then the following are equivalent.

(a) $f$ is $A$-continuous.
(b) $f$ is $A_I$-continuous.
(c) $f$ is both $f_I$-continuous and $ILC$-continuous.
(d) $f$ is both semicontinuous and $LC$-continuous.
Proof. (a) and (b) are equivalent, by Theorem 2.5. (b) and (c) are equivalent, by Theorem 2.9. (d) and (a) are equivalent by [5, Theorem 4(i)]. The proof will be over, if we prove (c) implies (d). From Lemma 1.3, it follows that every \( \mathcal{I} \)-locally closed set in \((X, \tau, \mathcal{I})\) is locally closed. Suppose \( A \) is an \( f_{\mathcal{I}} \)-set. Then \( A \subset (\text{int}(A))^* = \text{cl}(\text{int}(A)) \) and so \( A \in SO(X, \tau) \). This completes the proof. ■

The following Theorem 2.11 shows that the three collection of sets namely, \( f_{\mathcal{I}} \)-sets, \( \mathcal{A}_I \)-sets and \( \mathcal{I} \)-locally closed sets coincide for the collection of open sets. The Example 2.12 below show that the condition open cannot be dropped.

**Theorem 2.11.** Let \((X, \tau, \mathcal{I})\) be an ideal space and \( A \subset X \) be open. Then the following hold.

(a) \( A \) is an \( f_{\mathcal{I}} \)-set if and only if \( A \) is an \( \mathcal{A}_I \)-set.

(b) \( A \) is an \( \mathcal{I} \)-locally closed set if and only if \( A \) is an \( \mathcal{A}_I \)-set.

Proof. (a) Suppose \( A \) is an open, \( f_{\mathcal{I}} \)-set. Then \( A \subset (\text{int}(A))^* \subset A^* \) and so \( A^* = (\text{int}(A))^* \) which implies that \( A^* \) is regular \( \mathcal{I} \)-closed. Since \( A = A \cap A^* \), it follows that \( A \) is an \( \mathcal{A}_I \)-set. Conversely, if \( A \) is an \( \mathcal{A}_I \)-set, by Theorem 2.7, \( A \) is an \( f_{\mathcal{I}} \)-set.

(b) Suppose \( A \) is an open, \( \mathcal{I} \)-locally closed set. Then \( A = G \cap A^* \) for some \( G \in \tau \), by Lemma 1.3 and so \( A \subset A^* \). We prove that \( A^* \) is regular \( \mathcal{I} \)-closed. Since \( \text{int}(A^*) \subset A^* \), we have \( (\text{int}(A^*))^* \subset (A^*)^* \subset A^* \). Therefore, \( (\text{int}(A^*))^* \subset A^* \). On the other hand, since \( A \) is open and \(*\)-dense in itself, \( A^* = (\text{int}(A))^* \subset (\text{int}(A^*))^* \) and so \( A^* = (\text{int}(A^*))^* \) which implies that \( A^* \) is regular \( \mathcal{I} \)-closed. Therefore, \( A \) is an \( \mathcal{A}_I \)-set. Conversely, if \( A \) is an \( \mathcal{A}_I \)-set, by Theorem 2.7, \( A \) is an \( \mathcal{I} \)-locally closed set. ■

**Example 2.12.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\} \) and \( \mathcal{I} = \{\emptyset, \{c\}\} \). If \( A = \{a, c\} \), then \( \text{int}(A) = \{a\} \) and \( (\text{int}(A))^* = \{a, c\} \) and so \( A \) is an \( f_{\mathcal{I}} \)-set which is not open. Since \( X \) is the only open set containing \( A \) and \( \mathcal{R}_X C(X, \tau) = \{\emptyset, \{a, c, d\}, \{b, c, d\}, X\} \), \( A \) is not an \( \mathcal{A}_I \)-set. If \( B = \{d\} \), then \( B^* = \{c, d\} \) and \( B = \{a, b, d\} \cap B^* \) and so \( B \) is \( \mathcal{I} \)-locally closed which is not open. Since \( X \) and \( \{a, b, d\} \) are the only open sets containing \( B \), it follows that \( B \) is not an \( \mathcal{A}_I \)-set.

### 3. \( \mathcal{I} \)-locally closed sets

In this section, we characterize codense ideals in terms of \( \mathcal{I} \)-locally closed sets. Theorem 3.3 gives a decomposition of continuity. We deduce some results established in [5] as corollaries to Theorem 3.3.

**Theorem 3.1.** Let \((X, \tau, \mathcal{I})\) be an ideal space. Then \( \mathcal{ILC}(X, \tau) \cap \mathcal{I} = \{\emptyset\} \).

Proof. Let \( A \in \mathcal{ILC}(X, \tau) \). Then by Lemma 1.3, \( A \subset A^* \) and so by Lemma 1.4, \( \mathcal{ILC}(X, \tau) \cap \mathcal{I} = \{\emptyset\} \). ■

**Theorem 3.2.** Let \((X, \tau, \mathcal{I})\) be an ideal space. Then the following are equivalent.


Some subsets of ideal topological spaces

(a) $I$ is codense.
(b) $\tau = P_2O(X, \tau) \cap ILC(X, \tau)$.
(c) $\tau = \alpha_I O(X, \tau) \cap ILC(X, \tau)$.
(d) $\tau \subset ILC(X, \tau)$.

**Proof.** (a) implies (b) follows from [3, Proposition 4.1]. (b) implies (d) and (c) implies (d) are clear. (d) implies (a) follows from Theorem 3.1. Therefore, the proof will be over, if we prove (a) implies (c). Suppose $I$ is codense. If $A$ is open, then $A$ is $\alpha - I$-open and $A \subset A^*$. By Lemma 1.3, it follows that $A$ is $I$-locally closed. Conversely, suppose $A$ is both $\alpha - I$-open and $I$-locally closed. $A$ is $I$-locally closed implies $A = U \cap A^*$ for some open set $U$. $A$ is $\alpha - I$-open implies $A \subset int(cl^*(int(A))) \subset int(cl^*(U \cap A^*)) \subset int(cl^*(A^*)) = int(A^*)$. Since $A \subset U$, $A \subset U \cap int(A^*) = int(U \cap A^*) = int(A)$ and so $A$ is open. This completes the proof of the theorem.

A function $f: (X, \tau, I) \to (Y, \sigma)$ is said to be $\alpha - I$-continuous [6] (resp. pre-$I$-continuous [3], $\alpha$-continuous [16], pre-continuous [15]) if $f^{-1}(V)$ is an $\alpha - I$-open (resp. pre-$I$-open, $\alpha$-open, preopen) set in $X$ for every open set $V$ in $Y$. The following Theorem 3.3, which is a decomposition of continuous function in ideal topological spaces, follows from Theorem 3.2.

**Theorem 3.3.** Let $f: (X, \tau, I) \to (Y, \sigma)$ be a mapping and $I$ be codense. Then the following are equivalent.
(a) $f$ is continuous.
(b) $f$ is $\alpha$-$I$-continuous and $ILC$-continuous.
(c) $f$ is pre-$I$-continuous and $ILC$-continuous [3, Theorem 4.3].

**Corollary 3.4.** Let $f: (X, \tau) \to (Y, \sigma)$ be a mapping. Then the following are equivalent.
(a) $f$ is continuous.
(b) $f$ is $\alpha$-continuous and $L$-continuous [5, Theorem 4(ii)].
(c) $f$ is pre-continuous and $L$-continuous [5, Theorem 4(iv)].

**Proof.** Suppose $I = \emptyset$ in Theorem 3.3. If $I = \emptyset$, then $\alpha$-open sets coincide with $\alpha - I$-open sets, preopen sets coincide with pre-$I$-open sets and locally closed sets coincide with $I$-locally closed sets. Hence the proof follows from Theorem 3.3.

**Corollary 3.5.** Let $f: (X, \tau) \to (Y, \sigma)$ be a mapping. Then the following are equivalent.
(a) $f$ is continuous.
(b) $f$ is $\alpha$-continuous and $A$-continuous.
(c) $f$ is pre-continuous and $A$-continuous [5, Theorem 4(v)].

**Proof.** Suppose $I = \mathcal{N}$ in Theorem 3.3. If $I = \mathcal{N}$, then $\alpha$-open sets coincide with $\alpha - I$-open sets, preopen sets coincide with pre-$I$-open sets and $A$-sets coincide with $I$-locally closed sets [3]. Hence the proof follows from Theorem 3.3.
4. Almost strong $\mathcal{I}$-open sets

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be almost strong $\mathcal{I}$-open [7] if $A \subset \text{cl}^*(\text{int}(A^*))$. Every $\mathcal{I}$-open set is an almost strong $\mathcal{I}$-open set but not the converse [7]. We will denote the family of all almost strong $\mathcal{I}$-open sets by $\text{asIO}(X, \tau)$. The following Theorem 4.1 gives some properties of almost strong $\mathcal{I}$-open sets.

**Theorem 4.1.** (i) If $A$ is an almost strong $\mathcal{I}$-open set of an ideal space $(X, \tau, \mathcal{I})$, then the following hold:

(a) $A$ is $*$-dense in itself.
(b) $A^* = \text{cl}(A) = \text{cl}^*(A)$.
(c) $A^*(\mathcal{I}) = (\text{cl}^*(\text{int}(A^*))^* = (\text{cl}(\text{int}(A^*))^* = (A^*)^* = (A^*(\mathcal{I}))^*(\mathcal{I})$.
(d) $A^*$ is $*$-perfect, regular closed and $\mathcal{I}$-locally closed.
(e) $A^* = A^*(\mathcal{I})$.
(f) $(\text{cl}^*(\text{int}(A^*)))^*$ is $*$-perfect and $\mathcal{I}$-locally closed.

(ii) In any ideal space $(X, \tau, \mathcal{I})$, $\text{asIO}(X, \tau) \cap \{0\}$.

**Proof.** (i)(a) Since $A \subset \text{cl}^*(\text{int}(A^*)) \subset \text{cl}(\text{int}(A^*)) \subset \text{cl}(A^*) = A^*$, $A$ is $*$-dense in itself.

(b) Follows from Lemma 1.2 and (a).

(c) Follows from the inequality in (a) and the fact that $\text{cl}(\text{int}(A^*)) = A^*(\mathcal{I})$ [10, Theorem 3.2].

(d) $A^*$ is $*$-perfect by (c) and hence it is $\mathcal{I}$-locally closed. Since every almost strong $\mathcal{I}$-open set is $\beta$-open [7] and the closure of a $\beta$-open set is regular closed, by (b), $A^*$ is regular closed.

(e) Since $A^*$ is regular closed, by (d), we have $A^* = \text{cl}(\text{int}(A^*)) = A^*(\mathcal{I})$.

(f) Since $A \subset \text{cl}^*(\text{int}(A^*)) \subset A^*$, it follows that $\text{cl}^*(\text{int}(A^*))$ is $*$-perfect and hence it is $\mathcal{I}$-locally closed.

(ii) Follows from Lemma 1.4. ■

**Corollary 4.2.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be codense. If $A \subset X$ is almost strong $\mathcal{I}$-open, then $A^*$ is regular $\mathcal{I}$-closed and an $f_\mathcal{I}$-set. Moreover, if $A$ is $*$-closed, then $A$ is regular $\mathcal{I}$-closed and an $f_\mathcal{I}$-set.

**Proof.** By Theorem 4.1(d), $A^*$ is regular closed. By Theorem 2.4, $A^*$ is regular $\mathcal{I}$-closed and hence an $f_\mathcal{I}$-set. If $A$ is $*$-closed, then $A$ is $*$-perfect and so $A = A^*$. Therefore, $A$ is regular $\mathcal{I}$-closed and hence $A$ is an $f_\mathcal{I}$-set. ■

The following Theorem 4.3 gives a characterization of codense ideals in terms of almost strong $\mathcal{I}$-open sets. Theorem 4.4 below gives another property of almost strong $\mathcal{I}$-open sets.

**Theorem 4.3.** Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $\text{SI}(X, \tau) \subset \text{asIO}(X, \tau)$.

**Proof.** Suppose $\mathcal{I}$ is codense. If $A \in \text{SI}(X, \tau)$, then $A \subset \text{cl}^*(\text{int}(A)) \subset \text{cl}(\text{int}(A))$ and so $A$ is semiopen. By Lemma 1.1(d), $A \subset A^*$. Therefore, $A \subset
such that $B \subset \text{int} \cup \text{int} \cap \text{cl} \text{int} \star \text{cl}((\text{int}(A^*)))$. Since $\tau \subset S\tau O(X, \tau) \subset as\tau O(X, \tau)$, by Theorem 4.1(ii), $\tau \cap \mathcal{I} = \{\emptyset\}$ and so $\mathcal{I}$ is
codense. ■

**Theorem 4.4.** Let $(X, \tau, \mathcal{I})$ be an ideal space and $A$ and $B$ be subsets of $X$
such that $A \subset B \subset A^*$. If $A$ is almost strong $\mathcal{I}$-open, then $B$ is almost strong
$\mathcal{I}$-open and so $\text{cl}^*(\text{int}(A^*))$ is almost strong $\mathcal{I}$-open.

Proof. If $A \subset B \subset A^*$, then $A^* \subset B^* \subset (A^*)^* \subset A^*$ and so $A^* = B^*$ which
implies that $B$ is $\star$-dense in itself and $B^*$ is $\star$-perfect. If $A \in as\tau O(X, \tau)$, then
$A \subset \text{cl}^*(\text{int}(A^*)) = \text{cl}^*(\text{int}(B^*))$. Now $B \subset A^*$ implies $B \subset (\text{cl}^*(\text{int}(B^*)))^* \subset$
$\text{cl}^*(\text{cl}^*(\text{int}(B^*))) = \text{cl}^*(\text{int}(B^*))$ and so $B$ is an almost strong $\mathcal{I}$-open set. Since
$A \subset \text{cl}^*(\text{int}(A^*)) \subset A^*$, $\text{cl}^*(\text{int}(A^*))$ is an almost strong $\mathcal{I}$-open set. ■

We define the *almost strong $\mathcal{I}$-interior* of any subset $A$ of $X$ as the largest
almost strong $\mathcal{I}$-open set contained in $A$ and denote it by $as\text{int}(A)$. The following
Theorem 4.5 deals with the almost strong $\mathcal{I}$-interior of subsets of $X$. Moreover, Theorem 4.5(b)
is a generalization of Theorem 4.1(4) of [10].

**Theorem 4.5.** Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following hold.
(a) $as\text{int}(A) = A \cap \text{cl}^*(\text{int}(A^*))$.
(b) $as\text{int}(A) = \emptyset$ if and only if $A \in \mathcal{I}$.

Proof. (a) $A \cap \text{cl}^*(\text{int}(A^*)) \subset \text{cl}^*(\text{int}(A^*)) = \text{cl}^*(\text{int}(\text{int}(A^*))) = \text{cl}^*(\text{int}(A^* \cap$
$\text{int}(A^*))) \subset \text{cl}^*(\text{int}(A \cap \text{int}(A^*))^*) \subset \text{cl}^*(\text{int}(A \cap \text{cl}^*(\text{int}(A^*)))^*)$. Therefore,$
A \cap \text{cl}^*(\text{int}(A^*))$ is an almost strong $\mathcal{I}$-open set contained in $A$. Hence $A \cap$
$\text{cl}^*(\text{int}(A^*)) \subset as\text{int}(A)$. Since $as\text{int}(A)$ is almost strong $\mathcal{I}$-open,$as\text{int}(A) \subset$
$\text{cl}^*(\text{int}(as\text{int}(A))^*) \subset \text{cl}^*(\text{int}(A^*))$ and so $A \cap as\text{int}(A) \subset A \cap \text{cl}^*(\text{int}(A^*))$
which implies that $as\text{int}(A) \subset A \cap \text{cl}^*(\text{int}(A^*))$. Therefore, $as\text{int}(A) =$
$A \cap \text{cl}^*(\text{int}(A^*))$.

(b) $as\text{int}(A) = \emptyset$ implies $A \cap \text{cl}^*(\text{int}(A^*)) = \emptyset$ implies $A \cap \text{int}(A^*) = \emptyset$
implies $\text{int}(A) = \emptyset$ implies $A \in \mathcal{I}$, by [10, Theorem 4.1(4)]. Conversely, $A \in \mathcal{I}$
implies $\text{int}(A) = \emptyset$ implies $\text{cl}^*(\text{int}(A^*)) = \emptyset$ implies $A \cap \text{cl}^*(\text{int}(A^*)) = \emptyset$
implies $as\text{int}(A) = \emptyset$. ■

In [7], it is established that the intersection of an almost strong $\mathcal{I}$-open set
with an open set is always an almost strong $\mathcal{I}$-open set. The following
Theorem 4.6 shows that, in the above result, open set can be replaced by $\alpha$-$\mathcal{I}$-open set.

**Theorem 4.6.** Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A$ is almost strong $\mathcal{I}$-open
and $B$ is $\alpha$-$\mathcal{I}$-open, then $A \cap B$ is almost strong $\mathcal{I}$-open.

Proof. $A$ is almost strong $\mathcal{I}$-open implies $A \subset \text{cl}^*(\text{int}(A^*))$ and $B$ is $\alpha$-$\mathcal{I}$-open
implies $B \subset \text{int}(\text{cl}^*(\text{int}(B)))$. Now, $A \cap B \subset \text{cl}^*(\text{int}(A^*)) \cap \text{int}(\text{cl}^*(\text{int}(B)))$
$= (\text{int}(A^*) \cup (\text{int}(A^*))^* \cap \text{int}(\text{cl}^*(\text{int}(B))) = (\text{int}(A^*) \cap \text{int}(\text{cl}^*(\text{int}(B)))) \cup (\text{int}(A^*)^* \cap \text{int}(\text{cl}^*(\text{int}(B))) \subset \text{int}(\text{cl}^*(\text{int}(A^*) \cap \text{int}(B))) \cup (\text{int}(\text{cl}(\text{int}(A^*) \cap \text{int}(B))))^* \subset \text{int}(\text{cl}^*(\text{int}(A^*) \cap \text{int}(B)))^* \subset \text{int}(\text{cl}^*(\text{int}(A^*) \cap \text{int}(B)))^* \cup (\text{int}(\text{cl}^*(\text{int}(A^* \cap \text{int}(B))))^*)$. Hence, $A \cap B$ is almost strong $\mathcal{I}$-open.
\[ int(B) \supseteq int(cl^*(int((A \cap int(B))^*)) \cup (int(cl^*(int((A \cap int(B))^*)) = cl^*(int(cl^*(int((A \cap int(B))^*)) \subseteq cl^*(int((A \cap B)^*)). \text{ Therefore, } A \cap B \text{ is almost strong } I \text{-open.} \]

**Acknowledgement.** The authors sincerely thank Professor T. Noiri (Japan) for sending some of his reprints which were useful for the preparation of this manuscript and the referee for the valuable comments for a better presentation of the paper.

**References**


