RELATIONS BETWEEN SOME TOPOLOGIES

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Abstract. Generalizations of openness, such as semi-open, preopen, semi-pre-open, α-open, etc. are important in topological spaces and in particular in topological spaces on which ideals are defined. α-equivalent topologies and ∗-equivalent topologies with respect to an ideal have some common properties. Relations between these aforementioned notions of openness are investigated within the framework of α-equivalence and ∗-equivalence.

1. Introduction

The subject of ideals in general topological spaces was introduced by Kuratowski [8] and Vaidyanathaswamy [18]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies
(i) If A ∈ I and B ⊂ A, then B ∈ I,
(ii) If A ∈ I and B ∈ I, then A ∪ B ∈ I.

By (X, τ, I) we will denote a topological space (X, τ) with an ideal I on X. No separation properties are assumed on X. For a space (X, τ, I) and a subset A ⊂ X,

\[A^*(τ, I) = \{ x ∈ X : U ∩ A \notin I \text{ for every } U ∈ τ(x) \}\]

(where τ(x) = \{U ∈ τ : x ∈ U\}) is called the local function of A with respect to I and τ [8]. Note that cl^* (A) = A ∪ A^* defines a Kuratowski closure operator for a topology τ^* (I) [15] on X. If there is no chance of confusion, we simply write A^* or A^* (I) instead of A^* (τ, I), and τ^* instead of τ^* (I).

If I and J are ideals on X, then I ∨ J = \{I ∪ J : I ∈ I and J ∈ J\} is also an ideal on X [6].

In a topological space (X, τ), for any subset A, A^o, int A or τ int A will stand for the interior of A and A̅, cl A, or τ cl A will stand for the closure of A. A subset A of a space (X, τ) is said to be semi-open (pre-open, α-open, semi-pre-open, regular open, nowhere dense, codense) if A ⊂ A^o (A ⊂ A^α, A ⊂ A^s, A ⊂ A^2, A = A^α, A^2 = ∅, A^o = ∅), respectively.

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A point \( x \) of \( X \) is called a \( \theta \)-interior point of \( A \) if there exists an open set \( U \) such that \( x \in U \subset \overline{U} \subset A \). Also, \( \theta \)-int \( A \) will stand for the set of \( \theta \)-interior points of \( A \). \( A \) is \( \theta \)-open iff \( A \subset \theta \)-int \( A \) [10]. The family of all \( \alpha \)-open sets in \((X, \tau)\) is a topology on \( X \) which is finer than \( \tau \) and it is denoted by \( \tau^\alpha \). Topologies \( \tau \) and \( \sigma \) on \( X \) are called \( \alpha \)-equivalent if they have the same \( \alpha \)-open sets [14].

A supratopology \( \mathcal{A} \) on \( X \) is a nonempty collection of subsets of \( X \) which satisfies

(i) \( \emptyset \in \mathcal{A}, \ X \in \mathcal{A} \),

(ii) \( \mathcal{A} \) is closed under arbitrary unions [10].

The \( \mathcal{A} \)-interior (shortly, \( \mathcal{A} \)-int) of a subset \( A \) of \( X \) is defined as 

\[
\mathcal{A} \text{-int} \ A = \bigcup \{ U : U \subset A, U \in \mathcal{A} \} \]

[9]. It is well known that the family of semi-open (pre-open, semi-pre-open) sets of a topological space is a supratopology on this space. If \( \mathcal{A} \) is a supratopology on \( X \), then

\[
\mathcal{T}_\mathcal{A} = \{ A \subset X : A \cap B \in \mathcal{A} \text{ for each } B \in \mathcal{A} \}
\]

is a topology and \( \mathcal{T}_\mathcal{A} \subset \mathcal{A} \) [19].

We will use the following notational conventions:

- \( \tau \in \text{Top}(X) \iff \tau \) is a topology on \( X \),
- \( I \in \text{Id}(X) \iff I \) is an ideal on \( X \),
- \( A \in \text{D}(X) \iff A \) is dense in \( X \),
- \( A \in \text{CD}(X) \iff A^\circ = \emptyset \) (i.e. \( A \) is codense),
- \( A \in \text{NO}(X) \iff A^\sharp = \emptyset \) (i.e. \( A \) is nowhere dense),
- \( A \in \text{SO}(X) \iff A \subset A^\sharp \),
- \( A \in \text{PO}(X) \iff A \subset A^\circ \),
- \( A \in \alpha \text{O}(X) \iff A \subset A^\sharp \),
- \( A \in \text{SPO}(X) \iff A \subset A^\sharp \),
- \( A \in \text{RO}(X) \iff A = A^\sharp \),
- \( A \in \theta \text{O}(X) \iff A \) is \( \theta \)-open,
- \( A \in \text{SC}(X) \iff X - A \) is semi-open (i.e. \( A \) is semi-closed),
- \( A \in \text{SR}(X) \iff A \) is semi-open and semi-closed (i.e. \( A \) is semi-regular),
- \( \sigma \in \tau^\alpha \iff \sigma \in \text{Top}(X) \), and \( \sigma^\alpha = \tau^\alpha \) (i.e. \( \tau \) and \( \sigma \) are \( \alpha \)-equivalent).

\( I_n \) (or \( I_n(\tau) \)) and \( I_n(\sigma) \) will stand for the family of nowhere dense sets in \( X \) with respect to \( \tau \) and \( \sigma \), respectively. From now on, \( A^\circ \) and \( \overline{A} \) will be reserved for the interior and closure of \( A \) with respect to topology \( \tau \), respectively.

In a topological space scl, sint, pcl, spcl, etc. will stand for the operations semi-closure, semi-interior, pre-closure, semi-pre-closure, respectively. Where it is necessary to indicate the topology, we will write, for example, \( \tau \)-scl or \( \sigma \)-scl.
In the following theorem we recall some known results in the literature which will be used in this paper. They appear in [1, 2, 5, 6, 13–18].

**Theorem 1.1.** In any topological space $(X, \tau)$ we have the following results:

1. For any subset $A$ of $X$,
   
   $\text{scl} A = A \cup A^\omega,$

   $\text{sint} A = A \cap A^\omega,$

   $\text{pcl} A = A \cup A^\pi,$

   $\text{pint} A = A \cap A^\omega,$

   $\text{spcl} A = A \cup A^{\omega},$

   $\alpha\text{-cl} A = \tau^\alpha \text{cl} A = A \cup A^\omega,$

   $\alpha\text{-int} A = \tau^\alpha \text{int} A = A \cap A^\omega$ [1, 2].

2. $\tau \subset \tau_{SO(X)} \cap \tau_{PO(X)} \cap \tau_{SPO(X)}$ [1, 2].


In the remaining results given below, $\sigma \in \text{Top}(X)$ and $\mathcal{I}, \mathcal{J} \in \text{Id}(X)$.

3. If $\tau \subset \sigma \subset \tau^\omega$, then $\sigma \in [\tau]^\omega$ [14].

4. $\tau \subset \tau^*(\mathcal{I})$ and $(\tau^*(\mathcal{I}))^\sim = \tau^*(\mathcal{I})$.

5. $A^*(\tau, \mathcal{I})$ is $\tau$-closed and $A^*(\tau, \mathcal{I}) = A^*(\tau^*(\mathcal{I}), \mathcal{I})$ for each $A \subset X$,

6. $I \in \mathcal{I} \implies I$ is $\tau^*(\mathcal{I})$-closed and $I^* = \emptyset$.

7. $\mathcal{I} \subset \mathcal{J} \implies A^*(\tau, \mathcal{J}) \subset A^*(\tau, \mathcal{I})$ for each $A \subset X$ and $\tau^*(\mathcal{I}) \subset \tau^*(\mathcal{J})$.

8. $\tau \subset \sigma \implies A^*(\sigma, \mathcal{I}) \subset A^*(\tau, \mathcal{I})$ for each $A \subset X$ and $\tau^*(\mathcal{I}) \subset \sigma^*(\mathcal{I})$.

9. $\tau \cap \mathcal{I} = \{\emptyset\} \iff \tau^*(\mathcal{I}) \cap \mathcal{I} = \{\emptyset\} \iff X = X^* \iff U \subset U^*$ for each $U \in \tau$.

10. If $\tau \cap \mathcal{I} = \emptyset$, then for each $U \in \tau$ and for each $I \in \mathcal{I}$, we have

    $U^* = (U - I^*)^* = (U - I)^* = \tau^*\text{-cl}(U - I) = \tau^*\text{-cl} U$.

11. $\tau^*(\mathcal{I} \cup \mathcal{J}) = (\tau^*(\mathcal{I}))^\sim(\mathcal{J})$ [6].

12. $\tau^*(\mathcal{I}_n) = \tau^\omega, \tau \cap \mathcal{I}_n = \{\emptyset\}, \text{PO}(X, \tau) \cap \mathcal{I}_n = \{\emptyset\}$.

13. $A \subset B \implies A^*(\mathcal{I}) \subset B^*(\mathcal{I})$.

14. $(A - I^*)^* = A^*$ for each $A \subset X$ and each $I \in \mathcal{I}$.

15. $A^*(\mathcal{I}_n) = A^\omega$ for each $A \subset X$.

16. If $\tau \cap \mathcal{I} = \emptyset$, then for each $U \in \tau$ we have $\tau\text{-cl} U = \tau^*(\mathcal{I})\text{-cl} U$.

2. Relations between topologies and some special sets

Firstly, some relations between families such as $SO(X), PO(X), SPO(X)$, etc., on a set $X$ with two topologies are investigated. Then, these relations will be carried over to topological spaces on which ideals are defined. Some known results will be obtained by a different method.

**Theorem 2.1.** Let $\tau, \sigma, \omega \in \text{Top}(X)$. Then we have the following results.

1. If $\tau \subset \sigma \subset SPO(X, \tau)$, then: (a) $SPO(X, \sigma) \subset SPO(X, \tau)$, (b) $I_n(\tau) \subset I_n(\sigma)$.

2. If $\tau \subset \sigma \subset PO(X, \tau)$, then $PO(X, \sigma) \subset PO(X, \tau)$, and the relations (a) and (b) in (1) are valid.

3. If $\tau \subset \sigma$, and $\tau\text{-cl} U = \sigma\text{-cl} U$ for each $U \in \tau$, then
(a) For each \( A \subset X \) we have
\[
\tau \operatorname{cl}(\tau \operatorname{int} A) = \sigma \operatorname{cl}(\tau \operatorname{int} A) \subset \sigma \operatorname{cl}(\sigma \operatorname{int} A),
\]
\[
\sigma \operatorname{int}(\sigma \operatorname{cl} A) \subset \sigma \operatorname{int}(\tau \operatorname{cl} A) = \tau \operatorname{int}(\tau \operatorname{cl} A),
\]
\[
\tau \operatorname{int}(\tau \operatorname{cl}(\tau \operatorname{int} A)) \subset \tau \operatorname{int}(\sigma \operatorname{cl}(\sigma \operatorname{int} A)) \subset \sigma \operatorname{int}(\sigma \operatorname{cl}(\sigma \operatorname{int} A)),
\]
\[
\sigma \operatorname{cl}(\sigma \operatorname{int}(\sigma \operatorname{cl} A)) \subset \sigma \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} A)) = \tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} A)).
\]

(b) For each \( U \in SO(X, \tau) \), we have \( \tau \operatorname{cl} U = \sigma \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\sigma \operatorname{int} U) \),
\[
\begin{align*}
\text{(c)} & \quad SO(X, \tau) \subset SO(X, \sigma), \\
\text{(d)} & \quad PO(X, \sigma) \subset PO(X, \tau), \\
\text{(e)} & \quad SPO(X, \sigma) \subset SPO(X, \tau), \\
\text{(f)} & \quad \alpha O(X, \tau) \subset \alpha O(X, \sigma), \\
\text{(g)} & \quad \mathcal{I}_n(\tau) \subset \mathcal{I}_n(\sigma), \\
\text{(h)} & \quad RO(X, \tau) \subset RO(X, \sigma).
\end{align*}
\]

II. If \( \tau \subset \sigma \), and \( \tau \operatorname{cl} U = \sigma \operatorname{cl} U \) for each \( U \in \sigma \), then in addition to the results given in (3.I) above, we have \( RO(X, \tau) = RO(X, \sigma) \), \( SR(X, \tau) = SR(X, \sigma) \), and \( \theta O(X, \tau) = \theta O(X, \sigma) \),
\[
\begin{align*}
\text{(4)} & \quad \text{If } \tau \subset \sigma \subset SO(X, \tau) \text{ and } \tau \operatorname{cl} U = \sigma \operatorname{cl} U \text{ for each } U \in \tau, \text{ then } \tau \operatorname{cl} U = \sigma \operatorname{cl} U \text{ for each } U \in \sigma. \text{ So, the results in (3) are valid.}
\end{align*}
\]
\[
\begin{align*}
\text{(5)} & \quad \text{If } \tau \subset \omega \subset \sigma, \text{ and } \tau \operatorname{cl} U = \sigma \operatorname{cl} U \text{ for each } U \in \sigma, \text{ then } \tau \operatorname{cl} U = \sigma \operatorname{cl} U = \omega \operatorname{cl} U \text{ for each } U \in \sigma \text{ (hence for each } U \in \omega).}
\end{align*}
\]

So, results similar to those given in (3) are valid for \( \tau, \omega \) and \( \sigma \).

Proof. (1a) Let \( \tau \subset \sigma \subset SPO(X, \tau) \) and \( U \in SPO(X, \sigma) \). We have:
\[
U \subset \sigma \operatorname{cl}(\sigma \operatorname{int}(\sigma \operatorname{cl} U)) \subset \tau \operatorname{cl}(\sigma \operatorname{int}(\tau \operatorname{cl} U)) \subset \tau \operatorname{cl}(\tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} U)))
\]
\[
\subset \tau \operatorname{cl}(\tau \operatorname{int}(\tau \operatorname{cl} U)).
\]

(1b) The proof is clear from Corollary 2.13 below.

2. Let \( \tau \subset \sigma \subset PO(X, \tau) \) and \( U \in PO(X, \sigma) \). We have:
\[
U \subset \sigma \operatorname{int}(\sigma \operatorname{cl} U) \subset \sigma \operatorname{int}(\tau \operatorname{cl} U) \subset \sigma \operatorname{int}(\tau \operatorname{cl}(\sigma \operatorname{int}(\tau \operatorname{cl} U))) \subset \tau \operatorname{int}(\tau \operatorname{cl} U).
\]

So, \( U \in PO(X, \tau) \). Since \( PO(X, \tau) \subset SPO(X, \tau) \), the results in (1) are valid.

(3lb) Let \( U \in SO(X, \tau) \). Since \( U \subset \tau \operatorname{cl}(\tau \operatorname{int} U) \) and \( \sigma \operatorname{cl} U \subset \tau \operatorname{cl} U \), we have
\[
\sigma \operatorname{cl} U \subset \tau \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\tau \operatorname{int} U) \subset \sigma \operatorname{int}(\sigma \operatorname{cl} U) \subset \sigma \operatorname{cl} U.
\]

Hence we have \( \sigma \operatorname{cl} U = \tau \operatorname{cl} U = \tau \operatorname{cl}(\tau \operatorname{int} U) = \sigma \operatorname{cl}(\sigma \operatorname{int} U) \).

(3le-f) These are clear from (3la).

(3lg) This is clear from (1a) since \( \tau \subset \sigma \subset SPO(X, \tau) \).

(3lh) Let \( U \in RO(X, \tau) \). We have \( U = \tau \operatorname{int}(\tau \operatorname{cl} U) \) and \( U \in SO(X, \tau) \). Now if we use (3lb), we obtain that \( \sigma \operatorname{int}(\sigma \operatorname{cl} U) = \sigma \operatorname{int}(\tau \operatorname{cl} U) = \tau \operatorname{int}(\tau \operatorname{cl} U) = U \).
(3II) Under the hypothesis, since \( RO(X, \tau) = RO(X, \sigma)[11] \), it is clear that \( SR(X, \tau) = SR(X, \sigma) \) [7], and \( \theta O(X, \tau) = \theta O(X, \sigma)[12] \).

(4) The proof is clear from (3Ib).

**Corollary 2.2.** If \( \tau \cap \mathcal{I} = \emptyset \) for \( \tau \in \text{Top}(X) \) and \( \mathcal{I} \in \text{Id}(X) \), then

(a) The results (3) in Theorem 2.1. are valid by taking \( \tau^*(\mathcal{I}) \) instead of \( \sigma \).

(b) For \( \mathcal{J} \in \text{Id}(X) \) and \( \omega \in \text{Top}(X) \), if \( \omega \cap \mathcal{J} = \emptyset \) and \( \omega^*(\mathcal{J}) = \tau^*(\mathcal{I}) \), then \( (X, \tau), (X, \omega), (X, \omega^*(\mathcal{J})) \) and \( (X, \tau^*(\mathcal{I})) \) have the same \( RO(X), SR(X) \) and \( \theta O(X) \) sets.

(c) If \( \tau^*(\mathcal{I}) = \sigma^*(\mathcal{I}) \) for \( \sigma \in \text{Top}(X) \), then the results (3) in Theorem 2.1. are valid by taking \( \sigma^*(\mathcal{I}) \) instead of \( \sigma \), and then \( (X, \tau), (X, \tau^*(\mathcal{I}), (X, \sigma) \) and \( (X, \sigma^*(\mathcal{I})) \) have the same \( RO(X), SR(X) \) and \( \theta O(X) \) sets.

The following theorem and corollaries can be obtained by using Lemma 2.7 below and the results of Andrijević given in [1,2,3]. We note that Corollary 2.6.1 was given by Rose and Hamlett using a different method [16]. We will obtain these results by using the results given here.

**Theorem 2.3.** Let \( \tau, \sigma \in \text{Top}(X) \). If \( \tau \subset \sigma \subset \text{SO}(X, \tau) \), and \( \tau \text{ cl} U = \sigma \text{ cl} U \) for each \( U \in \sigma \), then we have the following results.

1. For each \( A \subset X \), we have
   (a) \( \tau \text{ int}(\tau \text{ cl} A) = \sigma \text{ int}(\sigma \text{ cl} A) \),
   (b) \( \tau \text{ cl}(\tau \text{ int} A) = \sigma \text{ cl}(\sigma \text{ int} A) \),
   (c) \( \tau \text{ cl}(\tau \text{ int} (\tau \text{ cl} A)) = \sigma \text{ cl}(\sigma \text{ int} (\sigma \text{ cl} A)) \),
   (d) \( \tau \text{ int}(\tau \text{ cl}(\tau \text{ int} A)) = \sigma \text{ int}(\sigma \text{ cl}(\sigma \text{ int} A)) \).

2. \( (X, \tau) \) and \( (X, \sigma) \) have the same \( \text{SO}(X), \text{PO}(X), \text{SPO}(X), \text{RO}(X), \text{SR}(X), \text{NO}(X), \text{D}(X), \alpha \text{O}(X), \text{CD}(X) \) and \( \theta O(X) \) sets.

**Proof.** (1a) Let \( A \subset X \). Then \( \sigma \text{ int}(\sigma \text{ cl} A) \subset \sigma \text{ int}(\tau \text{ cl} A) = \tau \text{ int}(\tau \text{ cl} A) \). Since \( \tau \text{ int}(\tau \text{ cl} A) \in \sigma \) and \( \tau \text{ int}(\tau \text{ cl} A) \subset A \cup \tau \text{ int}(\tau \text{ cl} A) = \tau \text{ scl} A \subset \sigma \text{ cl} A \), we have \( \tau \text{ int}(\tau \text{ cl} A) = \sigma \text{ int}(\tau \text{ int}(\tau \text{ cl} A)) \subset \sigma \text{ int}(\sigma \text{ cl} A) \). Hence \( \tau \text{ int}(\tau \text{ cl} A) = \sigma \text{ int}(\sigma \text{ cl} A) \).

The remaining proofs are clear. ■

**Corollary 2.4.** Let \( \tau \in \text{Top}(X), \mathcal{I} \in \text{Id}(X) \). If \( \tau \cap \mathcal{I} = \emptyset \) and \( \tau^*(\mathcal{I}) \subset \text{SO}(X, \tau) \), then the results of Theorem 2.3 are satisfied by taking \( \tau^*(\mathcal{I}) \) instead of \( \sigma \).

**Corollary 2.5.** Since \( \tau \cap \mathcal{I}_n = \emptyset \) and \( \tau^*(\mathcal{I}_n) = \tau^\alpha \subset \text{SO}(X, \tau) \), the results of the above theorem are satisfied by taking \( \tau^\alpha \) instead of \( \sigma \).

**Corollary 2.6.** If \( \sigma^\alpha = \tau^\alpha \) (i.e. if \( \sigma \in [\tau]^\alpha \) in the sense of Njastad [14]), then we have the following results.

1. For each \( A \subset X \), we have
   (a) \( \tau \text{ int}(\tau \text{ cl} A) = \tau^\alpha \text{ int}(\tau^\alpha \text{ cl} A) = \sigma^\alpha \text{ int}(\sigma^\alpha \text{ cl} A) = \sigma \text{ int}(\sigma \text{ cl} A) \),
   (b) \( \tau \text{ cl}(\tau \text{ int} A) = \tau^\alpha \text{ cl}(\tau^\alpha \text{ int} A) = \sigma^\alpha \text{ cl}(\sigma^\alpha \text{ int} A) = \sigma \text{ cl}(\sigma \text{ int} A) \),
   (c) \( \tau \text{ int}(\tau \text{ cl}(\tau \text{ int} A)) = \tau^\alpha \text{ int}(\tau^\alpha \text{ cl}(\tau^\alpha \text{ int} A)) = \sigma^\alpha \text{ int}(\sigma^\alpha \text{ cl}(\sigma^\alpha \text{ int} A)) \).
\( \tau \text{ cl}(\tau \text{ int}(\tau \text{ cl}A)) = \tau^\alpha \text{ cl}(\tau^\alpha \text{ int}A) = (\sigma^\alpha \text{ cl}A(\sigma^\alpha \text{ int}(\sigma^\alpha \text{ cl}A) = \\
\sigma \text{ cl}(\sigma \text{ int}(\sigma \text{ cl}A)). \)

(2) \( (X, \tau), (X, \tau^\alpha), (X, \sigma^\alpha) \text{ and } (X, \sigma) \) have the same \( \text{SO}(X), \text{PO}(X), \text{SPO}(X), \text{NO}(X), \text{D}(X), \text{CD}(X), \alpha \text{O}(X), \text{RO}(X), \text{SR}(X) \) and \( \theta \text{O}(X) \) sets \([3]\).

**Lemma 2.7.** Let \( \tau, \sigma \in \text{Top}(X) \) and \( \tau \subset \sigma \). Then, \( \sigma \subset \text{SO}(X, \tau) \) and \( \tau \text{ cl}U = \sigma \text{ cl}U \) for each \( U \in \sigma \) iff \( \sigma \in [\tau]^\alpha \).

Njåstad defined \( \alpha \)-equivalent topologies and \( * \)-equivalent topologies in \([14]\), \([15]\), respectively. Njåstad showed that if \( \tau \subset \sigma \subset \tau^\alpha \) for \( \tau, \sigma \in \text{Top}(X) \), then \( \tau \) and \( \sigma \) are \( \alpha \)-equivalent. For \( \tau, \sigma \in \text{Top}(X), I \in \text{Id}(X) \), if \( \tau^*(I) = \sigma^*(I) \), then we say that \( \sigma \) and \( \tau \) are \( *I \)-equivalent.

The \( \alpha \)-equivalence or \( *I \)-equivalence of topologies on a set on which ideals are defined is important.

For any ideal \( I \) on \( (X, \tau) \), \( \tau^*(I) \) and \( \tau^*(I)^\alpha \) are \( \alpha \)-equivalent. We know that \( \tau^*(I)^\alpha = (\tau^*(I)^\alpha)(\mathcal{I}_n(\tau^*(I)^\alpha)) = \tau^*(I \lor \mathcal{I}_n(\tau^*(I)^\alpha)). \) Hence, for each ideal \( J \) such that \( J \subset \mathcal{I}_n(\tau^*(I)^\alpha) \), \( \tau^*(I) \), \( \tau^*(I \lor J) \) and \( \tau^*(I \lor \mathcal{I}_n(\tau^*(I)^\alpha)) \) are \( \alpha \)-equivalent.

At the same time, for any ideal \( I \), since \( \tau^*(I) \cap \mathcal{I}_n(\tau^*(I)^\alpha) = \{\emptyset\} \) and \( \tau \subset \tau^*(I) \), we have that \( \tau \cap \mathcal{I}_n(\tau^*(I)^\alpha) = \{\emptyset\} \). Hence for any ideal \( J \) such that \( J \subset \mathcal{I}_n(\tau^*(I)) \) we have \( \tau \cap J = \{\emptyset\} \). And, if \( \tau \cap I = \{\emptyset\} \), then we know that \( \mathcal{I}_n(\tau) \subset \mathcal{I}_n(\tau^*(I)) \). Now, we can give the following result.

**Theorem 2.8.** Let \( I \) be an ideal on \( (X, \tau) \) and \( J \) any ideal such that \( J \subset \mathcal{I}_n(\tau^*(I)) \). Then the following are equivalent.

(a) \( \tau \cap I = \{\emptyset\} \)
(b) \( \mathcal{I} \subset \mathcal{I}_n(\tau^*(I)) \)
(c) \( \mathcal{I} \lor \mathcal{I}_n \subset \mathcal{I}_n(\tau^*(I)) \)
(d) \( \mathcal{I} \lor J \subset \mathcal{I}_n(\tau^*(I)) \)
(e) \( \mathcal{I} \lor \mathcal{I}_n \lor J \subset \mathcal{I}_n(\tau^*(I)) \)
(f) \( \tau \cap (\mathcal{I} \lor J) = \{\emptyset\} \)
(g) \( \tau \cap (\mathcal{I} \lor J \lor \mathcal{I}_n) = \{\emptyset\} \)
(h) \( \tau \cap (\mathcal{I} \lor \mathcal{I}_n) = \{\emptyset\} \).

**Proof.** (a) \( \Rightarrow \) (b) Let \( I \in \mathcal{I} \). Then \( I \) is \( \tau^*(I) \)-closed and since \( \tau^*(I) \cap \mathcal{I} = \{\emptyset\} \) we have that \( \tau^*(I) \)-int \( I = \emptyset \). So, \( \tau^*(I) \)-int(\( \tau^*(I) \)-cl \( I) = \emptyset \) and \( I \in \mathcal{I}_n(\tau^*(I)) \).

The remaining proofs are clear. \( \blacksquare \)

We deduce that if \( \tau \cap I = \{\emptyset\} \) for an ideal \( I \), then \( \mathcal{I} \lor \mathcal{I}_n(\tau^*(I)) = \mathcal{I}_n(\tau^*(I)) \), and \( (\tau^*(I))^\alpha = \tau^*(\mathcal{I}_n(\tau^*(I))). \)

**Corollary 2.9.** If \( \tau \cap I = \{\emptyset\} \), then for any ideal \( J \) satisfying \( J \subset \mathcal{I}_n(\tau^*(I)) \) we have that \( \tau^*(I), \tau^*(I \lor J), \tau^*(I \lor J \lor \mathcal{I}_n(\tau^*(I))) \) and \( \tau^*(\mathcal{I}_n(\tau^*(I))) \) are all \( \alpha \)-equivalent.

Several statements equivalent to \( \mathcal{I} \subset \mathcal{I}_n \) have been given in the literature. Since \( \tau \) and \( \tau^*(I) \) are \( \alpha \)-equivalent when \( \mathcal{I} \subset \mathcal{I}_n \), we give some further conditions for \( \mathcal{I} \subset \mathcal{I}_n \), in the following theorem.
Theorem 2.10. Let \( I \) be an ideal on \((X, \tau)\) and \( I_n \) the ideal of nowhere dense sets in \((X, \tau)\). Then the following are equivalent.

1. \( I \subset I_n \),
2. \( \tau \cap I = \{ \emptyset \} \) and \( \tau \) and \( \tau^\ast(I) \) are \( \alpha \)-equivalent,
3. \( \tau \cap I = \{ \emptyset \} \) and \( \tau^\ast(I) \subset \tau^\alpha \),
4. \( \tau \cap I = \{ \emptyset \} \) and \( \tau^\ast(I) \subset SO(X, \tau) \),
5. \( \tau \cap I = \{ \emptyset \} \) and \( SO(X, \tau^\ast(I)) \subset SO(X, \tau) \),
6. \( \tau \cap I = \{ \emptyset \} \) and \( PO(X, \tau) \subset PO(X, \tau^\ast(I)) \),
7. \( \tau \cap I = \{ \emptyset \} \) and \( SPO(X, \tau) \subset SPO(X, \tau^\ast(I)) \),
8. \( \tau \cap I = \{ \emptyset \} \) and \( D(X, \tau) \subset D(X, \tau^\ast(I)) \),
9. \( A^\ast(I_n) \subset A^\ast(I) \) for each \( A \subset X \),
10. \( A^\ast \subset A^\ast(I) \) for each \( A \subset D(X, \tau) \),
11. \( \tau \cap I = \{ \emptyset \} \) and \( A^\ast \subset \tau^\ast(I) \)-cl \( A \) for each \( A \subset X \),
12. \( \tau \cap I = \{ \emptyset \} \) and \( I \subset \sc(X, \tau) \).

Proof. (1) \( \implies \) (2) Let \( I \subset I_n \). We have, \( \tau \subset \tau^\ast(I) \subset \tau^\ast(I_n) = \tau^\alpha \), so \( \tau \) and \( \tau^\ast(I) \) are \( \alpha \)-equivalent from Theorem 1.1.(3).

(2) \( \implies \) (1) If \( \tau^\ast(I) \in [\tau]^\alpha \), then we have \( I_n(\tau) = I_n(\tau^\ast(I)) \). If \( \tau \cap I = \{ \emptyset \} \), then \( I \subset I_n(\tau^\ast(I)) \). Hence, \( I \subset I_n \), if \( \tau \cap I = \{ \emptyset \} \) and \( \tau^\ast(I) \in [\tau]^\alpha \).

(2) \( \iff \) (3) \( \iff \) (4) Clear.

(2) \( \iff \) (5) \( \iff \) (6) \( \iff \) (7) \( \iff \) (8) Clear from [3, Theorem 1] and Corollary 2.2

(1) \( \iff \) (9) Known from the literature.

(9) \( \implies \) (10) Clear.

(10) \( \implies \) (8) Since \( X \in D(X, \tau) \), we have \( X^\ast = X \subset X^\ast(I) \), \( X = X^\ast(I) \) and hence \( \tau \cap I = \{ \emptyset \} \). For \( A \in D(X, \tau) \) we deduce \( A = X, A^\ast = X, A^\ast(I) = X \) and \( \tau^\ast(I) \)-cl \( A = A \cup A^\ast(I) = X \). Hence we have \( D(X, \tau) \subset D(X, \tau^\ast(I)) \).

(4) \( \implies \) (11) We have \( \text{scl} A \subset \tau^\ast(I) \)-cl \( A \) for each \( A \subset X \). Since \( \text{scl} A = A \cup A^\ast \), the result is clear.

(11) \( \implies \) (3) Under the hypothesis of (11) we obtain \( A^\ast = \tau^\ast(I) \)-cl \( A^\ast \subset \tau^\ast(I) \)-cl \( A \) and \( \tau^\alpha \)-cl \( A = A \cup A^\ast \subset \tau^\ast(I) \)-cl \( A \) for each subset \( A \). Hence we have \( \tau^\ast(I) \subset \tau^\alpha \).

(4) \( \implies \) (12) We know that each \( I \in I \) is \( \tau^\ast(I) \)-closed. Since \( \tau^\ast(I) \subset SO(X, \tau) \), it follows that each \( I \in I \) is \( \tau \)-semiclosed.

(12) \( \implies \) (4) If \( I \subset \sc(X, \tau) \), then \( U - I \in SO(X, \tau) \) for any \( U \in \tau \) and any \( I \in I \). So, \( SO(X, \tau) \) contains a base of \( \tau^\ast(I) \) (from Theorem 1.1.(10)). Hence \( \tau^\ast(I) \subset SO(X, \tau) \). ■

Corollary 2.11. If \( I \subset I_n \), then we have the following results.

(a) \( I_n(\tau^\ast(I)) = I_n(\tau) \),
(b) \( \sigma \in [\tau]^\alpha \) \iff \( \sigma \in [\tau^\ast(I)]^\alpha \),
(c) If \( \sigma^\ast(I) = \tau^\ast(I) \), then \( \tau, \tau^\ast(I) \) and \( \sigma^\ast(I) \) are all \( \alpha \)-equivalent.
Some other statements equivalent to $I \subset I_n$ can be seen from Corollary 2.13. and Corollary2.15.

If $A$ is a supratopology and $I$ an ideal on X, then it is clear that $A \cap I = \emptyset$ iff $A \text{-int } I = \emptyset$ for each $I \in I$. In the following theorem, the results are clear and almost all of them are known.

**Theorem 2.12.** Let $(X, \tau)$ be a topological space. Then we have the following results for any $A \subset X$.

1. $A^\sharp = \emptyset \iff \text{pre-int } A = \emptyset \iff A^\sharp = \emptyset \iff \text{semi-pre-int } A = \emptyset$,
2. $A^\sharp = \emptyset \iff \alpha\text{-int } A = \emptyset \iff A^* = \emptyset \iff \text{semi-int } A = \emptyset$,
3. $A^\sharp = X \iff \text{pre-cl } A = X \iff A^\sharp = X \iff \text{semi-pre-cl } A = X$,
4. $A^\sharp = X \iff \alpha\text{-cl } A = X \iff A^\sharp = X \iff \text{scl } A = X$.

Clearly, in a topological space $(X, \tau)$, for any $x \in X$, $\{x\} \notin I$ iff pre-int \{$x\} \neq \emptyset$ iff semi-pre-int $\{x\} \neq \emptyset$ iff $\{x\}$ is semi-pre-open.

In the following corollary we assume that the necessary ideals are defined on the topological space $(X, \tau)$.

**Corollary 2.13.** We have the following results.

1. $I_n = \{A : A^\sharp = \emptyset\} = \{A : \text{pre-int } A = \emptyset\} = \{A : \text{semi-pre-int } A = \emptyset\} = \{A : A^\sharp = \emptyset\} = \{A : A^\sharp = \emptyset\} = \{A : A^\sharp = \emptyset\} = CD(X, \tau) \cap SC(X, \tau)$.
2. $I_n \cap PO(X, \tau) = \{\emptyset\}$ and $I_n \cap SO(X, \tau) = \{\emptyset\}$.
3. $I \subset I_n$ iff $I \cap PO(X, \tau) = \{\emptyset\}$ or $I \cap SO(X, \tau) = \{\emptyset\}$.
4. For $\sigma \in \text{Top}(X)$, if $PO(X, \sigma) \subset PO(X, \tau)$ or $SO(X, \sigma) \subset SO(X, \tau)$, then $I_n(\sigma) \subset I_n(\tau)$.
5. $CD(X) = \{A : A^\sigma = \emptyset\} = \{A : A^\sharp = \emptyset\} = \{A : \alpha\text{-int } A = \emptyset\} = \{A : \text{semi-int } A = \emptyset\} = \{A : A^\sharp = \emptyset\} = \{A : A^\sharp = \emptyset\} = CD(X, \sigma) \subset CD(X, \tau)$ and $CD(X, \sigma) \subset CD(X, \tau)$.
6. For $\sigma \in \text{Top}(X)$, if $SO(X, \tau) \subset SO(X, \alpha)$ or $\tau^\alpha \subset \sigma^\alpha$, then $D(X, \sigma) \subset D(X, \tau)$ and $CD(X, \sigma) \subset CD(X, \tau)$.
7. (a) $\tau \cap I = \{\emptyset\}$ iff $SO(X, \tau) \cap I = \{\emptyset\}$ or $SO(X, \tau) \cap I = \{\emptyset\}$ or $\tau^\alpha \cap I = \{\emptyset\}$ or $\tau^\alpha \cap I = \{\emptyset\}$

(b) Let $\sigma \in \text{Top}(X)$ and $\sigma^\alpha(\tau) = \tau^\alpha(\tau)$, then we have that $\tau \cap I = \{\emptyset\}$ iff $\sigma \cap I = \{\emptyset\}$.
(c) Let $\sigma \in \text{Top}(X)$ and $\sigma \in [\tau^\alpha(\tau)]^\alpha$, then we have that $\tau \cap I = \{\emptyset\}$ iff $\sigma \cap I = \{\emptyset\}$.

Now, by combining the results given above with the following facts,

(i) If $\tau \subset \sigma \subset \tau^\alpha(\tau)$ for $\sigma \in \text{Top}(X)$, then $\sigma^\alpha(\tau) = \tau^\alpha(\tau)$,
(ii) If $\tau^\alpha(\tau) \subset \sigma \subset (\tau^\alpha(\tau))^\sigma = (\tau^\alpha(\tau))^\sigma(I_n(\tau^\alpha(\tau)))$ then $\sigma \in [\tau^\alpha(\tau)]^\alpha$,

we can obtain several conditions equivalent to $\tau \cap I = \{\emptyset\}$.

For a supratopology $A$ on $X$, $T_A$ will stand for the topology

$$\{U : A \in A \implies U \cap A \in A\}$$

We know that $\tau \subset T_{PO(X, \tau)}$, $\tau \subset T_{SO(X, \tau)}$, $\tau \subset T_{SO(X, \tau)}$ and $\tau \subset \tau^\alpha = T_{\tau^\alpha} [1-3]$. 

Theorem 2.14. Let \((X, \tau)\) be a topological space, \(I \in Id(X)\) and \(A\) a supratopology on \(X\). Then we have the following results.

1. If \(\tau \cap I = \{\emptyset\}\) and \(A^\circ \neq \emptyset\) for each \(A \in A - \{\emptyset\}\), then \(A \cap I = \{\emptyset\}\).
2. If \(A \cap I = \{\emptyset\}\) and \(\tau \subset T_A\), then
   - \((a)\) \(A \subset A^*\) for each \(A \in A\),
   - \((b)\) \(\bar{A} = A^* \subset \tau^*\) for each \(A \in A\).
3. \((a)\) If \(A^*(I) \neq \emptyset\) for each \(A \in A - \{\emptyset\}\), then \(A \cap I = \{\emptyset\}\).
   - \((b)\) If \(\tau \subset T_A\), then \(A \cap I = \{\emptyset\}\) iff \(A \subset A^*(I)\) for each \(A \in A\).
4. \(A \cap I_n = \{\emptyset\}\) iff \(A^* \neq \emptyset\) for each \(A \in A - \{\emptyset\}\).
5. If \(A \cap I_n = \{\emptyset\}\) and \(\tau \subset T_A\), then \(A \subset SPO(X, \tau)\).
6. \(PO(X, \tau) \subset A \subset SPO(X, \tau)\), then \(A \cap I_n = \{\emptyset\}\) iff \(I \subset I_n\) iff \(A \subset A^*(I)\) for each \(A \in A\).
7. \(I = \emptyset\) for each ideal \(I\) and for each \(I \in I\). Now, result is clear.

Proof. \((a)\) Let \(A \in A\) and \(x \in A\). If \(x \in U \in \tau\), then since \(\tau \subset T_A\), we have \(\emptyset \neq U \cap A \in A\). Hence, \(U \cap A \notin I\). So, we have \(x \in A^*\).

\((b)\) We know that \(A^*\) is \(\tau\)-closed, \(A^* \subset \bar{A}\) and \(\tau^*\) \(\subset \tau\). Result is clear from \((a)\).

\((3a)\) It is known that \(I^* = \emptyset\) for any ideal \(I\) and for each \(I \in I\). Now, result is clear.

\((3b)\) Clear from \((3a)\) and \((2a)\)

\((4)\) Clear from Corollary 2.13.

\((5)\) Let \(A \in A\). From \((3b)\) we have \(A \subset A^*(I_n) = A^*\). Hence, \(A \subset SPO(X, \tau)\).

\((6)\) Let \(PO(X, \tau) \subset A \subset SPO(X, \tau)\). If \(A \cap I = \{\emptyset\}\) then \(PO(X, \tau) \cap I = \{\emptyset\}\) and hence \(I \subset I_n\). If \(I \subset I_n\), then \(SPO(X, \tau) \cap I = \{\emptyset\}\) and hence \(A \cap I = \{\emptyset\}\).

If \(I \subset I_n\), then \(SPO(X, \tau) \cap I = \{\emptyset\}\). Now, from Corollary 2.15.(1) below, and since \(A \subset SPO(X, \tau)\), we have \(A \subset A^*(I)\) for each \(A \in A\).

If \(A \subset A^*(I)\) for each \(A \in A\), then we have \(A \subset A^*(I)\) for each \(A \in PO(X, \tau)\). So, from Corollary 2.15.(1), we have \(I \subset I_n\).

\((7)\) Let \(\tau \subset A \subset SO(X, \tau^*(I))\). If \(A \cap I = \{\emptyset\}\), then \(\tau \cap I = \{\emptyset\}\). If \(\tau \cap I = \{\emptyset\}\), then from Corollary 2.13(7) we have \(SO(X, \tau^*(I)) \cap I = \{\emptyset\}\). So, \(A \cap I = \{\emptyset\}\).

If \(\tau \cap I = \{\emptyset\}\), then from Corollary 2.15(2) below we have \(A \subset A^*(I)\) for each \(A \in SO(X, \tau^*(I))\). So, \(A \subset A^*(I)\) for each \(A \in A\).

If \(A \subset A^*(I)\) for each \(A \in A\), then from Theorem 1.1.(9) we have \(U \subset U^*\) for each \(U \in \tau\), and \(\tau \cap I = \{\emptyset\}\).

Corollary 2.15. We have the following results.

1. \(PO(X, \tau) \cap I = \{\emptyset\}\) iff \(A \subset A^*(I)\) for each \(A \in PO(X, \tau)\) iff \(A \subset A^*(I)\) for each \(A \in SPO(X, \tau)\).
2. \(\tau \cap I = \{\emptyset\}\) iff \(A \subset A^*(I)\) for each \(A \in SO(X, \tau)\).
each $A \in \tau^o$ iff $A \subset A^*(I)$ for each $A \in SO(X, \tau^*(I))$ iff $A \subset A^*(I)$ for each $A \in \tau^*(I)^o$.

Proof. The proofs are clear from Theorem 1.1.(2), Theorem 2.14(3b) and Theorem 1.1(5).

Corollary 2.16. We have the following results.
(1) If $PO(X, \tau) \cap I = \{\emptyset\}$, then $\tau$-$cl A = A^* = \tau^*$-$cl A$ for each $A \in SPO(X, \tau)$.
(2) If $\tau \cap I = \{\emptyset\}$, then $\tau$-$cl A = A^* = \tau^*$-$cl A$ for each $A \in SO(X, \tau^*(I))$.

Lemma 2.17. Let $(X, \tau)$ be topological space, $\mathcal{A}$ a supratopology on $X$ such that $\tau \subset \mathcal{T}_\mathcal{A}$. If $\mathcal{A}$-$int B = \emptyset$ for a subset $B$, then $(A \cap B)^- \subset (A - B)^-$ and $\bar{A} = (A - B)^-$ for each $A \in \mathcal{A}$.

Proof. Let $B \subset X$, $\mathcal{A}$-$int B = \emptyset$, $A \in \mathcal{A}$, $x \in (A \cap B)^-$ and $x \in U \in \tau$. Then $U \cap A \cap B \neq \emptyset$ and $U \cap A \neq \emptyset$. Since $\tau \subset \mathcal{T}_\mathcal{A}$, we have $\emptyset \neq U \cap A \in \mathcal{A}$. So $U \cap A \not\subset B$, and $(A - B) \cap U = A \cap U - B \neq \emptyset$. Hence $x \in (A - B)^-$.

This now gives $\bar{A} = (A - B)^- \cup (A \cap B)^- = (A - B)^-$. ■

Corollary 2.18. Let $I \in Id(X)$ and $\mathcal{A}$ a supratopology on $X$. If $\mathcal{A} \cap I = \{\emptyset\}$, then we have the following results.
(a) If $\tau \subset \mathcal{T}_\mathcal{A}$, then $\bar{A} = (A - I)^-$ for each $A \in \mathcal{A}$ and for each $I \in I$.
(b) If $\tau^*(I) \subset \mathcal{T}_\mathcal{A}$, then $\bar{A} = (A - I)^-$ and $\tau^*$-$cl A = \tau^*$-$cl (A - I)$ for each $A \in \mathcal{A}$ and for each $I \in I$.

Proof. (a) If $I \in I$ we obtain $\mathcal{A}$-$int I = \emptyset$. The proof is now clear from Lemma 2.17.
(b) If $\tau^* \subset \mathcal{T}_\mathcal{A}$ then we obtain $\tau \subset \tau^* \subset \mathcal{T}_\mathcal{A}$. The proof is now clear from Lemma 2.17. ■

Corollary 2.19. Let $(X, \tau)$ be a topological space and $I \in Id(X)$. Then we have the following results.
(1) If $PO(X, \tau) \cap I = \{\emptyset\}$, then $\bar{A} = (A - I)^- = A^* = \tau^*$-$cl A = (A - I)^*$ for each $I \in I$ and for each $A \in SPO(X, \tau)$.
(2) If $\tau \cap I = \{\emptyset\}$, then $\bar{A} = (A - I)^- = A^* = \tau^*$-$cl A = \tau^*$-$cl (A - I) = (A - I)^*$ for each $I \in I$ and for each $A \in SO(X, \tau^*(I))$.
(3) If $\tau \cap I = \{\emptyset\}$, then we have $\tau$-$scl A = \tau^*$-$scl A$ for each $I \in I$ and for each $A \in SO(X, \tau^*(I))$.

Proof. (1),(2) Clear from Corollary 2.16, Corollary 2.18 and Theorem 1.1.(15).
(3) From (2) we will have $A^z = \tau$-$int (\tau^*$-$cl A) = \tau^*$-$int (\tau^*$-$cl A)$ and $\tau$-$scl A = A \cup A^z = A \cup \tau$-$int (\tau^*$-$cl A) = \tau^*$-$scl A$. ■

REFERENCES
Relations between some topologies


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