SIGNED DEGREE SETS IN SIGNED 3-PARTITE GRAPHS

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Abstract. If each edge of a 3-partite graph is assigned a positive or a negative sign then it is called a signed 3-partite graph. Also, signed degree of a vertex $x$ in a signed 3-partite graph is the number of positive edges incident with $x$ less than the number of negative edges incident with $x$. The set of distinct signed degrees of the vertices of a signed 3-partite graph is called its signed degree set. In this paper, we prove that every set of $n$ integers is the signed degree set of some connected signed 3-partite graph.

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graph is given by Harary [3]. Let $G$ be a signed graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The signed degree of $v_i \in V$ is $sdeg(v_i) = d_i^+ - d_i^-$, where $d_i^+$ ($d_i^-$) is the number of positive (negative) edges incident with $v_i$. A signed degree sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of a signed graph $G$ is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \ldots, d_n]$ is standard sequence if $\sigma$ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geq |d_n|$.

Chartrand et al. [1] obtained the necessary and sufficient conditions for an integral sequence to be s-graphical, which is similar to Hakimi’s result for degree sequences in graphs [2]. Another necessary and sufficient conditions for an integral sequence to be the signed degree sequence of a signed graph is given by Yan et al. [8].

The set of distinct signed degrees of the vertices of a signed graph is called its signed degree set. Pirzada et al. [6] proved that every set of positive(negative) integers is the signed degree set of some connected graph and determined the smallest possible order for such a signed graph.

A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let $G(U,V,W)$ be a signed 3-partite graph with

AMS Subject Classification: 05C22

Keywords and phrases: Signed graph, signed tripartite graph, signed degree, signed set.
$U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Then, signed degree of $u_i$ is $\text{sdeg} (u_i) = d_i = d_i^+ - d_i^-$, where $d_i^+$ ($d_i^-$) is the number of positive (negative) edges incident with $u_i$, signed degree of $v_j$ is $\text{sdeg} (v_j) = e_j = e_j^+ - e_j^-$, where $e_j^+$ ($e_j^-$) is the number of positive (negative) edges incident with $v_j$, and signed degree of $w_k$ is $\text{sdeg} (w_k) = f_k = f_k^+ - f_k^-$, where $f_k^+$ ($f_k^-$) is the number of positive (negative) edges incident with $w_k$. Then, the sequences $\alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q]$ and $\gamma = [f_1, f_2, \ldots, f_r]$ are called the signed degree sequences of $G(U, V, W)$. Also, the sequences of integers $\alpha$, $\beta$ and $\gamma$ are said to be standard sequences if $\alpha$ is non-zero and non-increasing, $\sum_{i=1}^p d_i + \sum_{j=1}^q e_j + \sum_{k=1}^r f_k$ is even, $d_1 > 0$, each $|d_i| \leq q + r$, each $|e_j| \leq r + p$, each $|f_k| \leq p + q$, $|d_1| \geq |e_j|$ and $|d_1| \geq |f_k|$ for each $j$ and $k$.

The following result is given by Pirzada et al. [5].

**Theorem 1.1.** Let $\alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q]$ and $\gamma = [f_1, f_2, \ldots, f_r]$ be standard sequences. Then, $\alpha$, $\beta$ and $\gamma$ are the signed degree sequences of a signed 3-partite graph if and only if there exists integers $g$ and $h$ with $d_1 = g - h$ and $0 \leq h \leq \frac{q + r - d_1}{2}$ such that $\alpha', \beta'$ and $\gamma'$ are the signed degree sequences of a signed 3-partite graph, where $\alpha'$ is obtained from $\alpha$ by deleting $d_1$ and $\beta'$ and $\gamma'$ are obtained from $\beta$ and $\gamma$ by reducing $g$ greatest entries of $\beta$ and $\gamma$ by 1 each and adding $h$ least entries of $\beta$ and $\gamma$ by 1 each.

The characterization of signed degree sequences in signed bipartite graphs can be found in [7]. That every set of integers is the signed degree set of some connected signed bipartite graph is proved in [4].

For any two sets $X$ and $Y$, we denote by $X \oplus Y$ to mean that each vertex of $X$ is joined to every vertex of $Y$ by a positive edge.

### 2. Main Results

A signed 3-partite graph $G(U, V, W)$ is said to be connected if each vertex of one partite set is connected to every vertex of other partite sets. The set $S$ of distinct signed degrees of the vertices of a signed 3-partite graph $G(U, V, W)$ is called its signed degree set.

First, we obtain the following result which shows that every set of positive integers is a signed degree set of some connected signed 3-partite graph.

**Theorem 2.1.** Let $d_1, d_2, \ldots, d_n$ be positive integers. Then, there exists a connected signed 3-partite graph with signed degree set $S = \{s_i| i = 1, 2, \ldots, n\}$, where $s_i = \sum_{j=1}^i d_j$.

**Proof.** If $n = 1$, $d_1 = 1$ then a signed 3-partite graph $G(U, V, W)$ with $U = \{u_1, u_2\}$, $V = \{v_1, v_2, v_3\}$, $W = \{w_1\}$ and in which $u_1v_1, u_2v_2, u_2v_3, v_1w_1$ are positive edges and $u_2v_1$ is negative edge has signed degree set $S = \{d_1\}$.

Also, if $n = 1$, $d_1 > 0$, then a signed 3-partite graph $G(U, V, W)$ with $|U| = 1, |V| = d_1 - 1, |W| = d_1, U \oplus W$ and $V \oplus W$ has signed degree set $S = \{d_1\}$. 
Now, assume that \( n \geq 2 \). Construct a signed 3-partite graph \( G(U, V, W) \) as follows.

Let \( U = X_1 \cup X_2 \cup \cdots \cup X_{n-1} \cup X_n, V = Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1}, W = Z_1 \cup Z_2 \cup Z'_1 \cup \cdots \cup Z'_{n-1} \cup Z_n \cup Z'_n \), with \( X_i \cap X_j = \phi, Y_i \cap Y_j = \phi, Z_i \cap Z_j = \phi, \)

\( Z_i \cap Z'_j = \phi, Z'_i \cap Z'_j = \phi \) (\( i \neq j \)), \( |X_i| = |Z_i| = d_i \) for all \( i, 1 \leq i \leq n \), \( |Y_i| = |Z'_i| = d_{i+1} \) for all \( i, 1 \leq i \leq n-1 \), \( X_i \cap Z_j \) whenever \( i \geq j \), \( Y_i \cap Z_{i+1} \) for all \( i, 1 \leq i \leq n-1 \), \( Y_i \cup Z'_{i+1} \) for all \( i, 1 \leq i \leq n-1 \). Then, the signed degrees of the vertices of \( G(U, V, W) \) are as follows.

For \( 1 \leq i \leq n \), \( sdeg(x_i) = \sum_{j=1}^n |Z_j| = \sum_{j=1}^n d_j = d_1 + d_2 + \cdots + d_i \), for all \( x_i \in X_i; \) for \( 1 \leq i \leq n-1 \), \( sdeg(y_i) = |Z_{i+1}| + |Z'_{i+1}| = d_{i+1} + d_1 + d_2 + \cdots + d_i \), \( d_1 + d_2 + \cdots + d_i + d_{i+1} \) for all \( y_i \in Y_i; \) for \( 1 \leq i \leq n \), \( sdeg(z_i) = (\sum_{j=1}^n |X_j|) + |Y_i| = \sum_{j=1}^n d_j + d_1 + d_2 + \cdots + d_{i-1} = d_1 + d_2 + \cdots + d_n \), \( z_i \in Z_i \); and for \( 2 \leq i \leq n \), \( sdeg(z'_i) = |Y_{i-1}| + |Z'_{i-1}| = d_1 + d_2 + \cdots + d_{i-1} \), for all \( z'_i \in Z'_i \).

Therefore, signed degree set of \( G(U, V, W) \) is \( S \). Clearly, all the signed 3-partite graphs constructed above are connected. Hence, the result.

The next result follows from Theorem 2.1 by interchanging positive edges with negative edges.

**Corollary 2.1.** Every set of negative integers is a signed degree set of some connected signed 3-partite graph.

Now we have the following result.

**Theorem 2.2** Every set of integers is a signed degree set of some connected signed 3-partite graph.

**Proof.** If \( S \) is a set of integers, then we have the following cases.

(i) \( S \) is a set of positive (negative) integers. Then, by Theorem 2.1 (Corollary 2.1), the result follows.

(ii) \( S = \{0\} \). Then, a signed 3-partite graph \( G(U, V, W) \) with \( U = \{u_1\}, V = \{v_1, v_2\}, W = \{w_1\} \) and in which \( u_1v_1, v_2w_1 \) are positive edges and \( u_1v_2, v_1w_1 \) are negative edges has signed degree set \( S \).

(iii) \( S \) is a set of non-negative (non-positive) integers. Let \( S = S_1 \cup \{0\} \), where \( S_1 \) is a set of negative (positive) integers. Then, by Theorem 2.1 (Corollary 2.1), there is a connected signed 3-partite graph \( G_1(U_1, V_1, W_1) \) with signed degree set \( S_1 \). Construct a new signed 3-partite graph \( G(U, V, W) \) as follows.

Let \( U = U_1 \cup \{u\}, V = V_1 \cup \{v\}, W = W_1 \cup \{w\} \), with \( U_1 \cap \{u\} = \phi, V_1 \cap \{v\} = \phi, W_1 \cap \{w\} = \phi \) and let \( uv, v_1w \) be positive edges and \( uw_1, v_1w \) be negative edges, where \( v_1 \in V_1 \). Then, \( G(U, V, W) \) has signed degree set \( S \). Also, note that addition of such edges do not effect the signed degrees of the vertices of \( G_1(U_1, V_1, W_1) \), and the vertices \( u, v, w \) have signed degrees zero each.

(iv) \( S \) is a set of non-zero integers. Let \( S = S_1 \cup S_2 \), where \( S_1 \) and \( S_2 \) are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary
2.1, there are connected signed 3-partite graphs $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$ with $S_1$ and $S_2$ respectively.

Construct a new signed 3-partite graph $G(U, V, W)$ as follows.

Let $U = U_1 \cup U_2, V = V_1 \cup V_2, W = W_1 \cup W_2$ with $U_1 \cap U_2 = \phi, V_1 \cap V_2 = \phi, W_1 \cap W_2 = \phi$ and let $u_1v_2, u_2w_1, v_1w_2$ be positive edges and $u_1w_2, u_2v_1, w_1v_2$ be negative edges where $u_i \in U_i, v_i \in V_i, w_i \in W$. Then, signed degree set of $G(U, V, W)$ is $S$. Clearly addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$.

(v) $S$ is a set of all integers. Let $S = S_1 \cup S_2 \cup \{0\}$, where $S_1$ and $S_2$ are sets of positive and negative integers respectively. Then, by Theorem 2.1 and Corollary 2.1, there exist connected signed 3-partite graphs $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$ with signed degree sets $S_1$ and $S_2$ respectively. Construct a new signed 3-partite graph $G(U, V, W)$ as follows.

Let $U = U_1 \cup U_2 \cup \{u\}, V = V_1 \cup V_2, W = W_1 \cup W_2$ with $U_1 \cap U_2 = \phi, U_1 \cap \{u\} = \phi, V_1 \cap V_2 = \phi, W_1 \cap W_2 = \phi$ and let $u_1w_1, u_2w_1$ be positive edges, where $u_2 \in U_2, v_1 \in V_1, w_1 \in W_1$. Then, signed degree set of $G(U, V, W)$ is $S$. We note that addition of such edges do not effect the signed degrees of the vertices of $G_1(U_1, V_1, W_1)$ and $G_2(U_2, V_2, W_2)$, and the signed degree of $u$ is zero. Clearly, by construction, all the above signed 3-partite graphs are connected. This proves the result. ■

REFERENCES


(received 13.02.2007, in revised form 10.07.2007)

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